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ON A CONSTRUCTION OF SOME CLASSES OF EXACT SOLUTIONS FOR A MODIFIED HIGGS MODEL

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Some modifications of nonlinear Higgs model are considered. The exact solutions are constructed in explicit form.

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Рассмотрены некоторые модификации нелинейной модели Хигса. Точные решения этих моделей найдены в явной форме.

1. Introduction

In this paper we consider methods of construction of some classes of exact solutions of nonlinear integrable models and an application of these methods to an example of modifications of known in quantum physics Higgs models [1].

In this section we briefly introduce basic notations.

Consider the algebra ζ of microdifferential operators [2] above the field $\mathbb{C}$,

$$\zeta := \left\{ A = \sum_{i=-\infty}^{n(A)} a_i \mathcal{D}^i : a_i = a_i(x,y,t),\ i, n(A) \in \mathbb{Z} \right\},$$

where the coefficients a_i are, in general, smooth $N \times N$ matrix-valued functions of the variables x, y, t. The operation of multiplication in the algebra ζ is induced by the generalized Leibnitz rule,

$$\mathcal{D}^n f := \sum_{j=0}^{\infty} \binom{n}{j} f^{(j)} \mathcal{D}^{n-j},\ n \in \mathbb{Z}, \quad f^{(j)} := \frac{\partial^j f}{\partial x^j},$$

$$\mathcal{D}^n \mathcal{D}^m := \mathcal{D}^m \mathcal{D}^n := \mathcal{D}^{n+m},\ n, m \in \mathbb{Z},$$

where the symbol f denotes the operator of multiplication $f\mathcal{D}^0$ by the function f from some functional space $\mathcal{A}$. The structure of Lie algebra in ζ is determined by the commutator $[\cdot,\cdot]\colon \zeta \times \zeta \to \zeta, [L_1, L_2] = L_1 L_2 - L_2 L_1$.

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Let an operator $W := \sum\limits_{i=0}^{\infty} w_i \mathcal{D}^{-i}$ be an element of the algebra ζ and the inverse operator W^{-1} exists. We shall call W a *dressing operator.* Further we shall consider an operator L of the type $L = \alpha\partial_y - \sum_{i=-\infty}^{n(L)} a_i\mathcal{D}^i$, where y is an evolution parameter. The operation of transposing of the operator L is defined as follows $L^\tau := -\alpha\partial_y - \sum_{i=-\infty}^{n(L)}(-1)^i\mathcal{D}^i a_i^\top$. The operator $L^* := \overline{L^\tau}$ is hermitian-conjugated for L.

Definition 1. We say that the operator L is σ-symmetric (σ-skew-symmetric), if $L^\tau = \pm\sigma L\sigma^{-1}$, where $\sigma \in \mathrm{Mat}_{N\times N}(\mathbb{C})$.

Remark 1. If the operator $L = L_+ := \alpha\partial_y - \sum_{i=0}^{n(L)} a_i\mathcal{D}^i$ is σ-symmetric (σ-skew-symmetric), then f is a solution of the equation $L(f) = 0$ if and only if $g := \sigma f$ is a solution of the equation $L^\tau(g) = 0$.

Definition 2. Let σ be a symmetric matrix ($\sigma = \sigma^\top$), then we say that a dressing operator W is *σ-orthogonal* if $W^{-1} = \sigma^{-1}W^\tau\sigma$.

We introduce the following notations. Let φ, ψ be smooth $(N \times K)$ matrix-valued functions of real variable $x \in \mathbb{R}$ and the improper integral $\int_{-\infty}^{s} \psi^\top\varphi ds$ converges absolutely $\forall s \in \mathbb{R}$. We define nonlinear nonlocal transformation $(\varphi,\psi) \to (\Phi,\Psi)$: $\Phi := \varphi\Omega^{-1}$, $\Psi^\top := \Omega^{-1}\psi^\top$, where $\Omega = C + \int_{-\infty}^{x}\psi^\top\varphi ds$, C is some complex $K \times K$ matrix.

Theorem 1. [3] *Let an operator W be given by the formula*

$$W = I - \varphi(C + \int_{-\infty}^{x}\psi^\top\varphi ds)^{-1}\mathcal{D}^{-1}\psi^\top = I - \Phi\mathcal{D}^{-1}\psi^\top, \tag{1}$$

then the inverse operator is defined as follows

$$W^{-1} = I + \varphi\mathcal{D}^{-1}\left(C + \int_{-\infty}^{x}\psi^\top\varphi ds\right)^{-1}\psi^\top = I + \varphi\mathcal{D}^{-1}\Psi^\top.$$

Proposition 1. *Operator (1) is σ–orthogonal if and only if $\psi = \sigma\varphi$ and matrix C is symmetric.*

Proof. $(\Rightarrow)$

$$W^{-1} = I + \varphi\mathcal{D}^{-1}(C + \int_{-\infty}^{x}\psi^\top\varphi ds)^{-1}\psi^\top,$$
$$W^\tau = I + \psi\mathcal{D}^{-1}(C^\top + \int_{-\infty}^{x}\varphi^\top\psi ds)^{-1}\varphi^\top,$$
$$\sigma^{-1}W^\tau\sigma = W^{-1} \Rightarrow \psi = \sigma\varphi, C = C^\top.$$

$(\Leftarrow)$

$$W^{-1} = I + \varphi\mathcal{D}^{-1}(C + \int_{-\infty}^{x}\varphi^\top\sigma^\top\varphi ds)^{-1}\varphi^\top\sigma^\top,$$
$$W^\tau = I + \psi\mathcal{D}^{-1}(C^\top + \int_{-\infty}^{x}\varphi^\top\psi ds)^{-1}\varphi^\top =$$
$$= I + \sigma\varphi\mathcal{D}^{-1}(C^\top + \int_{-\infty}^{x}\varphi^\top\sigma\varphi ds)^{-1}\varphi^\top \Rightarrow \sigma^{-1}W^\tau\sigma = W^{-1}.$$

□

Theorem 2. *Let the following conditions hold:*

a. *The operator L is σ–symmetric (σ-skew-symmetric).*

b. *The operator W is σ–orthogonal.*

Then the operator $\hat{L} := WLW^{-1}$ is σ–symmetric (σ–skew-symmetric).

Proof.

$$\hat{L}^\tau = (W^{-1})^\tau L^\tau W^\tau = (\sigma^{-1}W^\tau\sigma)^\tau L^\tau W^\tau =$$
$$= \sigma^\top W \sigma^{\top -1}\sigma L \sigma^{-1} W^\tau = \sigma^\top W L \sigma^{-1} W^\tau = \sigma^\top W L W^{-1}\sigma^{-1} \Rightarrow \hat{L}^\tau = \sigma \hat{L}\sigma^{-1}.$$

□

Definition 3. We say that an operator L is *σ-hermitian (σ-skew-hermitian)*, if $L^* = \pm\sigma L\sigma^{-1}$, where $\sigma \in \mathrm{Mat}_{N\times N}(\mathbb{C})$.

Remark 2. If an operator L is σ-hermitian (σ-skew-hermitian), then f is a solution of the equation $L(f) = 0$ if and only if $g := \bar{\sigma}\bar{f}$ is a solution of the equation $L^*(\bar{g}) = 0$.

Definition 4. If σ is an hermitian matrix ($\sigma = \sigma^*$), then we say that a dressing operator W is *σ-unital* if $W^{-1} = \sigma^{-1}W^*\sigma$.

Proposition 2. *Operator (1) is σ-unital if and only if $\psi = \sigma\bar{\varphi}$ and the matrix C is hermitian.*

The proof will is similar to that of Proposition 1.

Remark 3. Statements for σ-hermitian and σ-unital operators hold similarly to Theorem 2.

Theorem 3. *Let L be a differential operator of the form $L = \alpha\partial_y - \sum_{i=0}^{n} u_i\mathcal{D}^i$ and functions φ and ψ be solutions of equations $L(\varphi) = \varphi\Lambda$, $L^\tau(\psi) = \psi\hat{\Lambda}$, where $\Lambda, \hat{\Lambda}$ are some constant $K \times K$ matrices, then*

$$\hat{L} := WLW^{-1} = \alpha\partial_y - \sum_{i=0}^{n} \hat{u}_i\mathcal{D}^i + \Phi(C\Lambda - \hat{\Lambda}^\top C)\mathcal{D}^{-1}\Psi^\top.$$

Proof. The given theorem is a consequence of the theorem from [4]. After transformation the operator $\hat{L}$ has the form $\hat{L} = \hat{L}_+ + \hat{L}_-$ [4], where

$$\hat{L}_+ = \alpha\partial_y - \sum_{i=0}^{n} \hat{u}_i\mathcal{D}^i,$$

$$\hat{L}_- = \left\{L(\varphi) - \Phi\int_{-\infty}^{x} \psi^\top L(\varphi)ds\right\}\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}\left\{L^\tau(\psi) - \int_{-\infty}^{x} L^\tau(\psi)\varphi ds\Psi^\top\right\}.$$

Using the equalities $L(\varphi) = \varphi\Lambda$, $L^\tau(\psi) = \psi\hat{\Lambda}$ and the definition of the function Φ (see Def. 2), we come to the required operator $\hat{L}$. □

Corollary. *Let M be a differential operator of the form $M = \alpha\partial_y - \sum_{i=0}^{n} v_i\mathcal{D}^i$, and functions φ and ψ be solutions of the equations $M(\varphi) = 0$, $M^\tau(\psi) = 0$, then*

$$\hat{M} := WMW^{-1} = \alpha\partial_y - \sum_{i=0}^{n} \hat{v}_i\mathcal{D}^i,$$

that is $\hat{M}$ is a pure differential operator.

Most of nonlinear dynamical systems which are investigated with the aid of integral equations of Gelfand-Levitan-Marchenko admit a representation as the commutators equation, so-called Lax-Zakharov-Shabat representation:

$$[L, M] := LM - ML = 0, \tag{2}$$

where $L,\ M$ are differential operators

$$L := \alpha\partial_y - U := \alpha\partial_y - \sum_{i=0}^{n} u_i(x,y,t)\mathcal{D}^i,$$

$$M := \beta\partial_t - V := \beta\partial_t - \sum_{i=0}^{m} v_i(x,y,t)\mathcal{D}^i.$$

Equation (2) is a condition of compatibility of the linear system of equations

$$\begin{cases} \alpha f_y = U(f), \\ \beta f_t = V(f), \end{cases}$$

or

$$\begin{cases} \alpha g_y = -U^\tau(g), \\ \beta g_t = -V^\tau(g), \end{cases} \quad \alpha,\ \beta \in \mathbb{C}$$

and is equivalent to some closed nonlinear (in general case) system of partial differential equations of the form

$$\mathcal{F}[u_i^{(k)}, v_j^{(l)}, u_{i,t}, v_{j,y}] = 0;\ i,l \in \{0,\dots,n\};\ j,k \in \{0,\dots,m\}, \tag{3}$$

where $\mathcal{F}$ is a polynomial of variables in the brackets. Let $\varphi(x;y,t) := \varphi,\ \psi(x;y,t) := \psi$ be solutions of equations

$$\alpha\varphi_y = U(\varphi),\ \alpha\psi_y = -U^\tau(\psi),\ \beta\varphi_t = V(\varphi),\ \beta\psi_t = -V^\tau(\psi).$$

The operators $L,\ M$ after "dressing" by the operators W and W^{-1} from Corollary Theorem 3 become:

$$\hat{L} := WLW^{-1} = \alpha\partial_y - \sum_{i=0}^{n} \tilde{u}_i(x;y,t)\mathcal{D}^i, \tag{4}$$

$$\hat{M} := WMW^{-1} = \beta\partial_t - \sum_{i=0}^{m} \tilde{v}_i(x;y,t)\mathcal{D}^i, \tag{5}$$

and the coefficients

$$\tilde{u}_i = \tilde{u}_i[u_k, \varphi, \psi, \Phi, \Psi],\ \tilde{v}_j = \tilde{v}_j[v_l, \varphi, \psi, \Phi, \Psi],\ \ i,k \in \overline{0,n};\ j,l = \overline{0,m}$$

are differential polynomials of the functions in the brackets.

Proposition 3. [4] *Let $[L,M] = 0$, i.e. the functions $u_i,\ v_j$ satisfy the system of nonlinear equations (3). Then the functions $\tilde{u}_i,\ \tilde{v}_j$ are solutions of the same system,*

$$\mathcal{F}[\tilde{u}_i^{(k)}, \tilde{v}_j^{(l)}, \tilde{u}_{i,t}, \tilde{v}_{j,y}] = 0;\ i,l \in \{0,\dots,n\};\ j,k \in \{0,\dots,m\}.$$

2. Modifications of Higgs model

Consider (2×2)-matrix differential operators L, M of the form

$$L = \mathcal{D}^2 + 2\omega_x,\ \ M = \alpha\partial_y - \sigma_3\mathcal{D} - [\sigma_3, \omega], \tag{6}$$

where

$$\alpha \in \mathbb{R}\cup i\mathbb{R},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} := \operatorname{diag}(1,-1),$$

$$\omega = P + S = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} := \operatorname{off}(q,r) + \operatorname{diag}(s_1,s_2).$$

Proposition 4. *The condition of compatibility of the linear system*

$$L(f) = f\Lambda,\ M(f) = 0, \tag{7}$$

and also the system

$$L^\tau(g) = g\hat{\Lambda},\ M^\tau(g) = 0, \tag{8}$$

where Λ and $\hat{\Lambda}$ are constant "spectral" matrices with dimension as required, is an operational Lax-Zakharov-Shabat representation $[L, M] = 0$ of a system of differential equations of the form

$$\left\{ \begin{array}{l} \alpha\sigma_3 P_{xy} = -2[S_x, P], \\ \alpha\sigma_3 S_y - S_x = 2P^2. \end{array} \right) \tag{9}$$

The proof is obtained by direct calculations.

After additional reductions

$$r = \mu\bar{q},\ s_1 = \bar{s}_1,\ s_2 = \bar{s}_2, \quad \alpha\in\mathbb{R}, \tag{10}$$

where $\mu\in\mathbb{R}$ is a constant of association, system (9) is reduced to the form

$$\left\{ \begin{array}{l} \alpha q_{xy} = 2(s_2 - s_1)_x q \\ s_{1x} - \alpha s_{1y} = -2\mu|q|^2 \\ s_{2x} + \alpha s_{2y} = -2\mu|q|^2. \end{array} \right. \tag{11}$$

Introducing a new function $s := (s_1 - s_2)_x$, as a differential consequence of system (11) we obtain a modification of Higgs model (H-I), which is known in the quantum theory of gauge fields and describes a mechanism of spontaneous infringement of symmetry:

$$\left\{ \begin{array}{l} \alpha q_{xy} = -2sq, \\ \alpha^2 s_{yy} - s_{xx} = 4\alpha\mu|q|^2_{xy}. \end{array} \right. \tag{12}$$

We define operators $L_0 = \mathcal{D}^2$, $M_0 = \alpha\partial_y - \sigma_3\mathcal{D}$ and the analogs of systems (7)–(8) for these operators,

$$L_0(f) = f\Lambda,\ M_0(f) = 0 \tag{13}$$

and

$$L_0^{\tau}(g) = g\hat{\Lambda},\ M_0^{\tau}(g) = 0. \tag{14}$$

From Theorem 3 and Corollary it follows that under the conditions $C\Lambda = \hat{\Lambda}^{\top} C$ and if φ, ψ are $(2\times K)$-matrix solutions of systems (13)–(14), then the differential operators L_0, M_0 after transformations (4)–(5) have the form

$$\hat{L} = \mathcal{D}^2 + 2\tilde{\omega}_x,\ \ \hat{M} = \alpha\partial_y - \sigma_3\mathcal{D} - [\sigma_3, \tilde{\omega}], \tag{15}$$

where

$$\tilde{\omega} = \varphi\Psi^{\top} = \varphi\Omega^{-1}\psi^{\top} = \varphi\Bigg(C + \int\limits_{-\infty}^{x} \psi^{\top}\varphi d\tau\Bigg)^{-1}\psi^{\top}.$$

Proposition 5. *Operator equation $[L, M] = 0$ (where the kind of the operators is given by formula (6)) is equivalent to system (11) if and only if $L^* = \sigma L\sigma^{-1}, M^* = -\sigma M\sigma^{-1}$ where $\sigma = \sigma(\mu) := \operatorname{diag}(\mu, 1)$.*

The proof is carried out by direct verification.

Remark 4. According to Remark 2 for solutions f and g of systems (7), (8) the equality

$$g = \sigma(\mu)\bar{f},\ \ \sigma(\mu) := \operatorname{diag}(\mu, 1), \tag{16}$$

holds.

Theorem 4. *Let φ and ψ be solutions of systems (13)–(14) for $\hat{\Lambda} = \bar{\Lambda}$, that satisfy reductions (16), $\varphi_{i\cdot}$ the i-row of the matrix φ, $i \in \{1, 2\}$. Then the functions*

$$\tilde{q} = \varphi_{1\cdot}\Bigg(C + \int\limits_{-\infty}^{x} \varphi^*\sigma(\mu)\varphi d\tau\Bigg)^{-1}\varphi_{2\cdot}^* = -\frac{\det\begin{pmatrix} \Omega & \varphi_{2\cdot}^* \\ \varphi_{1\cdot} & 0 \end{pmatrix}}{\det\Omega},$$

$$\tilde{s}_1 = \mu\varphi_{1\cdot}\Omega^{-1}\varphi_{1\cdot}^* = -\mu(\det\Omega)^{-1}\det\begin{pmatrix} \Omega & \varphi_{1\cdot}^* \\ \varphi_{1\cdot} & 0 \end{pmatrix}, \tag{17}$$

$$\tilde{s}_2 = \varphi_{2\cdot}\Omega^{-1}\varphi_{2\cdot}^* = -(\det\Omega)^{-1}\det\begin{pmatrix} \Omega & \varphi_{2\cdot}^* \\ \varphi_{2\cdot} & 0 \end{pmatrix},$$

are solutions of system (11) for $\alpha \in \mathbb{R}$ if and only if C is an hermitian matrix ($C = C^*$) and $\Lambda^* C = C\Lambda$.

Proof. For $\hat{\Lambda} = \bar{\Lambda}$ the operator L_0 is σ-hermitian and the operator M_0 is σ-skew-hermitian, therefore from Remark 3 we see that the operators $\hat{L}, \hat{M}$ (15) are also σ-hermitian and σ-skew-hermitian. Using Proposition 5, we obtain that $[\hat{L}, \hat{M}] = 0 \Leftrightarrow$ (11), whence the form of coefficients of the operators $\hat{L}$ and $\hat{M}$ follows. In proving we carry out reduction (16) in (15) and obtain that

$$\tilde{\omega} = \varphi\Bigg(C + \int\limits_{-\infty}^{x} \varphi^*\sigma(\mu)\varphi d\tau\Bigg)^{-1}\varphi^*\sigma(\mu).$$

Using the well-known algebraic equality for framed determinant,

$$\det \begin{pmatrix} \Omega & \psi_{j\cdot}^{\top} \\ \varphi_{i\cdot} & \gamma \end{pmatrix} = \gamma \det \Omega - \varphi_{i\cdot} \Omega^{C} \psi_{j\cdot}^{\top},$$

where Ω^C is the matrix of co-factors, we obtain the components of the matrix $\tilde{\omega}$ of kind (17). □

Thus, the wide class of exact solutions of nonlinear model (12) is received,

$$\tilde{q} = -(\det \Omega)^{-1} \det \begin{pmatrix} \Omega & \varphi_{2\cdot}^{*} \\ \varphi_{1\cdot} & 0 \end{pmatrix},$$

$$\tilde{s} = \frac{\partial}{\partial x} \left\{ (\det \Omega)^{-1} \left(\det \begin{pmatrix} \Omega & \varphi_{2\cdot}^{*} \\ \varphi_{2\cdot} & 0 \end{pmatrix} - \mu \det \begin{pmatrix} \Omega & \varphi_{1\cdot}^{*} \\ \varphi_{1\cdot} & 0 \end{pmatrix} \right) \right\},$$

where $\varphi_{1\cdot}(x,y) := \varphi_1(\alpha x + y)$, $\varphi_{2\cdot}(x,y) := \varphi_2(\alpha x - y)$ are K-component vector-rows of the matrix φ, which is a solution of equation $\varphi_{xx} = \varphi \Lambda$, where Λ is some $K \times K$ matrix and $\Lambda^* C = C\Lambda$, $\alpha \in \mathbb{R}$.

For example, in the case $K = 1$ solutions will be of the form

$$\tilde{q} = \frac{\varphi_1 \bar{\varphi}_2}{C + \int_{-\infty}^{x} \mu |\varphi_1|^2 + |\varphi_2|^2 ds}, \tag{18}$$

$$\tilde{s} = \left(\frac{\mu |\varphi_1|^2 - |\varphi_2|^2}{C + \int_{-\infty}^{x} \mu |\varphi_1|^2 + |\varphi_2|^2 ds} \right)_x, \; C \in \mathbb{R}. \tag{19}$$

Consider the following particular case: $k = 1$, $\mu, C \in \mathbb{R}, \Lambda = \lambda^2, \lambda \in \mathbb{R}_+$; $\varphi_1 = ae^{\lambda(x+y)}$; $\varphi_2 = be^{\lambda(x-y)}$ we obtain the following result

$$q = \frac{2\lambda e^{2\lambda x}}{2\lambda C + e^{2\lambda x}(\mu |a|^2 e^{2\lambda y} + |b|^2 e^{-2\lambda y})} = \frac{2\lambda}{2\lambda C e^{-2\lambda x} + \mu |a|^2 e^{2\lambda y} + |b|^2 e^{-2\lambda y}},$$

$$s = \frac{8\lambda^3 C e^{-2\lambda x}(\mu |a|^2 e^{2\lambda y} - |b|^2 e^{-2\lambda y})}{(2\lambda C e^{-2\lambda x} + \mu |a|^2 e^{2\lambda y} + |b|^2 e^{-2\lambda y})^2}.$$

If one of the conditions holds

1. $C < 0$ and $\mu > 0$;
2. $C < 0, \mu < 0$ and $y > \frac{1}{4\lambda} \ln \left\{ -\frac{|b|^2}{\mu |a|^2} \right\}$;
3. $C > 0, \mu < 0$ and $y < \frac{1}{4\lambda} \ln \left\{ -\frac{|b|^2}{\mu |a|^2} \right\}$,

then $x = -\frac{1}{2\lambda} \ln \left\{ -\frac{1}{2\lambda C} (\mu |a|^2 e^{2\lambda y} + |b|^2 e^{-2\lambda y}) \right\}$ is the curve of discontinuity. In the case $C = 0, \mu < 0$ the curve of discontinuity has the form: $y = \frac{1}{4\lambda} \ln \left\{ -\frac{|b|^2}{\mu |a|^2} \right\}$.

Describe behavior at infinity for some cases when solutions are nonsingular:

1. $C > 0, \mu > 0$.
   a. $x \to +\infty, y$ fixed: $q \to \frac{2\lambda}{\mu|a|^2 e^{2\lambda y} + |b|^2 e^{-2\lambda y}}, s \to 0$.
   b. $x \to -\infty, y$ fixed: $q \to 0, s \to 0$.
   c. x fixed, $y \to \pm\infty$: $q \to 0, s \to 0$.
   d. $x \to \pm\infty, y \to \pm\infty$: $q \to 0, s \to 0$.
2. $C < 0, \mu < 0$ and $y < \frac{1}{4\lambda} \ln \left\{ -\frac{|b|^2}{\mu|a|^2} \right\}$.
   a. $x \to +\infty, y$ fixed: $q \to \frac{2\lambda}{\mu|a|^2 e^{2\lambda y} + |b|^2 e^{-2\lambda y}}, s \to 0$.
   b. $x \to -\infty, y$ fixed: $q \to 0, s \to 0$.
   c. x fixed, $y \to -\infty$: $q \to 0, s \to 0$.
   d. $x \to \pm\infty, y \to -\infty$: $q \to 0, s \to 0$.
3. $C > 0, \mu < 0$ and $y > \frac{1}{4\lambda} \ln \left\{ -\frac{|b|^2}{\mu|a|^2} \right\}$.
   a. $x \to +\infty, y$ fixed: $q \to \frac{2\lambda}{\mu|a|^2 e^{2\lambda y} + |b|^2 e^{-2\lambda y}}, s \to 0$.
   b. $x \to -\infty, y$ fixed: $q \to 0, s \to 0$.
   c. x fixed, $y \to +\infty$: $q \to 0, s \to 0$.
   d. $x \to \pm\infty, y \to +\infty$: $q \to 0, s \to 0$.

In particular, choosing $K = 1, \alpha = \mu = C = \Lambda = 1, \varphi_1 = e^{x+y}, \varphi_2 = e^{x-y}$ from formulas (18)–(19) respectively, we obtain

$$q = \frac{e^{2x}}{1 + e^{2x}\cosh 2y}, \quad s = \frac{4e^{-2x}\sinh 2y}{(e^{-2x} + \cosh 2y)^2}.$$

If $K = 2$ and $\varphi_{1\cdot} = (\varphi_1(\alpha x + y), 0), \varphi_{2\cdot} = (0, \varphi_2(\alpha x - y))$, $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then the solutions can be written as follows

$$\tilde{q} = \frac{\varphi_1 \bar{\varphi}_2}{1 - \mu \int_{-\infty}^{x} |\varphi_1|^2 dx \int_{-\infty}^{x} |\varphi_2|^2 dx},$$

$$\tilde{s}_1 = \frac{|\varphi_1|^2 \int_{-\infty}^{x} |\varphi_2|^2 dx}{1 - \mu \int_{-\infty}^{x} |\varphi_1|^2 dx \int_{-\infty}^{x} |\varphi_2|^2 dx},$$

$$\tilde{s}_2 = \frac{\mu|\varphi_2|^2 \int_{-\infty}^{x} |\varphi_1|^2 dx}{1 - \mu \int_{-\infty}^{x} |\varphi_1|^2 dx \int_{-\infty}^{x} |\varphi_2|^2 dx}.$$

For $\mu < 0$ the solutions are nonsingular in $\mathbf{R}^2(x, y)$.

For the case $K = 1, \alpha = C = \Lambda = 1, \mu = -1, \varphi_1 = e^{x+y}, \varphi_2 = e^{2(x-y)}$ from previous formulas,

$$\tilde{q} = \frac{8e^{3x-y}}{8 + e^{2(3x-y)}}, \quad \tilde{s}_1 = -\frac{1}{2}\tilde{s}_2 = \frac{2e^{2(3x-y)}}{8 + e^{2(3x-y)}}.$$

3. Second type of modified Higgs model

System (9) also supposes restrictions of the form

$$r = \mu \bar{q}, \;\; s_2 = \bar{s}_1, \;\; \alpha \in i\mathbb{R}. \tag{20}$$

The reduced nonlinear system (12) is called a modified Higgs model of the second type (H-II). We notice, that for the first model the corresponding Lax M-operator is hyperbolic, and for the second one it is of elliptic type and in this case $s = 2i\mathrm{Im} s_{1x}$.

Proposition 6. *Solutions f and g of system (7)–(8) admit reductions*

$$f = \sigma(\mu)\bar{f}, \;\;\; g = \sigma^\top(\mu)\bar{g}, \;\;\; \sigma(\mu) = \begin{pmatrix} 0 & \frac{1}{\sqrt{\mu}} \\ \sqrt{\mu} & 0 \end{pmatrix}, \;\;\; \mu > 0, \tag{21}$$

if and only if the coefficients of the operators L, M (6) satisfy restriction (20).

The proof of Proposition 6 can be carried out by direct verification.

Note that the functions f, g which satisfy reductions (21) have the form

$$f = \begin{pmatrix} f_{1\cdot} \\ \sqrt{\mu}\bar{f}_{1\cdot} \end{pmatrix}, \;\;\; g = \begin{pmatrix} g_{1\cdot} \\ \frac{1}{\sqrt{\mu}}\bar{g}_{1\cdot} \end{pmatrix}.$$

Theorem 5. *Let φ and ψ be solutions of systems (13)–(14) that satisfy reductions (21). Then the functions*

$$\tilde{q} = \varphi_{1\cdot}\left(C + \int_{-\infty}^{x} \psi^\top \varphi d\tau\right)^{-1} \psi_{2\cdot}^\top = \frac{1}{\sqrt{\mu}}\varphi_{1\cdot}\left(C + 2Re\int_{-\infty}^{x} \psi_{1\cdot}^\top \varphi_{1\cdot} d\tau\right)^{-1}$$

$$\psi_{1\cdot}^\top = \frac{1}{\sqrt{\mu}}\varphi_{1\cdot}\Omega^{-1}\bar{\psi}_{1\cdot}^\top, \tag{22}$$

$$\tilde{s} = \frac{\partial}{\partial x}\left(\varphi_{1\cdot}\Omega^{-1}\psi_{1\cdot}^\top - \bar{\varphi}_{1\cdot}\Omega^{-1}\bar{\psi}_{1\cdot}^\top\right)$$

are solutions of nonlinear modified Higgs model (H-II) for $\alpha \in i\mathbb{R}$, $\mu > 0$ if and only if $C, \Lambda, \hat{\Lambda}$ are real matrices and $\hat{\Lambda}^\top C = C\Lambda$.

The proof is similar to that of Theorem 4 with using of Theorem 3. In formula (22) reductions (21) are taken into account,

$$\Omega = C + \int_{-\infty}^{x} \psi^\top \varphi d\tau = C + \int_{-\infty}^{x} \left(\psi_{1\cdot}^\top, \frac{1}{\sqrt{\mu}}\bar{\psi}_{1\cdot}^\top\right) \begin{pmatrix} \varphi_{1\cdot} \\ \sqrt{\mu}\bar{\varphi}_{1\cdot} \end{pmatrix} d\tau$$

$$= C + \int_{-\infty}^{x} \left(\psi_{1\cdot}^\top \varphi_{1\cdot} + \bar{\psi}_{1\cdot}^\top \bar{\varphi}_{1\cdot}\right) d\tau = C + 2Re\int_{-\infty}^{x} \psi_{1\cdot}^\top \varphi_{1\cdot} d\tau.$$

Solutions of model H-II can be rewritten as

$$\tilde{q} = -\frac{1}{\sqrt{\mu}} \det \begin{pmatrix} \Omega & \psi_{1\cdot}^* \\ \varphi_{1\cdot} & 0 \end{pmatrix} (\det \Omega)^{-1},$$

$$\tilde{s} = \frac{\partial}{\partial x}\left\{\left(\det\begin{pmatrix}\Omega & \psi_{1\cdot}^{*}\\ \bar{\varphi}_{1\cdot} & 0\end{pmatrix} - \det\begin{pmatrix}\Omega & \psi_{1\cdot}^{\top}\\ \varphi_{1\cdot} & 0\end{pmatrix}\right)(\det\Omega)^{-1}\right\},$$

where $\varphi_{1\cdot}(x,y) := \varphi_1(\alpha x + y)$, $\psi_{1\cdot}(x,y) := \psi_1(\alpha x + y)$ are K-component functions which are the first rows of the linear matrix system of equations $\varphi_{xx} = \varphi\Lambda, \psi_{xx} = \psi\hat{\Lambda}$, where $\Lambda, \hat{\Lambda}$ are some constant real $K \times K$ matrices and $\hat{\Lambda}^{\top}C = C\Lambda$, $\alpha \in i\mathbb{R}$.

In the case $K = 1$ the solutions have the form

$$\tilde{q} = \frac{1}{\sqrt{\mu}}\frac{\varphi\bar{\psi}}{C + 2Re\int_{-\infty}^{x}\psi\varphi ds},\quad \tilde{s} = \left(\frac{\varphi\psi - \bar{\varphi}\bar{\psi}}{C + 2Re\int_{-\infty}^{x}\psi\varphi ds}\right)_x, \quad C \in \mathbb{R}.$$

In the case $\mu > 0, \mu, C \in \mathbb{R}, \Lambda = \lambda^2, \lambda \in \mathbb{R}_+$. $\varphi = \psi = ae^{\lambda(-x+iy)}, a \in \mathbb{R}$ the solutions of (12) are of the form

$$\tilde{q} = \frac{1}{\sqrt{\mu}}\frac{a^2\lambda e^{-2\lambda x}}{\lambda C - a^2e^{-2\lambda x}\cos(2\lambda y)} = \frac{1}{\sqrt{\mu}}\frac{a^2\lambda}{\lambda Ce^{2\lambda x} - a^2\cos(2\lambda y)},$$

$$\tilde{s} = -\frac{4ia^2\lambda^3Ce^{2\lambda x}\sin(2\lambda y)}{(\lambda Ce^{2\lambda x} - a^2\cos(2\lambda y))^2}.$$

The solution q is periodic by y with period $\frac{\pi}{\lambda}$. If $x \le \frac{1}{2\lambda}\ln\frac{a^2}{\lambda|C|}$ then $y = \pm\frac{1}{2\lambda}\arccos(\frac{\lambda C}{a^2}e^{2\lambda x}) + \frac{\pi}{\lambda}n, n \in \mathbb{Z}$ is the curve of discontinuity and if $C = 0$ then $y = \frac{\pi}{4\lambda} + \frac{\pi}{2\lambda}n, n \in \mathbb{Z}$ are the points of discontinuity.

In the simplest case $\mu = K = 1, \alpha = i, C = \Lambda = 1, \varphi = \psi = e^{-x+iy}$ we obtain

$$\tilde{q} = \frac{e^{-2x}}{1 - e^{-2x}\cos 2y},\quad \tilde{s} = -\frac{4ie^{-2x}\sin 2y}{(1 - e^{-2x}\cos 2y)^2}.$$

If we replace the operator L from (6) by the operator $\widetilde{L}$ of the form

$$\widetilde{L} = \sigma_3\mathcal{D}^2 + 2\sigma_3P\mathcal{D} + \sigma_3P_x + \alpha P_y + \operatorname{diag}(s_1, s_2),$$

then we obtain an analogue of system (9):

$$\begin{cases} \alpha^2\sigma_3P_{yy} = -\sigma_3P_{xx} - 2[S, P], \\ \alpha\sigma_3S_y - S_x = 2\Big[\alpha(P^2)_y - \sigma_3(P^2)_x\Big], \end{cases} \tag{23}$$

where $S = \operatorname{diag}(s_1, s_2)$.

Introducing a function $s := s_1 - s_2$ under reduction $r = \mu\bar{q}$ we receive the differential consequence of system (23)

$$\begin{cases} \alpha^2q_{yy} = -q_{xx} - 2sq, \\ \alpha^2s_{yy} - s_{xx} = 4\mu\Big(\alpha^2|q|^2_{yy} + |q|^2_{xx}\Big), \end{cases} \tag{24}$$

which is also a modification of nonlinear Higgs model and has essentially different two cases, H-I (for $\alpha \in \mathbb{R}$) or H-II (for $\alpha \in i\mathbb{R}$).

The methods of finding the exact solutions of system (24) are similar to those given above, but some complications require further research.

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