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ON WEAKLY ADDITIVE FUNCTIONALS

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The functor O of order-preserving functionals extended to the category Tych. The main result (Theorem 2) states that this new functor O_β preserves weight of Tychonoff spaces.

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Функтор O, сохрпняющий порядок функционалов, продолжается в категорию Tych. Основной результат (теорема 2) утверждает, что новый функтор O_β сохраняет вес тихоновских пространств.

Recall a construction introduced by T. Radul [1]. Let *Comp* be the category of all Hausdorff compact spaces (compacta) and their continuous mappings (maps). For $X \in$ *Comp*, by $C(X)$ we denote the set of all continuous functions $\varphi\colon X \to \mathbb{R}$ with ordinary sup-norm:

$$\|f\| = \sup\{|\varphi(x)| : \ x \in X\}.$$

For every $c \in \mathbb{R}$, by c_X we denote the constant function on X with value c. Let $\varphi, \psi \in C(X)$. We say that $\varphi \le \psi$ iff $\varphi(x) \le \psi(x)$ for every $x \in X$. A mapping $v\colon C(X) \to \mathbb{R}$ is said to be a *functional*. We say that a functional $v\colon C(X) \to \mathbb{R}$ is *weakly additive*, if

$$v(\varphi + c_X) = v(\varphi) + c$$

for every $\varphi \in C(X)$ and $c \in \mathbb{R}$. A functional $v\colon C(X) \to \mathbb{R}$ is said to be *order preserving*, if for every $\varphi, \psi \in C(X)$, provided $\varphi \le \psi$, we have that $v(\varphi) \le v(\psi)$.

We say that a functional $v\colon C(X) \to \mathbb{R}$ is *normed* if $v(1_X) = 1$.

For every normed weakly additive functional v we have

$$v(0_X) = 0. \tag{1}$$

Indeed,

$$v(0_X) = v(1_X - 1_X) = v(1_X + (-1)_X) =$$
$$= v(1_X) + v((-1)_X) = 1 - 1 = 0.$$

For a compactum X, by $O(X)$ we denote the set of all weakly additive, order preserving, normed functional on X. The set $O(X)$ can be considered as a subspace of $C_p(C(X))$, which

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is the set of all continuous functions on $C(X)$ equipped with the pointwise convergence topology (look at [1]).

Hence, a topology base of $O(X)$ consists of the following sets:

$$O(v, \varphi_1, \dots, \varphi_n, \varepsilon) = \{v' \in O(X) : |v'(\varphi_i) - v(\varphi_i)| < \varepsilon\}$$

for every $i \in \{1, \dots, n\}$, where

$$v \in O(X); \quad \varphi_1, \dots, \varphi_n \in C(X); \quad \varepsilon > 0.$$

Proposition 1 ([1], Theorem 1). *If X is a compactum, then $O(X)$ is a compactum as well.*

Let $X, Y \in Comp$ and $f\colon X \to Y$ be a map. Define a mapping $O(f)\colon O(X) \to O(Y)$ by the formula

$$(O(f)(v))(\varphi) = v(\varphi \circ f), \tag{2}$$

where $v \in O(X)$ and $\varphi \in C(Y)$.

From this definition we get that $O(f)(v) \in O(Y)$. It is also clear that the mapping $O(f)$ is continuous. Moreover, the following assertion holds:

$$O(f \circ g) = O(f) \circ O(g), \tag{3}$$

$$O(\mathrm{id}_X) = \mathrm{id}_{O(X)} \text{ for every } X \in Comp. \tag{4}$$

This yields

Theorem 1 [1]. *O is a covariant functor in $Comp$.*

T. Radul [1] has also shown that functor O is *weakly normal*, i.e. O satisfies all conditions of normality [2] except preservation of preimages. In particular,

Proposition 2 ([1], Proposition 3). *Functor O preserves weight of infinite compacta.*

Corollary 1. *Functor O preserves metrizable compacta.*

Clearly, every linear non-negative functional preserves order. This yields

Proposition 3 ([1]). *The functor $P\colon Comp \to Comp$ of probability measures is a subfunctor of O.*

The aim of this article is to discuss some properties of the spaces $O(X)$ for compacta X, and to extend the functor O to category $Tych$ of all Tychonoff spaces and their maps.

If k is some cardinal invariant, then by hk we denote a new cardinal invariant defined by

$$hk(X) = \sup\{k(Y) : Y \subset X\}.$$

By $c(X)$, $d(X)$, $t(X)$ and $\pi w(X)$ we denote respectively the Souslin number (or cellularity), density, tightness, and π-weight of X. Let us recall that *π-weight* of X is the minimum of cardinalities of π-bases of X, where a system $\mathcal{B}$ of non-empty open subsets of X is said to be a *π-base* of X if for an arbitrary non-empty open set $U \subset X$ there is an element $V \in \mathcal{B}$ such that $V \subset U$. Recall also that a regular space X is said to be a space of a *point-wise countable type*, if for every $x \in X$ there is a compactum $F \subset X$ such that $x \in F$ and F is a G_δ-set in X. In particular, every compactum is a space of a point-wise countable type.

Proposition 4. *If X is an infinite compactum, then*

$$hd(O(X)) \le 2^{hd(X)}.$$

Proof. B. E. Shapirovski [3] proved that if X is a regular space of a point-wise countable type, then $w(X) \le 2^{c(X)t(X)}$. Since $t(X) \le hd(X)$ and $c(X) \le hd(X)$, we have

$$hd(O(X)) \le w(O(X)) \overset{(\text{byProp.2})}{=} w(X) \le 2^{c(X)t(X)} = 2^{hd(X)}.$$

Proposition 4 is proved. □

It is known that $hd(X) = h\pi w(X)$ for every compactum X (see [3]). Hence, Proposition 4 yields

Corollary 2. *If X is an infinite compactum, then $h\pi w(O(X)) \le 2^{h\pi w(X)}$.*

The next statement is well known.

Proposition 5. *For every compactum Y its square Y^2 is topologically contained in $P(Y)$.*

Example 1. *There exists an infinite first countable compactum X^* such that*

$$hd(O(X^*)) = 2^{hd(X^*)}. \tag{5}$$

Proof. As an X^* we take "two Alexandroff's arrows" (see [4]). It is known that X^* is hereditarily separable space. Further, $O(X^*) \supset P(X^*)$ by Proposition 3. Hence, $O(X^*) \supset (X^*)^2$ in accordance with Proposition 5. Moreover, $O(X^*) \supset A^2$, where A is a "one arrow": an open base of $A = [0, 1)$ consists of half-intervals $[a, b)$, $0 \le a < b < 1$. The space A^2 contains a discrete closed set $F = \{(t, 1-t) : t \in (0, 1)\}$ of cardinality of the continuum. Consequently, $hd(O(X^*)) \ge 2^{\omega_0}$. Finally,

$$hd(O(X^*)) \le w(O(X^*)) \overset{(\text{Prop.2})}{=} w(X^*) = 2^{\omega_0}.$$

Equality (5) is proved. □

Remark 1. Example 1 shows that one cannot strengthen the assertion of Proposition 4 for an arbitrary compactum X. But there are compacta X with

$$hd(O(X)) < 2^{hd(X)}. \tag{6}$$

Indeed, as an example one can take any X with $w(X) = hd(X)$. Except metrizable compacta, a lot of non-metrizable compacta have the last property, for example every Tychonoff cube I^τ.

Remark 2. Example 1 and Proposition 5 show that the functor O does not preserve the class of hereditarily normal compacta. In fact, $X^* \times X^*$ is not a hereditarily normal compactum, but $X^* \times X^* \subset O(X^*)$ and X^* is hereditarily normal.

Now we are going to extend the functor O to the category *Tych*. We follow Chigogidze's scheme [5]. For an arbitrary monomorphic functor $F\colon Comp \to Comp$, a compactum X, and its closed subset A we may consider the compactum $O(A)$ as a closed subset of $F(X)$. We say that an element $a \in F(X)$ *is supported on* A if $a \in F(A)$. For $a \in F(X)$, let

$$M_a = \cap\{A : A \subset X,\ a \in F(A)\}.$$

If F is a monomorphic intersection-preserving functor, then $a \in F(M_a)$. Clearly, M_a is the smallest closed subset of X such that $a \in F(M_a)$. This set M_a is called the *support* of a and is denoted by $\operatorname{supp}_{F,X} a$ or simply $\operatorname{supp} a$.

We put

$$F_\beta(X) = \{a \in F(\beta X) : \operatorname{supp} a \subset X\}$$

for a monomorphic intersection-preserving functor $F\colon Comp \to Comp$ and a Tychonoff space X. If $f\colon X \to Y$ is a morphism in $Tych$, we set

$$F_\beta(f) = F(\beta f)|F_\beta(X),$$

where $\beta f\colon \beta X \to \beta Y$ is the Čech-Stone extension of f.

Proposition 6. *Let $F\colon Comp \to Comp$ be a monomorphic intersection-preserving functor. Then for any morphism $f\colon X \to Y$ of $Comp$ and $a \in F(X)$ we have*

$$f(\operatorname{supp} a) \supset \operatorname{supp} F(f)(a).$$

Proof. Let $L = \operatorname{supp} a$ and $M = f(L)$. Let $i_L\colon L \to X$ and $i_M\colon M \to Y$ be the identity embeddings. Then

$$f \circ i_L = i_M \circ (f|L).$$

Hence,

$$F(f) \circ F(i_L) = F(i_M) \circ F(f|L).$$

In particular,

$$F(f) \circ F(i_L)(a) = F(i_M) \circ F(f|L)(a) \tag{7}$$

Since F is a monomorphic functor, $F(i_L)\colon F(L) \to F(X)$ is the identity embedding. Thus $F(i_L)(a) = a$. Consequently, (7) implies that

$$\begin{aligned} F(f)(a) \in \operatorname{Im}(F(i_M) \circ F(f|L)) = F(i_M)(\operatorname{Im}(F(f|L)) = \\ = F(i_M)(F(f(L)) = F(i_M)(F(M)) = F(M). \end{aligned}$$

So, $\operatorname{supp} F(f)(a) \subset M$. But $M = f(\operatorname{supp} a)$ by the definition. □

From this proposition we get that $F(\beta f)(F_\beta(X)) \subset F_\beta(Y)$. So we defined the map

$$F_\beta(f)\colon F_\beta(X) \to F_\beta(Y).$$

Clearly, the construction F_β is a covariant functor in $Tych$.

Lemma 1. *Let $f\colon X \to Y$ be a closed map from a Tychonoff space X onto a normal space Y. Let B be a closed subset of Y and $A = f^{-1}(B)$. Then for every bounded continuous function $\varphi\colon X \to \mathbb{R}$ such that $\varphi(A) = 0$ there is a bounded continuous function $\psi\colon Y \to \mathbb{R}$ such that $\psi(B) = 0$ and $\varphi \leq \psi \circ f$.*

Proof. Going from φ to $|\varphi|$ we may assume that $\varphi \geq 0$. Since φ is bounded, without loss of generality we may also assume that $0 \leq \varphi \leq 1$. Let

$$C_k = \{x \in X \colon \varphi(x) \geq 2^{-k-1}\}, \quad k \in \omega.$$

The set C_k is closed in X and doesn't meet A. Hence, $f(C_k)$ is closed in Y and does not meet B. By Urysohn's lemma there is a continuous function $\psi_k \colon Y \to [0, 2^{-k}]$ such that

$$\psi_k(B) = 0, \tag{8}$$

$$\psi_k(f(C_k)) = 2^{-k}. \tag{9}$$

Let $\psi = \sum_{k=0}^{\infty} \psi_k$. Clearly, ψ is continuous. Further, ψ is bounded: $0 \leq \psi \leq 2$. From (8) we get that $\psi(B) = 0$. It remains to check that

$$\varphi(x) \leq (\psi \circ f)(x) \tag{10}$$

for any $x \in X$.

Since $\psi \geq 0$, it suffices to consider points x with $\varphi(x) > 0$. There exists the smallest k such that $x \in C_k$. Then

$$2^{-k-1} \leq \varphi(x) \leq 2^{-k}.$$

Consequently,

$$\psi(f(x)) \geq \psi_k(f(x)) \overset{(9)}{=} 2^{-k} \geq \varphi(x).$$

Hence, condition (10) holds. □

Proposition 7. *Let $f \colon X \to Y$ be a closed map from a Tychonoff space X onto a normal space Y. Let B be a closed subset of Y and $A = f^{-1}(B)$. Let $f|A \colon A \to B$ be a homeomorphism. Then for every bounded continuous function $\varphi \colon X \to \mathbb{R}$ there are bounded continuous functions $\psi_1, \psi_2 \colon Y \to \mathbb{R}$ such that*

$$\varphi|A = \psi_1 \circ f|A = \psi_2 \circ f|A, \tag{11}$$

$$\psi_1 \circ f \leq \varphi \leq \psi_2 \circ f. \tag{12}$$

Proof. Since $f|A$ is a homeomorphism, there is a function $\chi_0 \colon B \to \mathbb{R}$ such that

$$\varphi|A = \chi_0(f|A)\ (\chi_0(y) = \varphi(f^{-1}(y))). \tag{13}$$

In view of a normality of Y we may apply Brouwer-Tietze-Urysohn theorem. Hence, there is a bounded continuous function $\chi \colon Y \to \mathbb{R}$ such that $\chi|B = \chi_0$. Let $\alpha = \varphi - \chi \circ f$. Then $\alpha|A = 0$, because of (13). According to Lemma 1 there exists a bounded continuous function $\beta \colon Y \to \mathbb{R}$ such that $\beta(B) = 0$ and $\alpha \leq \beta \circ f$. Set $\psi_2 = \beta + \chi$. Then

$$\psi_2 \circ f|A = \beta \circ f|A + \chi \circ f|A \overset{\beta(B)=0}{=} \chi \circ f|A \overset{(13)}{=} \varphi|A.$$

Further, from $\alpha \leq \beta \circ f$ we get

$$\varphi \leq \beta \circ f + \chi \circ f = \psi_2 \circ f.$$

So, we have a constructed function ψ_2 satisfying conditions (11) and (12). Now we repeat the previous proof for the function $-\varphi$ and construct a function ψ_2^- satisfying (11) and (12) for $-\varphi$. It remains to put $\psi_1 = -\psi_2^-$. □

Proposition 8. (Lemma 4 [1]) *Let X be a compactum, A be a closed subset of X, and $u \in O(X)$. Then u is supported on A iff for each $\varphi_1, \varphi_2 \in C(X)$ with $\varphi_1|A = \varphi_2|A$ we have $u(\varphi_1) = u(\varphi_2)$.*

Proposition 9. *Let X be a Tychonoff space, bX be its compactification, and $\pi\colon \beta X \to bX$ be the natural projection. Then*

$$O(\pi)|O_\beta(X)\colon O_\beta(X) \to O_b(X)$$

is a homeomorphism, where

$$O_b(X) = \{v \in O(bX) : \ \operatorname{supp} v \subset X\}.$$

Proof. First, we show that

$$(O(\pi))^{-1}(O_b(X)) = O_\beta(X). \tag{14}$$

Proposition 6 yields the inclusion $\supset$. Now let $u \in O({}_\beta X)$ and let

$$v = O(\pi)(u) \in O_b(X).$$

Put $B = \operatorname{supp} v$. By the definition, $B \subset X$. To check the inclusion $\subset$, it suffices to prove that

$$\operatorname{supp} u \subset B = \pi^{-1}(B).$$

In accordance with Proposition 7 there are functions $\psi_{ij} \in C(bX)$, $i, j \in \{1, 2\}$ such that

$$\psi_{ij}|B = \varphi_i|B, \quad i, j \in \{1, 2\}; \tag{15}$$

$$\psi_{11} \circ \pi \le \varphi_1 \le \psi_{12} \circ \pi; \tag{16}$$

$$\psi_{21} \circ \pi \le \varphi_2 \le \psi_{22} \circ \pi. \tag{17}$$

Proposition 8 and equality (15) imply

$$v(\psi_{11}) = v(\psi_{12}) = v(\psi_{21}) = v(\psi_{22}). \tag{18}$$

From the definition of $O(\pi)$ we get

$$v(\psi_{ij}) = u(\psi_{ij} \circ \pi), \quad i, j \in \{1, 2\}. \tag{19}$$

Since u is an order preserving functional, conditions (16)–(19) yield $u(\varphi_1) = u(\varphi_2)$. Then $\operatorname{supp} u \subset B$ by Proposition 8. Thus equality (14) is proved. This equality implies that $O(\pi)|O_\beta(X)$ is a closed map.

It remains to show that $O(\pi)|O_\beta(X)$ is a bijection. Let $u_1, u_2 \in O_\beta(X)$ and $u_1 \ne u_2$. Denote $\operatorname{supp} u_i$ by K_i and set $K = K_1 \cup K_2$. Since O is a monomorphic functor, $u_i \in O(K)$ and $O(\pi)(u_i) = O(\pi|K)(u_i)$. But $\pi|K$ is a homeomorphism. Hence, $O(\pi|K)$ is a homeomorphism too. Consequently,

$$O(\pi)(u_1) = O(\pi|K)(u_1) \ne O(\pi|K)(u_2) = O(\pi)(u_2).$$

□

Remark 3. In ([5], Proposition 1) A. Ch. Chigogidze formulated (without proof) a statement similar to Proposition 9 for any normal functor $F\colon Comp \to Comp$. It seems that a normality of F is essential. We do not see how one can prove this statement for an arbitrary weakly normal but non-normal functor.

Propositions 2 and 9 imply

Theorem 2. *The functor O_β preserves weight of infinite Tychonoff spaces.*

Corollary 3. *The functor O_β preserves the class of separable metrizable spaces.*

Since O is a weakly normal functor, Theorem 1 from [6] implies

Proposition 10. *For every infinite compactum X we have $dO(X) \le dX$.*

Corollary 4. *The functor O preserves the class of separable compacta.*

Recall that a topological space X has a *weak density* $\le \tau$ [7] if τ is the smallest infinite cardinal such that there is a π-base of X which is a union of τ centered systems. In this case we write $wdX \le \tau$. It should be mentioned that in 1977 the weak density property for Tychonoff spaces was implicitly considered by E. K. van Douwen [8]. A space X is said to be *weakly separable*, if $wdX \le \omega_0$.

The next statement was actually proved by E. K. van Douwen [8].

Proposition 11. *For an arbitrary Tychonoff space X the following conditions are equivalent:*

1) $wdX \le \tau$;

2) $d(bX) \le \tau$ for some compactification bX;

3) $d(bX) \le \tau$ for any compactification bX.

Proposition 12. *For every Tychonoff space X we have $wdO_\beta(X) \le wdX$.*

Proof. Let $wdX \le \tau$. Then $d(\beta X) \le \tau$ by Proposition 11. Hence, $dO(\beta X) \le \tau$ in view of Proposition 10. But $O_\beta(X)$ is dense in $O(\beta X)$ according to Proposition 3 from [9]. Thus, applying Proposition 11 once more we get $wd(O_\beta(X)) \le \tau$. □

Corollary 5. *The functor O_β preserves the class of weakly separable Tychonoff spaces.*

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