УДК 517.95

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INVERSE PROBLEM OF DETERMINATION OF THE SOURCE IN A GENERAL PARABOLIC EQUATION

I. B. Bereznyts'ka. *Inverse problem of determination of the source in a general parabolic equation*, Matematychni Studii, **18** (2002) 169–176.

Existence and uniqueness theorems for the inverse problem of the determination of the source in a general parabolic equation are proved. The first and second boundary problems are considered. Overdetermination condition is of nonlocal and integral type.

И. Б. Березницкая. *Обратная задача определения источника в общем параболическом уравнении* // Математичні Студії. – 2002. – Т.18, №2. – С.169–176.

Доказаны теоремы существования и единственности для обратной задачи определения источника в общем параболическом уравнении. Рассмотрены краевые задачи первого и второго рода, а условия переопределения нелокального и интегрального типа.

Many authors have investigated problems of determination of unknown source. In [1, 2, 3] problems with local overdetermination condition were considered. The next point in the investigation of such problems was consideration of a nonlocal overdetermination condition [4, 5].

In this paper we use a nonlocal and integral condition for general parabolic equation. Two problems with boundary data of the first and second kind were considered. In [5] such problem was investigated but with a nonlocal overdetermination condition. In existence and uniqueness theorems we use properties of a Volterra integral equation of the second kind.

Results of these problems may be used in the investigation of the uniqueness of the coefficient inverse problems.

In the domain $Q_T = \{(x,t) : 0 < x < h,\ 0 < t < T\}$ consider the equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u + f(t)g_0(x,t) + g_1(x,t),\ (x,t) \in Q_T, \tag{1}$$

the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,h], \tag{2}$$

the boundary data of the second kind

$$u_x(0,t) = \mu_1(t),\ u_x(h,t) = \mu_2(t),\ t \in [0,T], \tag{3}$$

2000 *Mathematics Subject Classification:* 35R30.

and the overdetermination condition of the nonlocal and integral type

$$\nu_1(t)u(0,t)+\nu_2(t)u(h,t)+\nu_3(t)\int\limits_0^h u(x,t)dx=\mu_3(t),\ t\in[0,T]. \tag{4}$$

Definition 1. A pair of functions $(f(t),u(x,t))\in H^{\alpha/2}[0,T]\times H^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ [6, p.16], $0<\alpha<1$, which satisfies conditions (1)–(4), is called a solution of problem (1)–(4).

Theorem 1. *Suppose that the problem data satisfy the following conditions:*

(A_1) $\varphi(x)\in H^{2+\alpha}[0,h],\ \mu_i(t)\in H^{(1+\alpha)/2}[0,T],\ i\in\{1,2\},$
$\nu_i(t),\ \mu_3(t)\in H^{1+\alpha/2}[0,T],\ i\in\{1,2,3\},$
$a(x,t),\ b(x,t),\ c(x,t),\ g_i(x,t)\in H^{\alpha,\alpha/2}(\overline{Q}_T),\ i\in\{0,1\};$

(A_2) $a(x,t)>0,\ (x,t)\in\overline{Q}_T;$

(A_3) $\nu_1(t)g_0(0,t)+\nu_2(t)g_0(h,t)+\nu_3(t)\int\limits_0^h g_0(x,t)dx\neq 0;$

(A_4) $\varphi'(0)=\mu_1(0),\ \varphi'(h)=\mu_2(0),$

$$\nu_1(0)\varphi(0)+\nu_2(0)\varphi(h)+\nu_3(0)\int\limits_0^h\varphi(x)dx=\mu_3(0).$$

Then problem (1)–(4) possesses a unique solution for $x\in[0,h],\ t\in[0,T]$.

Proof. Let

$$u(x,t)=v(x,t)+w(x,t),$$

be a solution of the problem, where the function $v(x,t)$ satisfies

$$v_t=a(x,t)v_{xx}+b(x,t)v_x+c(x,t)v+g_1(x,t),\ (x,t)\in Q_T, \tag{5}$$

$$v(x,0)=\varphi(x),\ x\in[0,h], \tag{6}$$

$$v_x(0,t)=\mu_1(t),\ v_x(h,t)=\mu_2(t),\ t\in[0,T]. \tag{7}$$

If conditions (A_1) are satisfied then direct problem (5)–(7) has a unique solution $v(x,t)\in H^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ [6, p.364]. The function $w(x,t)$ is a solution of the problem

$$w_t=a(x,t)w_{xx}+b(x,t)w_x+c(x,t)w+f(t)g_0(x,t),\ (x,t)\in Q_T, \tag{8}$$

$$w(x,0)=0,\ x\in[0,h], \tag{9}$$

$$w_x(0,t)=w_x(h,t)=0,\ t\in[0,T], \tag{10}$$

$$\nu_1(t)w(0,t)+\nu_2(t)w(h,t)+\nu_3(t)\int\limits_0^h w(x,t)dx=\mu_4(t),\ t\in[0,T], \tag{11}$$

where

$$\mu_4(t)=\mu_3(t)-\nu_1(t)v(0,t)-\nu_2(t)v(h,t)-\nu_3(t)\int\limits_0^h v(x,t)dx.$$

Let a pair of functions $(f(t),\ w(x,t))$ be a solution of problem (8)–(10). By using equation (8) and boundary conditions (10) we differentiate (11) with respect to t,

$$\nu_1'(t)w(0,t)+\nu_1(t)(a(0,t)w_{xx}(0,t)+c(0,t)w(0,t)+f(t)g_0(0,t))+$$

$$+\nu_2'(t)w(h,t)+\nu_2(t)(a(h,t)w_{xx}(h,t)+c(h,t)w(h,t)+f(t)g_0(h,t))+$$

$$+\nu_3'(t)\int\limits_0^h w(x,t)dx+\nu_3(t)\Bigg(\int\limits_0^h a(x,t)w_{xx}(x,t)dx+\int\limits_0^h b(x,t)w_x(x,t)dx+$$

$$+\int\limits_0^h c(x,t)w(x,t)dx+f(t)\int\limits_0^h g_0(x,t)dx\Bigg)=\mu_4'(t).$$

Then we have the following equation for the function $f(t)$:

$$f(t)\Bigg(\nu_1(t)g_0(0,t)+\nu_2(t)g_0(h,t)+\nu_3(t)\int\limits_0^h g_0(x,t)dx\Bigg)+$$

$$+\nu_1(t)a(0,t)w_{xx}(0,t)+\nu_2(t)a(h,t)w_{xx}(h,t)+(\nu_1'(t)+\nu_1(t)c(0,t))w(0,t)+$$

$$+(\nu_2'(t)+\nu_2(t)c(h,t))w(h,t)+\nu_3'(t)\int\limits_0^h w(x,t)dx+$$

$$+\nu_3(t)\Bigg(\int\limits_0^h a(x,t)w_{xx}(x,t)dx+\int\limits_0^h b(x,t)w_x(x,t)dx+$$

$$+\int\limits_0^h c(x,t)w(x,t)dx\Bigg)=\mu_4'(t). \tag{12}$$

Write the solution of problem (8)–(10) by means of the Green function of the second boundary problem $G_2(x,t,\xi,\tau)$ [5] as follows

$$w(x,t) = f(\tau)d\tau \int_0^h G_2(x,t,\xi,\tau)g_0(\xi,\tau)d\xi. \tag{13}$$

Then according to [7, p.21] we have

$$w_{xx}(x,t) = f(\tau)d\tau \int_0^h G_{2xx}(x,t,\xi,\tau)g_0(\xi,\tau)d\xi. \tag{14}$$

Substitute (13), (14) in (12) and obtain the following equation

$$f(t)\left(\nu_1(t)g_0(0,t) + \nu_2(t)g_0(h,t) + \nu_3(t)\int_0^h g_0(x,t)dx\right) + K(t,\tau)f(\tau)d\tau = \mu_4'(t), \tag{15}$$

where for function $K(t,\tau)$ the following estimate holds [7, p.23]:

$$|K(t,\tau)| \leq \frac{\text{const}}{(t-\tau)^{\mu}}, \quad 1-\frac{\alpha}{2} < \mu < 1.$$

If condition (A_3) is satisfied then equation (15) is a Volterra integral equation of the second kind and it has a unique solution $f(t) \in H^{\alpha/2}[0,T]$ [8, p.422]. Since the function $f(t) \in H^{\alpha/2}[0,T]$ is known, the direct problem (8)–(10) has a unique solution $w(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ [6, p.364]. □

Let us consider boundary data of the first kind

$$u(0,t) = \mu_1(t), \ u(h,t) = \mu_2(t), \tag{16}$$

and an overdetermination condition

$$\nu_1(t)u_x(0,t) + \nu_2(t)u_x(h,t) + \nu_3(t)\int_0^h u(x,t)dx = \mu_3(t), \ t \in [0,T]. \tag{17}$$

Definition 2. A pair of function $(f(t),\, u(x,t)) \in H^{\alpha/2}[0,T] \times H^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$, $0<\alpha<1$, which satisfies conditions (1), (2), (16), (17) is called a solution of problem (1), (2), (16), (17).

Theorem 2. *Suppose that the problem data satisfy the following conditions:*

(A_1) : $\varphi(x) \in H^{2+\alpha}[0,h]$, $\mu_i(t) \in H^{1+\alpha/2}[0,T]$, $i \in \{1,2\}$,
$\nu_i(t)$, $\mu_3(t) \in H^{(1+\alpha)/2}[0,T]$, $i \in \{1,2,3\}$,
$a(x,t)$, $g_i(x,t) \in H^{1+\alpha,\alpha/2}(\overline{Q}_T)$, $i \in \{0,1\}$, $b(x,t)$, $c(x,t) \in H^{\alpha,\alpha/2}(\overline{Q}_T)$;

(A_2) : $a(x,t) > 0$, $(x,t) \in \overline{Q}_T$;

(A_3) : $\dfrac{\nu_1(t)g_0(0,t)}{\sqrt{a(0,t)}} - \dfrac{\nu_2(t)g_0(h,t)}{\sqrt{a(h,t)}} \neq 0;$

(A_4) : $\varphi(0) = \mu_1(0),\ \varphi(h) = \mu_2(0),$

$\nu_1(0)\varphi'(0) + \nu_2(0)\varphi'(h) + \nu_3(0)\int\limits_0^h \varphi(x)dx = \mu_3(0).$

Then problem (1), (2), (16), (17) possesses a unique solution for $x \in [0,h],\ t \in [0,T]$.

Proof. Consider the problem with $\varphi(x) \equiv 0,\ x \in [0,h],\ \mu_1(t) \equiv \mu_2(t) \equiv 0,\ t \in [0,T]$, $g_1(x,t) \equiv 0,\ (x,t) \in \overline{Q}_T$. Let the function $f(t)$ be known. Write the solution of the direct problem (1), (2), (16) by using the Green function of the first boundary problem $G_1(x,t,\xi,\tau)$ as

$$u(x,t) = f(\tau)d\tau \int\limits_0^h G_1(x,t,\xi,\tau)g_0(\xi,\tau)d\xi$$

and substitute it in the overdetermination condition (17):

$$I_1(t) + I_2(t) = \mu_3(t), \tag{18}$$

where

$$I_1(t) = f(\tau)d\tau \int\limits_0^h (\nu_1(t)G_{1x}(0,t,\xi,\tau) + \nu_2(t)G_{1x}(h,t,\xi,\tau))g_0(\xi,\tau)d\xi, \tag{19}$$

$$I_2(t) = \nu_3(t)\int\limits_0^h dx f(\tau)d\tau \int\limits_0^h G_1(x,t,\xi,\tau)g_0(\xi,\tau)d\xi. \tag{20}$$

Since the Green function can be written [5] as

$$G_1(x,t,\xi,\tau) = Z(x,t,\xi,\tau) + G(x,t,\xi,\tau)$$

with the principal part

$$Z(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi a(\xi,\tau)(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4a(\xi,\tau)(t-\tau)}\right) \tag{21}$$

and

$$|G(x,t,\xi,\tau)| \le \frac{C}{(t-\tau)^{(1-\alpha)/2}}\exp\left(-\frac{(x-\xi)^2}{4a_0(t-\tau)}\right),\ a_0 = \max_{\overline{Q}_T} a(x,t),$$

substitute (21) in (19), (20). Consider

$$\widetilde{I}_{11}(t) = \frac{1}{4\sqrt{\pi}}\nu_1(t)\frac{f(\tau)}{(t-\tau)^{3/2}}d\tau \times$$

$$\int\limits_0^h \frac{g_0(\xi,\tau)}{a^{3/2}(\xi,\tau)}\xi\exp\left(-\frac{\xi^2}{4a(\xi,\tau)(t-\tau)}\right)d\xi. \tag{22}$$

Let us change the variable t into σ, multiply () by $\dfrac{1}{\sqrt{t-\sigma}}$, integrate from 0 to t and change the order of integration

$$\frac{\widetilde{I}_{11}(\sigma)}{\sqrt{t-\sigma}}d\sigma = \frac{1}{4\sqrt{\pi}}f(\tau)d\tau\int_{\tau}^{t}\frac{\nu_1(\sigma)d\sigma}{\sqrt{(t-\sigma)(\sigma-\tau)^3}}\times$$

$$\times\int_0^h \frac{g_0(\xi,\tau)}{a^{3/2}(\xi,\tau)}\xi\exp\left(-\frac{\xi^2}{4a(\xi,\tau)(\sigma-\tau)}\right)d\xi.$$

Use the substitutions

$$\eta = \frac{\xi}{2\sqrt{\sigma-\tau}},$$

$$z = \frac{\sigma-\tau}{t-\tau}, \tag{23}$$

and obtain

$$\frac{\widetilde{I}_{11}(\sigma)}{\sqrt{t-\sigma}}d\sigma = \frac{1}{\sqrt{\pi}}f(\tau)d\tau\int_0^1\frac{\nu_1(z(t-\tau)+\tau)}{\sqrt{(1-z)z}}dz\times$$

$$\times\int_0^{\frac{h}{2\sqrt{z(t-\tau)}}}\frac{g_0(2\eta\sqrt{z(t-\tau)},\tau)}{a^{3/2}(2\eta\sqrt{z(t-\tau)},\tau)}\eta\exp\left(-\frac{\eta^2}{a(2\eta\sqrt{z(t-\tau)},\tau)}\right)d\eta. \tag{24}$$

Differentiate (24) with respect to t:

$$\frac{d}{dt}\left(\frac{\widetilde{I}_{11}(\sigma)}{\sqrt{t-\sigma}}d\sigma\right) = \frac{\sqrt{\pi}}{2}f(t)\nu_1(t)\frac{g_0(0,t)}{\sqrt{a(0,t)}}+$$

$$+\frac{1}{\sqrt{\pi}}f(\tau)\frac{\partial}{\partial t}\left(\int_0^1\frac{\nu_1(z(t-\tau)+\tau)}{\sqrt{(1-z)z}}dz\times\right.$$

$$\left.\times\int_0^{\frac{h}{2\sqrt{z(t-\tau)}}}\frac{g_0(2\eta\sqrt{z(t-\tau)},\tau)}{a^{3/2}(2\eta\sqrt{z(t-\tau)},\tau)}\eta\exp\left(-\frac{\eta^2}{a(2\eta\sqrt{z(t-\tau)},\tau)}\right)d\eta\right)d\tau. \tag{25}$$

Transformation $\widetilde{I}_{12}(t)$:

$$\frac{d}{dt}\left(\frac{\widetilde{I}_{12}(\sigma)}{\sqrt{t-\sigma}}d\sigma\right) = -\frac{\sqrt{\pi}}{2}f(t)\nu_2(t)\frac{g_0(h,t)}{\sqrt{a(h,t)}}-$$

$$-\frac{1}{\sqrt{\pi}}f(\tau)\frac{\partial}{\partial t}\Big(\int\limits_0^1 \frac{\nu_2(z(t-\tau)+\tau)}{\sqrt{(1-z)z}}dz \times$$

$$\times \int\limits_0^{\frac{h}{2\sqrt{z(t-\tau)}}} \frac{g_0(h-2\sqrt{z(t-\tau)},\tau)}{a^{3/2}(h-2\sqrt{z(t-\tau)},\tau)}\eta \exp\Big(-\frac{\eta^2}{a(2\sqrt{z(t-\tau)},\tau)}\Big)d\eta\Big)d\tau. \qquad (26)$$

Similarly, using (23), for $I_2(t)$ we obtain

$$\frac{d}{dt}\Big(\frac{I_2(\sigma)}{\sqrt{t-\sigma}}d\sigma\Big) = f(\tau)d\tau \int\limits_0^h \frac{g_0(\xi,\tau)d\xi}{\sqrt{a(\xi,\tau)}} \int\limits_0^h \frac{\partial}{\partial t}\Big(\int\limits_0^1 \frac{\nu_3(z(t-\tau)+\tau)}{\sqrt{(1-z)z}} \times$$

$$\times \exp\Big(-\frac{(x-\xi)^2}{4a(\xi,\tau)z(t-\tau)}\Big)dz\Big)dx. \qquad (27)$$

Taking into account (25), (26), (27) we obtain from (18) the following equation for $f(t)$:

$$f(t)\Big(\frac{\nu_1(t)g_0(0,t)}{\sqrt{a(0,t)}} - \frac{\nu_2(t)g_0(h,t)}{\sqrt{a(h,t)}}\Big) + K(t,\tau)f(\tau)d\tau = \frac{2}{\sqrt{\pi}}\frac{d}{dt}\Big(\frac{\mu_3(\sigma)}{\sqrt{t-\sigma}}d\sigma\Big). \qquad (28)$$

If conditions (A_1) are satisfied then for the function $K(t,\tau)$ the following estimate holds [9, Lemma 4]

$$|K(t,\tau)| \le \frac{\text{const}}{(t-\tau)^{(2-\alpha)/2}}.$$

Equation (28) possesses a unique solution $f(t) \in H^{\alpha/2}[0,T]$ [8]. Then the direct problem (1), (2), (16) has a unique solution $u(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$. □

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Received 20.09.2002