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ON SOLUTIONS OF EULER-POISSON'S EQUATIONS WHICH ARE LINEAR COMBINATIONS OF ζ- AND $\wp$-FUNCTIONS OF WEIERSTRASS

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We consider solutions of Euler-Poisson's equations which are linear combinations of ζ- and $\wp$-functions of Weierstrass.

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Рассматриваются решения уравнений Эйлера-Пуассона, которые являются линейной комбинацией ζ- и $\wp$-функций Вейерштрасса.

§1. Introduction.

The problem of finding explicit solutions for Euler-Poisson's equations is the most interesting one in the mechanics of solid ([1]). In spite of the fact that many authors (see [2]) have been investigated this problem one cannot be convinced that all solutions are discovered.

In [3] some restrictions on parameters of solid were found such that Euler-Poisson's equations have single-valued solutions. Some explicit solutions satisfying these restrictions were presented in [4]. The present paper strengthens results of [3] and [4].

Let us write Euler-Poisson's equations in the following form:

$$\begin{cases} A\dot{p} = & Ap \times p + \gamma \times r, \\ \dot{\gamma} \; = & \gamma \times p, \end{cases} \tag{1.1}$$

here $p = (p_1, p_2, p_3) \in \mathbb{C}^3$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3$, $A = \mathrm{diag}(A_1, A_2, A_3)$, $r = (r_1, r_2, r_3) \in \mathbb{R}^3$, $A_i > 0$ $(i \in \{1, 2, 3\})$.

In order to investigate solutions of this system it is necessary to know solutions of the characteristic system.

Definition 1. The system

$$\begin{cases} A\tilde{p}^0 \times \tilde{p}^0 + \tilde{\gamma}^0 \times r + A\tilde{p}^0 = 0, \\ \tilde{\gamma}^0 \times \tilde{p}^0 + 2\tilde{\gamma}^0 = 0, \end{cases} \tag{1.2}$$

is called the *characteristic system* for Euler-Poisson's equations.

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Definition 2. Characteristic system (1.2) has two types of solutions: $(\tilde{p}^0, 0)$ and $(\tilde{p}^0, \tilde{\gamma}^0)$, $\tilde{\gamma}^0 \neq 0$ which are called α- and β-*solutions*, respectively.

The following theorem is proved in [3]. Below we use the notations $B_{ij} = A_i - A_j$ and the cyclic replacement of indices $\sigma = (1, 2, 3)$ for writing products or sums (e.g. $\sum_\sigma = A_1 A_2 + A_2 A_3 + A_3 A_1$, $\prod_\sigma A_1 = A_1 A_2 A_3$). Define also the $\mathbb{C}$-scalar product in $\mathbb{C}^3$: $\langle x, y\rangle = \sum_{i=1}^3 x_i y_i$.

Theorem 1. *If there exist the single-valued solutions of Euler-Poisson's equations (1.1) then the parameters a_i, r_i satisfy one of the following conditions:*

1) $\prod_\sigma B_{12} \sum_\sigma r_1 \sqrt{B_{23}} = 0$;

2) $\sum_\sigma A_1 r_1 \sqrt{\frac{A_2 A_3}{B_{12} B_{31}}} = 0$;

3) $\lambda_0 \in \mathbb{Z}$; *where* $\lambda_0 = \dfrac{1}{2} - \sqrt{\dfrac{1}{4} - S}$,

$$S = \frac{\dfrac{(2\langle A\gamma, \delta\rangle + \langle Ap, p\rangle)(\langle A\gamma, \delta\rangle + \langle Ap, p\rangle)}{\langle A\gamma, \gamma\rangle} - \dfrac{3\langle p, r\rangle^2}{\langle \gamma, r\rangle} - 2\langle A\delta, \delta\rangle + 2\langle \delta, r\rangle}{\dfrac{\langle p, r\rangle^2}{2\langle \gamma, r\rangle} - \dfrac{\langle A\gamma, \delta\rangle^2}{\langle A\gamma, \gamma\rangle} + \langle A\delta, \delta\rangle.}$$

(p, γ) is any β-solution of the characteristic system and δ is a vector such that $p \times \delta = -2$, $\langle \gamma, \delta\rangle = 2$.

4) the solution $(p(t), \gamma(t))$ has the representation

$$\begin{cases} p(t) = \sum_{k=1}^n \tilde{p}_k^0 \zeta(t - t_k) + p_0, \\ \gamma(t) = \sum_{k=1}^n \tilde{\gamma}_k^0 \wp(t - t_k) + \gamma_0, \end{cases} \tag{1.3}$$

where $\{(\tilde{p}_k^0, \tilde{\gamma}_k^0)\}$ are the β-solutions of the characteristic system and $\zeta(t)$, $\wp(t)$ are the Weierstrass functions ([5]).

Remark 1. As it follows from [3] the number n in (1.3) may be equal to 2, 4, 6, and 8, because the sum in the first equation of (1.3) contains $\tilde{p}_k^0$ with $\overline{\tilde{p}_k^0}$. Moreover all coefficients $\tilde{p}_k^0$, $k \in \{1, \dots, n\}$, are different.

We shall prove that if $n \in \{2, 4\}$ then there exist solutions of form (1.3). These are solutions of the Bobiliov-Steklov ([6], [7]) and Steklov ([8]) cases. Moreover, if $n = 6$ then there is no solutions of (1.3).

§2. Relations between the coordinates of the singular points.

According to [3], [9] an asymptotics of the solutions $p(t)$ of Euler-Poisson's equations at singular points is the following

$$p(t) = \frac{\tilde{p}^0}{t} + \beta_2 u_2 t + \beta_3 u_3 t^2 + \beta_4 u_4 t^3 + \dots. \tag{2.1}$$

Let the solution p be of form (1.3). By comparing the coefficients at t^0 of the asymptotic behaviour of $p(t)$ at the singular points and (2.1) we get the following relations

$$\sum_{k=1,k\neq i}^{n} \tilde{p}_k^0 \zeta(t_i - t_k) + p_0 = 0, \quad i \in \{1, \ldots, n\}.$$

For convenience let t_n be equal to 0, then we have for $i = n$

$$-\sum_{k=1}^{n-1} \tilde{p}_k^0 \zeta(t_k) + p_0 = 0, \tag{2.2}$$

and for $i \neq n$

$$\sum_{k=1,k\neq i}^{n-1} \tilde{p}_k^0 \zeta(t_i - t_k) + p_0 + \tilde{p}_n^0 \zeta(t_i) = 0. \tag{2.3}$$

Since

$$\zeta(u - v) = \zeta(u) - \zeta(v) + \frac{\wp'(u) - \wp'(v)}{2(\wp(u) - \wp(v))},$$

(2.3) takes the form

$$\sum_{k=1,k\neq i}^{n-1} \tilde{p}_k^0 \big(\zeta(t_i) - \zeta(t_k) + \frac{\rho_{ik}}{2}\big) + p_0 + \tilde{p}_n^0 \zeta(t_i) = 0,$$

where

$$\rho_{ik} = \frac{\wp'(t_i) + \wp'(t_k)}{\wp(t_i) - \wp(t_k)}. \tag{2.4}$$

If $\wp(t_i) = \wp(t_j)$ for some $i \neq j$ we take

$$\rho_{ik} = -\frac{\wp''(t_i)}{\wp(t_i)}, \tag{2.5}$$

because

$$\lim_{\varepsilon \to 0} \frac{\wp'(t) + \wp'(-t-\varepsilon)}{\wp(t) - \wp(-t-\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\wp'(t+\varepsilon) - \wp'(t)}{\wp(t+\varepsilon) - \wp(t)} = -\frac{\wp''(t)}{\wp(t)}.$$

In formula (2.5) $\wp(t_i) \neq 0$, because $\wp(t_i) = 0$, if and only if $t_i \equiv t_j$(mod period of $\wp(t)$) that is impossible. The sum of residues at the poles of the doubly periodic functions is equal to zero therefore

$$\sum_{k=1}^{n-1} \tilde{p}_k^0 = -\tilde{p}_n^0, \tag{2.6}$$

and by (2.2) we obtain

$$\sum_{k=1}^{n-1} \tilde{p}_k^0 \rho_{ik} = 0.$$

For convenience, we define $\rho_{ii} = 0$. So we have

$$\rho \tilde{p}^0 = 0, \tag{2.7}$$

where ρ is an antisymmetric $(n-1) \times (n-1)$-matrix and $\tilde{p}^0$ is an $(n-1) \times 3$-matrix.

§3. Solutions of Euler-Poisson's equations in the cases $\mathbf{n} \in \{\mathbf{2}, \mathbf{4}\}$.

Let $n = 4$ in representation (1.3). It follows from properties of the characteristic system [10] that $\tilde{p}^0 \neq 0$. Consequently, $\rho\tilde{p}^0 = 0$ if and only if $\rho = 0$. Then

$$\wp'(t_1) + \wp'(t_2) = 0, \quad \wp'(t_1) + \wp'(t_3) = 0, \quad \wp'(t_2) + \wp'(t_3) = 0,$$

and hence $\wp'(t_1) = \wp'(t_2) = \wp'(t_3) = 0$.

This fact means that the coordinates of the singular points t_1, t_2, t_3, t_4 are coordinates of the singular points of the Jacobi functions $\mathrm{sn}(t, \kappa)$, $\mathrm{cn}(t, \kappa)$, $\mathrm{dn}(t, \kappa)$ with the suitable κ. Hence one can replace representation (1.3) by the following one

$$\begin{cases} p(t) = b_1 \,\mathrm{sn}(ka, t) + b_2 \,\mathrm{cn}(\kappa, t) + b_3 \,\mathrm{dn}(\kappa, t), \\ \gamma(t) = a_1 \,\mathrm{cn}(\kappa, t)\,\mathrm{dn}(\kappa, t) + a_2 \,\mathrm{sn}(\kappa, t)\,\mathrm{dn}(\kappa, t) + a_3 \,\mathrm{sn}(\kappa, t)\,\mathrm{cn}(\kappa, t) + a_4 \,\mathrm{sn}(\kappa, t)^2 + a_0. \end{cases} \tag{3.1}$$

This representation was studied in [4], and the following theorem was proved there

Theorem 2. *All solutions of Euler-Poisson's equations (1.1) which has the form*

$$\begin{cases} p(t) = b_1 \,\mathrm{sn}(ka, t) + b_2 \,\mathrm{cn}(\kappa, t) + b_3 \,\mathrm{dn}(\kappa, t), \\ \gamma(t) = a_1 \,\mathrm{cn}(\kappa, t)\,\mathrm{dn}(\kappa, t) + a_2 \,\mathrm{sn}(\kappa, t)\,\mathrm{dn}(\kappa, t) + a_3 \,\mathrm{sn}(\kappa, t)\,\mathrm{cn}(\kappa, t) + a_4 \,\mathrm{sn}(\kappa, t)^2 + a_0 \end{cases}$$

are partial solutions of the Steklov and Bobiliov-Steklov cases, and moreover, they are partial solutions of the Grioli and Lagrange cases if one admit $\kappa = 0$.

Remark 2. If $\kappa = 0$, solutions (3.1) of the Lagrange case ($A_1 = A_2$, $r_1 = r_2 = 0$, $r_3 \neq 0$) and the Grioli case ($\sum_\sigma r_1\sqrt{B_{23}} = 0$) have no singular points. In these cases condition 1) of Theorem 1 is realized.

Remark 3. The solutions of the Steklov case have representation (1.3) with $n = 4$. In this case the vectors b_1, b_2m b_3 form an orthogonal basis.

Remark 4. Solutions of the Bobiliov-Steklov case have the following form: $r_3 = 0$ and

$$\begin{cases} p_1 = p_2 = \gamma_3 = 0, \\ A_3\dot{p}_3 = r_2\gamma_1 - r_1\gamma_2, \\ \dot{\gamma}_1 = p_3\gamma_2, \\ \dot{\gamma}_2 = -p_3\gamma_1, \end{cases} \tag{3.2}$$

The first integrals of system (1.1) are

$$\mathcal{H} = \frac{1}{2}p_3^2 + r_1\gamma_1 + r_2\gamma_2,$$
$$\mathcal{M} = \sum_\sigma A_1 p_1 = 0,$$
$$\mathcal{T} = \gamma_1^2 + \gamma_2^2.$$

The function p_3 is a Jacobi elliptic function, because

$$A_3^2(\dot{p}_3)^2 = (r_1^2 + r_2^2)\mathcal{T} - \Big(\mathcal{H} - \frac{1}{2}p_3^2\Big)^2. \tag{3.3}$$

The functions γ_1, γ_2 are given by

$$\begin{aligned} \gamma_1 &= \frac{A_3\dot{p}_3 r_2 + (\mathcal{H} - \frac{1}{2}p_3^2)r_2}{r_1^2 + r_2^2}, \\ \gamma_2 &= -\frac{A_3\dot{p}_3 r_1 + (\mathcal{H} - \frac{1}{2}p_3^2)r_1}{r_1^2 + r_2^2}. \end{aligned} \tag{3.4}$$

These solutions have representation (1.3) with $n = 2$.

Remark 5. There exist solutions of the Bobiliov-Steklov case which differ from (3.2)–(3.) and satisfy the conditions $A_2 = 2A_1$, $r_1 = r_3 = 0$, $p_2 = p_{20} = \text{const}$, $p_3 = 0$. These solutions have no form (1.3) although $p_1(t)$ is a Jacobi elliptic function and $\gamma_i(t)$ are doubly periodic single-valued functions. In this case the characteristic system has the solution

$$\tilde{p}^0 = (\pm 2i, 0, 0), \quad \tilde{\gamma}^0 = \Big(0, \frac{A_2}{r_2}, \pm\frac{A_2 i}{r_2}\Big), \quad \lambda_0 = 0,$$

i.e. condition 3) of Theorem 1 is fulfilled.

Let $n = 2$ in (1.3). In this case the doubly periodic functions $p(t)$, $\gamma(t)$ have 2 singular points in a parallelogram of periods. We can consider this case as a partial case of $n = 4$. Then all reasoning of §3 can be repeated and representation (3.1) will be obtained again.

So it is natural that we got the solutions which have form (1.3) with $n = 2$ in Theorem 2 as the solutions of the Bobiliov-Steklov case.

§4. Solutions of Euler-Poisson's equations in the non-degenerate case $n = 6$.

Let $n = 6$ and the matrix ρ be of maximal rank, i.e. $\operatorname{rank} p = 3$.

Proposition 1. *Let $a, b \in \mathbb{C}^5$, $a \nparallel b$ and let a, b be orthogonal to the vectors $\tilde{p}_j^0$, $j \in \{1, 2, 3\}$. Then $\rho_{ik} = \lambda(a_i b_k - a_k b_i)$, $\lambda \in \mathbb{C}$.*

Proof. The proposition has invariant character, consequently it may be considered in the convenient coordinates. In view of this circumstance the proof becomes evident. □

So we have a collection of the equalities,

$$\rho_{ij} = a_i b_j - a_j b_i, \quad 1 \le i, j \le 5. \tag{4.1}$$

Let us put the origin of coordinates at the point t_k. Naturally the above reasoning for the case $t_6 = 0$ can be repeated in a new situation again. In this case the values $\wp(t_i)$, $\wp'(t_i)$ do not vary. It follows from (2.6) that the vectors a and b vary, by the following rule $a_i \to a_k$, $b_i \to b_k$, $i \ne k$, and the coefficients a_6, b_6 are equal to $-a_k$ and $-b_k$, respectively. Formulae (4.1) take the form

$$\rho_{ij} = \lambda_k((a_i - a_k)(b_j - b_k) - (a_j - a_k)(b_i - b_k)), \tag{4.2}$$

or

$$\rho_{ij} = \lambda_k(\rho_{ij} + \rho_{jk} + \rho_{ki}). \tag{4.3}$$

Now replace the index j by 1 in (4.3)

$$\rho_{il} = \lambda_k(\rho_{il} + \rho_{lk} + \rho_{ki})$$

and eliminate λ_k:

$$\rho_{ij}(\rho_{lk} + \rho_{ki}) = \rho_{il}(\rho_{jk} + \rho_{ki}).$$

After substitution (4.1) the last equality is reduced to the relation

$$(\rho_{ij} + \rho_{jl} + \rho_{li})\rho_{ik} = 0.$$

Since the indices i, j, k, l are arbitrary, it follows from (4.3), (4.4) that $\rho_{ij} = 0$ for any i, j. Evidently, this is impossible in the case $n = 6$.

§5. The solutions of Euler-Poisson's equations in the degenerate case with n = 6.

We assume that $n = 6$ in (1.3) and the rank of the matrix ρ is equal to 2. Moreover $\Pi_\sigma B_{12} \neq 0$ because in the opposite case there exist only 4 solutions of the characteristic system (1.2) if we do not want to consider the Lagrange and Kovalevskaya cases. The Lagrange and Kovalevskaya cases are solved explicitly and we think that it is not a problem to find solutions (1.3) in these cases.

Theorem 3. *The sum of all β-solutions $\tilde{p}^0$ of the characteristic system (1.2) is equal to zero.*

Proof. According to [10] there exists the representation

$$\begin{aligned} \tilde{p}_1^0 &= \frac{B_{23}\sqrt{-1-\xi^2}}{B_{12}\xi^2 - B_{23}}, \\ \tilde{p}_2^0 &= \frac{B_{31}\xi}{B_{12}\xi^2 - B_{23}}, \\ \tilde{p}_3^0 &= \frac{B_{12}\xi\sqrt{-1-\xi^2}}{B_{12}\xi^2 - B_{23}}, \end{aligned} \tag{5.1}$$

where ξ is one of the roots $\xi_1, \dots, \xi_8$ of the polynomial

$$\begin{aligned} &(\xi^2+1)(r_1 B_{23}(C_{21}B_{31} + C_{31}B_{12} + C_{31}B_{12}\xi^2) - \\ &-r_3 B_{12}\xi(C_{13}B_{23} + C_{13}B_{23}\xi^2 + C_{23}B_{31}\xi^2))^2 + \\ &+(r_2 B_{31}\xi(C_{32}B_{12}\xi^2 - C_{12}B_{23}))^2 = 0. \end{aligned} \tag{5.2}$$

It is enough to prove that $\sum\limits_{k=1}^{8} \tilde{p}_{k2}^0 = 0$ because the formulae analogous to (5.1) can be also obtained for $\tilde{p}_1^0$ and $\tilde{p}_3^0$.

Further we shall use the notation $\sum\limits_{\delta}$ for writing the sum which contains similar terms with all possible collections of k indices, $k \leq 8$. For example, the reduction coefficients $a_0, \dots, a_7$ of polynomial (5.2) can be represented in the following form:

$$a_7 = -\sum_{\delta} \xi_1, \quad a_6 = \sum_{\delta} \xi_1 \xi_2, \quad \dots, \quad a_1 = -\sum_{\delta} \xi_1 \dots \xi_7.$$

We want to prove that

$$\sum_{i=1}^{8} \frac{\xi_i}{b{\xi_i}^2 - 1} = 0, \tag{5.3}$$

where $b = B_{12}/B_{23}$. It is clear that (5.3) is equivalent to

$$\sum_{i=1}^{8} \xi_i \prod_{j=1, j\neq i}^{8} (b{\xi_i}^2 - 1) = 0,$$

For the convenience let us define the parameters

$$q_1 = -\sum_{\delta} \xi_1, \; q_2 = \sum_{\delta} \xi_1 {\xi_2}^2, \; q_3 = -\sum_{\delta} \xi_1 {\xi_2}^2 {\xi_3}^2, \dots, \; q_8 = -\sum_{\delta} \xi_1 {\xi_2}^2 \dots {\xi_8}^2.$$

There exist simple connections between a_i and q_i :

$$q_1 = -a_1,$$

$$q_2 = -a_1 a_2 + 3a_3,$$

$$q_3 = -a_2 a_3 + 3a_1 a_4 - 5a_5,$$

$$q_4 = -a_3 a_4 + 3a_2 a_5 - 5a_1 a_6 + 7a_7,$$

$$q_5 = -a_4 a_5 + 3a_3 a_6 - 5a_1 a_6 + 7a_8 a_1,$$

$$q_6 = -a_5 a_6 + 3a_4 a_7 - 5a_3 a_8,$$

$$q_7 = -a_6 a_7 + 3a_2 a_8,$$

$$q_8 = -a_7 a_8.$$

Condition (5.3) is equivalent to the following

$$b^7 q_8 + b^6 q_7 + \dots + b q_2 + q_1 = 0. \tag{5.4}$$

Simple but awkward testing shows that condition (5.4) is true. □

Since $\sum_{i=1}^{6} \tilde{p}_k^0 = 0$, it follows from Theorem 3 that there exist two β-solutions of the characteristic system (1.2) such that $\tilde{p}_7^0 + \tilde{p}_8^0 = 0$.

Proposition 2. *If $\tilde{p}^0$ and $-\tilde{p}^0$ are β-solutions of the characteristic system and $\Pi_\sigma B_{12} \neq 0$ then $r_1 r_2 r_3 = 0$.*

Proof. For convenience we write (p,γ) instead of $(\tilde{p}^0,\tilde{\gamma}^0)$ in this proof. Let (p,γ) and $(-p,\gamma')$ be β-solutions of the characteristic system. It follows from (1.2) that

$$\langle Ap\times p, r\rangle = \langle Ap, r\rangle = 0 \tag{5.5}$$

Let $Ap\times p\neq 0$. It is easy to prove that

$$\langle Ap,p\rangle+\langle\gamma,r\rangle=0,\quad \langle\gamma,\gamma\rangle=0,\quad \langle Ap,\gamma\rangle=0,$$

consequently $\gamma\parallel Ap\times p$, and $\langle Ap,p\rangle=0$. By equations (5.5) we get

$$r\parallel (Ap\times p)\times p=\langle Ap,Ap\rangle p-\langle Ap,p\rangle Ap=\langle Ap,Ap\rangle p\parallel p,$$

but this is impossible because $\langle Ar,r\rangle\neq 0$. So we see that $Ap\times p=0$ and it is possible only if $p=(\pm 2i,0,0)$ or $(0,\pm 2i,0)$ or $(0,0,\pm 2i)$. Here we used the equality $\langle p,p\rangle=-4$. Since $\langle Ap,r\rangle=0$, we get $r_1=0$, or $r_2=0$, or $r_3=0$. □

Let $r_2=0$. Then the characteristic system (1.2) has the symmetry

$$S_3\colon (p_1,p_2,p_3,\gamma_1,\gamma_2,\gamma_3)\longleftrightarrow(-p_1,p_2,-p_3,\gamma_1,-\gamma_2,\gamma_3).$$

In this case the matrix $\tilde{p}^0$ has the following form:

$$\begin{pmatrix} p_1 & p_2 & p_3\\ -p_1 & p_2 & -p_3\\ p_1' & p_2' & p_3'\\ -p_1' & p_2' & -p_3'\\ p_1'' & p_2'' & p_3''\\ -p_1'' & p_2'' & -p_3'' \end{pmatrix}, \tag{5.6}$$

where p_1, p_2', p_3'' are found by (5.2) and ξ is the root of the polynomial

$$r_1B_{23}(C_{21}B_{31}+C_{31}B_{12}+C_{31}B_{12}\xi^2)-r_3B_{12}\xi(C_{13}B_{23}+C_{13}B_{23}\xi^2+C_{23}B_{31}\xi^2). \tag{5.7}$$

The rank of matrix (5.6) is equal to 2 and coincides with the rank of the matrix:

$$\begin{pmatrix} p_1 & 0 & p_3\\ 0 & p_2 & 0\\ p_1' & 0 & p_3'\\ 0 & p_2' & 0\\ p_1'' & 0 & p_3''\\ 0 & p_2'' & 0 \end{pmatrix},$$

consequently,

$$\frac{p_1}{p_3}=\frac{p_1'}{p_3'}=\frac{p_1''}{p_3''},$$

i.e. all 3 roots of polynomial (5.7) are equal. Polynomial (5.7) can be reduced,

$$\xi^3+\frac{r_1B_{23}C_{31}}{r_3A_3B_{12}}\xi^2-\frac{B_{23}C_{31}}{A_3B_{12}}\xi-\frac{r_1A_1B_{23}^2}{r_3A_3B_{12}^2}=0,$$

and the necessary condition can be expressed by the following relation:

$$9\left(\frac{r_1 B_{23} C_{31}}{r_3 A_3 B_{12}}\right)^{-1} \frac{r_1 A_1 B_{23}^2}{r_3 A_3 B_{12}^2} = \left(\frac{B_{23} C_{31}}{A_3 B_{12}}\right)^2.$$

or, equivalently,

$$(A_1 + A_3)^2 = 0.$$

Clearly, this is impossible. So we can formulate a strengthening of Theorem 1.

Theorem 4. *If there exist single-valued solutions of Euler-Poisson's equations (1.1) which differ from the solutions of Bobilov-Steklov [6, 7] and Steklov [8] cases, then the parameters A_i, r_i satisfy one of the following conditions:*

1. $\prod_\sigma B_{12} \sum_\sigma r_1 \sqrt{B_{23}} = 0$;
2. $\sum_\sigma A_1 r_1 \sqrt{\dfrac{A_2 A_3}{B_{12} B_{31}}}$;
3. $\lambda_0 \in \mathbb{Z}$; *where* $\lambda_0 = \dfrac{1}{2} - \sqrt{\dfrac{1}{4} - S}$,

$$S = \frac{\dfrac{(2\langle A\gamma,\delta\rangle + \langle Ap,p\rangle)(\langle A\gamma,\delta\rangle + \langle Ap,p\rangle)}{\langle A\gamma,\gamma\rangle} - \dfrac{3\langle p,r\rangle^2}{\langle\gamma,r\rangle} - 2\langle A\delta,\delta\rangle + 2\langle\delta,r\rangle}{\dfrac{\langle p,r\rangle^2}{2\langle\gamma,r\rangle} - \dfrac{\langle A\gamma,\delta\rangle^2}{\langle A\gamma,\gamma\rangle} + \langle A\delta,\delta\rangle},$$

   (p,γ) *is any* β*-solution of the characteristic system and* δ *is a vector such that* $p \times \delta = -2\delta, \langle\gamma,\delta\rangle = 2$;

4. *the solution* $(p(t),\gamma(t))$ *has the representation*

$$\begin{cases} p(t) = \sum_{k=1}^{8} \tilde{p}_k^0 \zeta(t - t_k) + p_0, \\ \gamma(t) = \sum_{k=1}^{8} \tilde{\gamma}_k^0 \wp(t - t_k) + \gamma_0, \end{cases}$$

   where $\{(\tilde{p}_k^0, \tilde{\gamma}_k^0)\}, k \in \{1,\dots,8\}$ *are the* β*-solutions of the characteristic system (1.2) and* $\zeta(t), \wp(t)$ *are the Weierstrass functions ([5]).*

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