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GEOMETRIC ASPECTS OF INVARIANTS OF FINITE TYPE OF KNOTS AND LINKS IN S^3

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In this paper we briefly review some well-known results on geometric properties of finite type invariants of knots and links in S^3 and announce several new results. We also formulate several open problems on the topics discussed.

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В данной работе делается краткий обзор известных результатов о геометрических свойствах инвариантов конечного типа узлов и зацеплений в S^3, а также анонсируются некоторые новые результаты. Сформулировано несколько открытых проблем относительно геометрии инвариантов конечного типа.

1. n-equivalent knots and "geometric" n-trivial knots

Here we consider oriented knots in S^3, up to a certain equivalence relation defined by ambient isotopy. Under a singular knot we shall mean an immersion of S^1 in $\mathbb{R}^3$ with only a finite number of transverse self-intersections called the singularities of this knot. The singular knots are considered up to the rigid vertex isotopy [4]. A Vassiliev invariant v of finite type n is an isotopy knot invariant which takes values in an abelian group Q and satisfies the two axioms [1]:

A1. For any three (singular) knots K_+, K_- and $K_\times$ which differ only inside some disc, where they look as in Fig. 1, there holds

$$v(K_+) - v(K_-) = v(K_\times);$$

A2. For any singular knot K with more than n singularities $v(K) = 0$.

The smallest number n for which v satisfies axiom A2 is called the order of v. The Vassiliev knot invariants are also called the invariants of finite type of knots. Two knots which cannot be distinguished by Vassiliev invariants (additive Vassiliev invariants) of order $\leq n$, the invariants taking values in any abelian group, are called V_n-equivalent (n-equivalent, respectively). It was an important and interesting problem in the knot theory to describe in combinatorial, algebraic or geometric terms the relationship between any two V_n-equivalent (n-equivalent) knots. Gusarov [7] was the first who described in combinatorial terms the notion of "n-equivalence" for knots. Moreover, Gusarov showed [7] that the classes of

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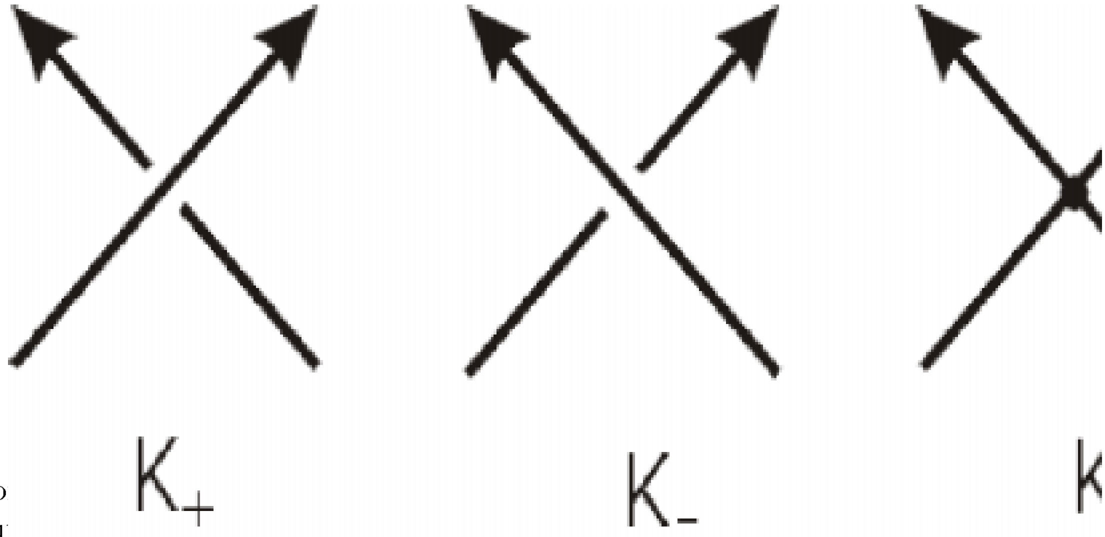
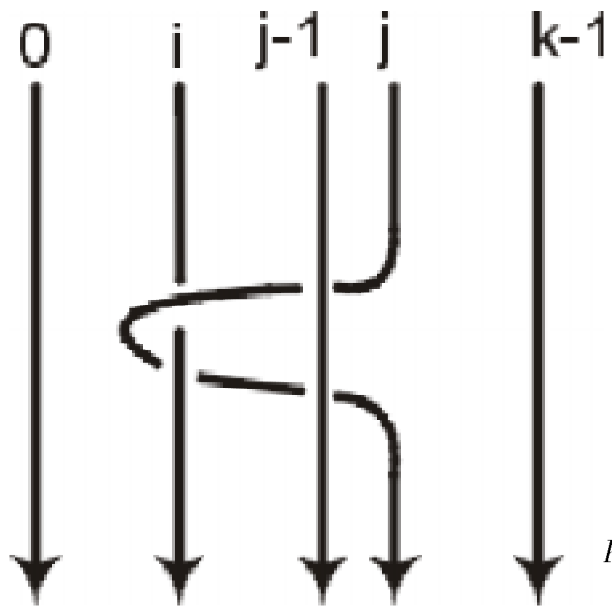


n-equivalent knots co
Later on, it turned ou
for the knots in S^3 [24]. Let us consider now another algebraic and combinatorial descriptions of V_n-equivalence for knots given by Stanford and Habiro.

Let B_k be the braid group on k strands and let P_k be its subgroup of pure braids. Denote by $\hat{b}$ the closure of a braid b in a standard way, via the permutation $(0, 1, \ldots, n-1)$. Let $LCS_n(P_k)$ be the n-th group of the lower central series of the group P_k. The knots K_1 and K_2 are said to be LCS_n-equivalent if there exists an integer $k > 0$ and the braids $p \in LCS_n(P_k)$ and $b \in B_k$ such that $K_1 = \hat{b}$ and $K_2 = \widehat{pb}$. Stanford showed (Theorem 0.2 of [24]) that any two knots are LCS_{n+1}-equivalent if and only if they are V_n-equivalent. Habiro [8] has described the V_n-equivalence in the terms of local moves on knots, called the C_n-moves. Two knots K and K' are called C_n-equivalent, where $n > 1$, if and only if one can pass from one to another by a finite sequence of C_n-moves and isotopies. Habiro showed that for each integer $n > 0$ any two knots K and K' in S^3 are V_n-equivalent if and only if they are C_{n+1}-equivalent. It follows (see Theorem 6.18 of [8]) that for each $n > 1$ the C_n-equivalence and LCS_n-equivalence relations coincide for knots in S^3. For $0 \le i < j \le k-1$ let $p_{i,j} \in P_k$ be the braid that links the ith and jth strands behind the others (see Fig. 2).



In [22], it is shown that each $K, n > 1$, is equivalent to a replacement in a knot the trivial braid 1_{n+1} with the pure braid n-commutator of the following particular form: $p_n = [p_{n-1,n}, [p_{n-2,n-1}, \dots, [p_{1,2}, p_{0,1}] \dots] \in P_{n+1}$.

It is still not clear what kind of geometric knot properties can be detected by the invariants of finite type. In [9], Kalfagianni and Lin interpret these invariants as obstructions to a knot's bounding a regular Seifert surface whose complement looks, modulo the lower central series of its fundamental group, like the complement of a null-isotopy. The purpose of the paper [9] was to give a geometric description of n-trivial knots (i.e. knots which are V_n-equivalent to a trivial knot). For this, Kalfagianni and Lin have defined for each $n \in \mathbb{N}$ several classes of geometric knots. Let us recall the definitions of some of them.

Let K be an oriented knot in S^3. A *Seifert surface* of K is an oriented, compact, connected surface S, embedded in S^3 and such that $\partial S = K$. A *spine* of S is a bouquet of circles $\Sigma \subset S$, which is a deformation retract of S. A Seifert surface S of a knot K is called *regular* if it has a spine Σ whose embedding in S^3, induced by the embedding $S \subset S^3$, is isotopic to the standard embedding of a bouquet of circles. Then Σ will be called a regular spine of S. We shall represent Seifert surfaces in the *disc-band form*, i.e. as a union of the disc D^2 and flat bands glued to D^2, to which some full twists are added if necessary. Pick a basepoint $p \in D^2 \subset S$, and let $\Sigma_n, n = 2g$, where g is the genus of S, be a regular spine of S such that p is the point of Σ_n where all circles in Σ_n meet. Denote by $\gamma_1, \beta_1, \dots, \gamma_g, \beta_g$ the circles in Σ_n (the cores of the bands), oriented so that they form a symplectic basis in $H_1(S)$. Let K be a knot in S^3 and let S be a Seifert surface of K of genus g. A collection of g non-separating, disjoint, simple closed curves $\gamma_1\gamma_2, \dots, \gamma_g$ in S is said to be a *half basis* if S, cut along these curves, is a disc with $2g$ holes.

Definition 1.1. (Kalfagianni and Lin, [9]) A regular Seifert surface S is called *n-hyperbolic*, if it has a half basis $\gamma_1\gamma_2, \dots, \gamma_g$, represented by circles in a regular spine Σ such that for every $i = 1, \dots, g$, either $[\gamma_A^+]$ or $[\gamma_A^-]$ lies in $\pi^{(n+2)}$, where $\pi^{(n+2)}$ is the $(n+2)$-th term of the lower central series of $\pi = \pi_1(S^3 \backslash S)$. The boundary of such a surface will be called an *n-hyperbolic knot.*

Theorem. (Kalfagianni and Lin, [9]). *If K is n-hyperbolic, for some $n \in \mathbb{N}$, then K is n-trivial.*

Kalfagianni and Lin also noticed [9] that 0-hyperbolic knots are 2-trivial, because they have the trivial Alexander polynomial. They asked whether each n-hyperbolic knot is $(n+2)$-trivial. This question has been negatively answered as follows.

Theorem. [23]. *For each odd $m \geq 1$ there exists an m-hyperbolic knot which is not $(m+2)$-trivial.*

Kalfagianni and Lin have also introduced for each $n \in \mathbb{N}$ another classes of "geometric" knots, called n-elliptic and n-parabolic knots, and proved that all such knots are n-trivial (Theorem 5.4 of [9]). Moreover, Kalfagianni and Lin showed that all n-elliptic and n-hyperbolic knots, $n \in \mathbb{N}$, have the trivial Alexander polynomial, while there is a 1-parabolic knot which has a non-trivial Alexander polynomial. The classes of n-hyperbolic, n-parabolic and n-elliptic knots however do not exhaust all n-trivial knots.

Problem 1. Are n-elliptic knots $(n+2)$- or $(n+1)$-trivial for each $n \geq 1$?

In [21], we consider Seifert surfaces for knots (not necessarily regular) in S^3 represented in the disc-band form and some specific moves on them, the band-analogies of insertions in knots of pure braids commutators. Such moves on Seifert surfaces allow, by analogy with LCS_n-equivalence, to introduce a new equivalence relation on knots, called LCS_n-band equivalence, where $n \geq 2$ (see [21]).

Proposition. (Corollary 3.1 of [21]). *If a knot K is LCS_n-band-equivalent to the trivial one, where $n \geq 2$, then K is $(n-1)$-trivial and has the trivial Alexander polynomial.*

The latter proposition yields a class of "geometric" $(n-1)$-trivial knots with the trivial Alexander polynomial.

Question 1. How does the class of "geometric" n-trivial knots described by Corollary 3.1 of [21] relate to the classes of n-hyperbolic and n-elliptic knots?

Conant and Teichner have given in [6] a geometric interpretation of knot invariants of finite type in terms of grope cobordism between two knots. The approach to the study of geometric properties of finite type of knots by means of grope cobordism turns out to be closely related to the Habiro theory of claspers in a 3-manifold. Moreover, the gropes in a 3-manifold, defined by Conant an Teichner, generalize in some sense the "cobordisms " on knots used by Kalfagianni and Lin in [9] for the description of the several classes of geometric n-trivial knots.

Note also that, using Habiro's theory of claspers, Murakami and Ohtsuki have given in [18] the filtration of the vector space $\mathcal{V}$ of rational Vassiliev invariants via Seifert matrices and described the rational Vassiliev knot invariants coming from the Alexander-Conway polynomial.

By definition, the Gusarov groups $\mathcal{G}_n$ are quotients of the semigroup of knots, so it is natural to ask what relationship these might have to the knot cobordism group $\mathcal{C}$, the other known quotient of the knot semigroup. It is known that the Arf invariant, which is a knot cobordism invariant, is the coefficient a_2 of the Conway polynomial modulo 2 (so it is the $\mathbb{Z}_2$-valued Vassiliev invariant of order 2) whereas the signature of a knot is not a Vassiliev invariant of any order (Dean and Trapp). Denote by $\mathcal{W}_n$ the torsion-free part of $\mathcal{G}_n$, i.e. the group of equivalence classes of knots which cannot be distinguished by rational Vassiliev invariants of order $\leq n$. Ng also showed [19] that any element of the group $\mathcal{W}_n$, whose Arf invariant is zero, can be represented by a slice knot. It follows that any finite number of rational Vassiliev invariants cannot distinguish between slice knots and non-slice knots having Arf invariant zero. Therefore, in fact, rational Vassiliev invariants and knot cobordism are not related to each other [20].

2. Geometric n-equivalence of links

There are several natural equivalence relations for oriented links in S^3 which are indicated below.

1) Links are considered in S^3 up to ambient (PL) isotopy of S^3;

2) Links are considered up to (PL)homotopy (in the sense of Milnor). In other words, two links are homotopic if one can be transformed into the other through a sequence of ambient isotopies of S^3 and crossing changes of component with itself (but *not* crossing changes of different components);

3) (non-locally flat) PL isotopy, i.e. ambient isotopy plus birth and death of local knots;

Notice also that there are at least two different concepts in defining the finite type invariants of links in S^3 following from the distinct notions of n-singular links;

a) in the first (usual) case, in addition to the self-intersections in each component, the intersections (singular points) between different components are allowed;

b) in the second case each component of a link may have self-intersections, but the different components do not intersect.

The finite type link invariants are defined in a similar way as for knots, via the corresponding Gusarov-Stanford decreasing filtrations of the abelian groups, freely generated by the equivalence classes of links, with the appropriate choice of an equivalence relation on links in S^3. As a result, we obtain at least six different definitions of link invariants of finite type combining items 1)-3) with the items describing the type singularities in a) or b), which are allowed in singular links. Notice that, in opposite to the situation with knots, neither combinatorial nor algebraic description of n-equivalence for links in S^3 is known in any of encountered cases, like the Habiro–Gusarov or Stanford characterizations of n-equivalent knots.

The link invariants of finite type in case 1a) have been studied by many authors. Among the other we would like to mention the paper [11]. The link invariants of finite type in case 2a), called also link homotopy invariants, have been studied by Lin [13], Mellor [14], Bar-Natan [2] and other authors. Denote by V_d (V_d^h) the vector space of $\mathbb{Q}$-valued finite type isotopy invariants (homotopy invariants, respectively) of type d. The main problem was to give a description of the quotients V_d/V_{d+1} (V_d^h/V_{d+1}^h, respectively) or, more generally, to describe the graded vector space associated with the filtered vector space V of rational finite type invariants of links, in both the cases. This has been done successfully in terms of uni-trivalent diagrams via the corresponding construction of the Kontsevich integral, like in the case of knots (see [11], in case 1a), and [3], in case of 2a)).

In particular, Mellor and Thurston [15] showed that for links with at most 5 components the only finite type homotopy invariants (i.e. link invariants of finite type defined by 1b)) are products of the linking numbers, whereas for links with at most 9 components there exist finite type invariants which are not products of the linking numbers (see also [13]). It follows that there exist finite type link concordance invariants other than the linking numbers.

The link invariants of finite type in case 1b) have been studied by Kirk and Livingston [10]. Kirk and Livingston used the Casson-Walker invariant of 3-manifolds to define a Vassiliev invariant λ of two-component links. More precisely, λ turns out to be a type 1 invariant of singular links on the space $\mathcal{L}_n$ of singular links of two disjoint components with linking number n. Moreover, λ is an isotopy link invariant of order 3 in the usual sense, i.e. in the case 1a) [10].

Recently S. Melikhov and D. Repovš [16, 17] have defined for each integer $k \geq 0$ an equivalence relation, called k-quasi-isotopy, on the set of oriented links in $\mathbb{R}^3$, as follows:

Definition 2.1. (Melikhov and Repovš, [17]). Let k be any nonnegative integer. A PL-map $f\colon S_1^1 \sqcup \cdots \sqcup S_m^1 \to \mathbb{R}^3$ with precisely one double point $f(p) = f(q), p, q \in S_i^1$ is called a *strong k-quasi-embedding*, if in addition to the singleton $B_0 = \{f(p)\}$ there is a sequence of closed PL 3-balls $B_1 \subset \cdots \subset B_k$ in the complement C to all other components $S_j^1, j \neq i$, such that each B_{n+1}, where $0 \leq n \leq k$, contains the f-image of an arc $J_n \subset S_i^1$ such that $J_n \supset f^{-1}(B_n)$. Also, all PL embeddings $f\colon S_1^1 \sqcup \cdots \sqcup S_m^1 \to \mathbb{R}^3$ are to be thought of as contained in the class of strong k-quasi-embeddings.

If the above balls B_n are replaced by arbitrary compact polyhedra $P_n \subset \mathbb{R}^3$, where $P_0 = \{f(p)\}$, such that each inclusion $P_n \cup f(J_n) \subset P_{n+1}$ induces the trivial homomorphism of fundamental groups, then f is called a *k-quasi-embedding*. The notions of k-quasi-embedding and strong k-quasi embedding can be thought of as arising from Penrose-Whitehead-Zeeman trick and the engulfing procedure connected with it. Replacing in the definition of k-quasi-embedding the induced homomorphism of fundamental groups by the induced homomorphism of the first homology groups, one obtains the definition of a *weak k-quasi-embedding.* Notice that the homomorphism of fundamental groups induced by the inclusion $P_j \cup J_j \subset P_n$ sends the group $\pi_1(P_j \cup J_j)$ into $LCS_{n-j}\pi(P_n)$.

Definition 2.2. (Melikhov and Repovš, [17]). Let $f_0, f_1 \colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^3$ be the two links. We say that they are *(weakly, strongly) k-quasi-isotopic*, if they are PL-homotopic through maps f_t with at most single transversal self-intersections of the components, all of which are (weak, strong) k-quasi-embeddings.

Note that the PL-homotopy in Definition 2.2 can be assumed to be locally flat since the self-intersections appeared when introducing a local knot can be performed by a PL-isotopy [17].

In this setting, 0-quasi-isotopy coincides with certain link homotopy, whereas 1-quasi-isotopy does not follow from the link concordance. Therefore, k-quasi-isotopy is not completely described by the lower central series quotients of the fundamental group [17].

The following assertion clarifies relationship between the notions of k-quasi-isotopy and topological isotopy of links:

Corollary 2.1. (Melikhov and Repovš, [16])

(a) *All sufficiently close approximations of any topological link are strongly k-quasi-isotopic for each $k \geq 1$;*

(b) *Topologically isotopic PL links (i.e. in the sense of Milnor) are strongly k-quasi-isotopic for all finite k.*

Consider the setting of finite type invariants introduced as in [10] (case 1b)). Let $\mathcal{LM}$ (respectively, $\mathcal{LM}^m$) denote the subspace of the space of all link maps $f \colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^3$ where m is arbitrary (respectively, all link maps $f \colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^3$ with m fixed), with only singularities being transversal double points. Note that the only singularities of the same component are allowed here. Let $\mathcal{LM}^m_n$ (respectively, $\mathcal{LM}^m_{\geq n}$) denote its subspace consisting of link maps with precisely (respectively, at least) n singularities.

Given an invariant $v \colon \mathcal{LM}^m_0 \to G$ on embedding links, taking values in an abelian group G, it can be extended to $\mathcal{LM}^m$ inductively by the formula

$$v(L_s) = v(L_+) - v(L_-)$$

where $L_+, L_- \in \mathcal{LM}^m_n$ differ by a single crossing change, and $L_s \in \mathcal{LM}^m_{n+1}$ is the intermediate link map with one more singular point, defined as in the case of knots (see Fig.1). If v vanishes on $\mathcal{LM}^m_{k+1}$ for some k, then v is called of *finite type k invariant in $\mathcal{LM}$* (i.e. in the sense of Kirk and Livingston).

For $m = 1$ these coincide with the knot invariant of finite type in the usual sense (Vassiliev invariants), while for $m \geq 1$ any type k invariant in the usual sense (i.e. in the space $\mathcal{L}$ of all singular links, case 1a)) is a type k invariant in $\mathcal{LM}$ but not *vice versa.* For example, the linking number lk and the generalized Sato-Levine invariant $\tilde{\beta}$ are invariants of types 0 and 1 in $\mathcal{LM}$ but of types 1 and 3, respectively, in $\mathcal{L}$ [10]. Moreover all higher Milnor $\bar{\mu}$-invariants (except for lk) are not of finite type in $\mathcal{L}$, because they are even not well-defined on all links.

Note that the type k invariants $v\colon \mathcal{LM}_0^m \to G$ taking values in an abelian group G form an abelian group, denoted by G_k^m. It is known [10] that $G_1^2 \simeq \mathbb{Z}$ and conjectured that G_r^2 is not finitely generated for $r > 1$, in contrast to the situation in the case of usual Vassiliev link invariants i.e. in case 1a). Denote by $\tilde{G}_k^m$ the subgroup of G_k^m consisting of invariants which remain unchanged under tying local knots, i.e. descent to PL-isotopy invariants. In particular, $\tilde{\mathbb{Q}}_k^m$ denotes the subspace of the vector space $\mathbb{Q}_k^m$ of rational Vassiliev invariants which descent to PL-isotopy invariants.

Problem 2. Describe the quotient vector spaces $\tilde{\mathbb{Q}}_k^m/\tilde{\mathbb{Q}}_{k+1}^m$ in combinatorial terms, like the finite type invariants in cases 1a)-2a)

For each $n > 0$, let $\mathcal{LM}_{n,0}^m$ denote the subspace of $\mathcal{LM}_n^m$ consisting of the link maps l such that all singularities of l are contained in a ball B such that $l^{-1}(B)$ is an arc. The link maps $l, l' \in \mathcal{LM}_n^m$ are called *geometrically k-equivalent* ([17]) if they are homotopic in the space $\mathcal{LM}_n^m \cup \mathcal{LM}_{n+1,k}^m$, where $\mathcal{LM}_{i,k}^m$ for $k > 0, i > 0$, is the space of all links maps with i singularities which are geometrically $(k-1)$-equivalent to a link map in $\mathcal{LM}_{i,0}^m$.

Theorem 2.1. (Melikhov and Repovš, [17]) *If two links L and L' are k-quasi-isotopic then any Vassiliev invariant of type $\leq k$ with respect to the space of link maps, which is well defined up to PL-isotopy, has the same values on L and L'.*

The latter theorem shows that any two geometrically k-equivalent links are k-equivalent with respect to the Vasiliev invariants in the sense of Kirk and Livingston, well defined up to PL-isotopy.

Problem 3 (Melikhov and Repovš, [17]). Is the converse to the above theorem true?
Now we concentrate on the relationship between k-quasi-isotopy and some classical geometric invariants of links in S^3. First, recall that Cochran's invariants [5, 17] are defined inductively in the following way. Let $L = K_+ \cup K_-$ be a two-component link with lk(L) $= 0$ and let K_*, where $*$ stands for either '+' or '−', be a fixed component, which is called *active* and the other one is called *passive*. Let $D_*(L)$ be the $*$-derivative of L defined by substituting K_* with the transversal intersection of oriented Seifert surfaces of the components in the link exterior, provided that it is a connected curve (the latter can be always achieved). Then $\beta^1{}_*(L) = \beta(L)$, the Sato-Levin invariant, and $\beta_*^{i+1}(L)$ is defined to be $\beta_*^i(D_*(L))$, $i \in \mathbb{N}$. Therefore, the two invariants, β_+^i and β_-^i, are defined for each i.

Now let us mention the definition of higher $\bar{\mu}$-invariants, introduced by J.Milnor. Given a link L, its link group $\pi(S^3 - L)$ has a Wirtinger presentation, generated by the arcs of the link diagram. We also have a presentation of the link group modulo qth subgroup of its lower central series:

$$\pi_1(S^3 - L)/LCS_q\pi_1(S^3 - L) = \langle m_i | m_i l_i m_i^{-1} l_i^{-1} = 1, A_q\rangle$$

where the generators are the meridians m_i of the components of the link, l_i denote the longitudes of the components of the link, and A_q denotes the qth subgroup in the lower central series of the free group on $\{m_i\}$. Each generator of the Wirtinger presentation (hence each longitude) can be written in $\pi_1(S^3 - L)/LCS_q\pi_1(S^3 - L)$ as a word in the m_i's. Next, we look at the Magnus expansion of the longitudes, which means replacing m_i with $1 + K_i$ and m_i^{-1} with $1 - K_i + K_i^2 - \dots$. We define $\mu(i_1, \dots, i_n, j)$ to be the coefficient of $K_{i_1} \dots K_{i_n}$ in the Magnus expansion of the word for jth longitude in $\pi_1(S^3 - L)/LCS_q\pi_1(S^3 - L)$, $q > n$. Now we define $\bar{\mu}_(i_1, \dots, i_n, j)$ to be $\mu(i_1, \dots, i_n, j)$, considered modulo Δ, which is the greatest common divisor of all μ-invariants whose indices are a cyclic permutation of a proper subsequence of $(i_1 \dots i_n, j)$. This is now a well-defined invariant of links up to concordance, as long as $q > n$. If the indices $i_1, \dots, i_n, j$ are all distinct, this is a well-defined link-homotopy invariant.

Corollary 2.2. (Melikhov and Repovš, [17]) *$\bar{\mu}$-invariants of length $\leq 2k+3$ are invariant under k-quasi-isotopy.*

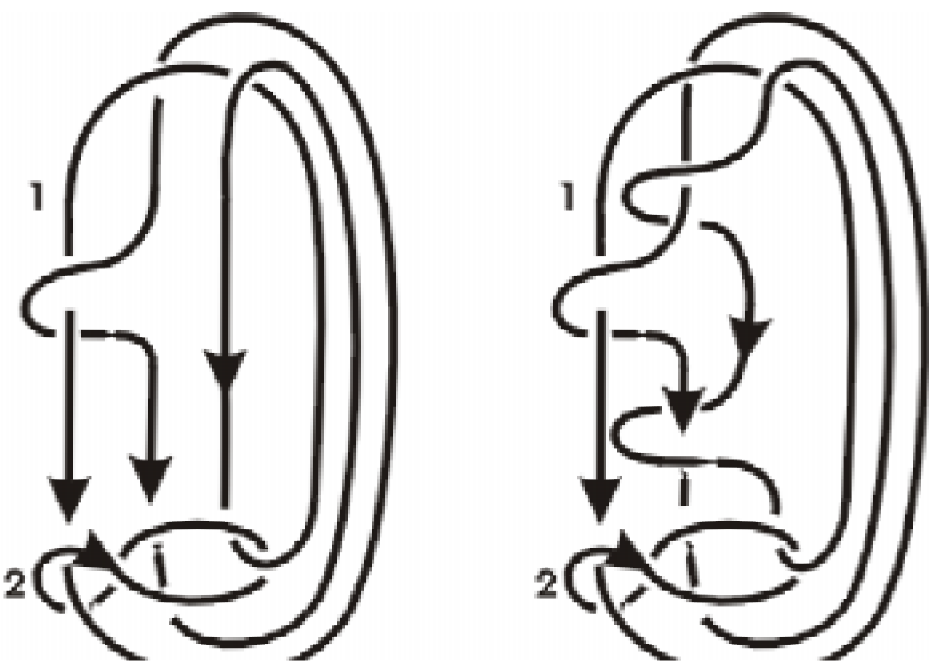
It follows that, in the terminology of [12], geometrically k-equivalent links are $(k+1)$-cobordant.

Corollary 2.3. (Melikhov and Repovš, [17]) *Cochran's invariants $\beta i, i \leq k$, of each two-component sublink with vanishing linking number are invariant under k-quasi isotopy.*

It turns out that Problem 3 has the negative solution. To show this let us consider the two 2-components links L_1 and L_2 indicated in Fig. 3, with the components denoted by 1 and 2. One can easily verify that L_1 and L_2 are 1-equivalent in the sense of Kirk and Livingston, well defined up to PL isotopy, because component 1 of the link L_2 is obtained from component 1 of L_1, by applying to it in $S^3 \setminus \{2\}$ a simple C_2-move [22]. On the other hand, the direct computation shows that L_1 and L_2 have the different Milnor $\bar{\mu}$-invariants of length 5. More precisely, we have $\bar{\mu}(1,1,1,2,2)(L_2) = -2, \bar{\mu}(1,1,2,1,2)(L_2) = 6, \bar{\mu}(1,2,1,1,2)(L_2) = -6, \bar{\mu}(2,1,1,1,2)(L_2) = 2$, while all Milnor $\bar{\mu}$-invariants of length 5 of the link L_1 and all Milnor $\bar{\mu}$-invariants of length ≤ 4 of the link L_2 vanish. Therefore, by Corollary 2.2, L_1 and L_2 are not geometrically equivalent in the sense of Melikhov and Repovš. Note that L_1 and L_2 are homotopy equivalent, so they have the same Milnor $\bar{\mu}$-invariants of any length with distinct indices. The details of the proof and some generalization of the above example will be treated in a forthcoming paper.

Finally we formulate an open problem concerning the relationship between n-equivalence and geometrical n-equivalence (n-quasi-isotopy) of links.

Problem 4. Find conditions on links under which two n-equivalent in the sense of Kirk and Livingston links L_1 and L_2, well defined up to PL-isotopy, are geometrically n-equivalent in the sense of Melikhov and Repovš.



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