УДК 511.33

O. V. Savastru

THE NON-SYMMETRIC DIVISOR FUNCTION $\tau(1,1,2;N)$ IN ARITHMETIC PROGRESSION

O. V. Savastru. *The non-symmetric divisor function $\tau(1,1,2;n)$ in arithmetic progression*, Matematychni Studii, **18** (2002) 115–124.

For the function $\tau(1,1,2;n)$ that defines the number of different representations of n in the form $n = n_1 n_2 n_3^2$, we construct an asymptotic formula of its summatory function in arithmetic progression, nontrivial for $q \ll x^{\frac{14}{27}}$, where q is the difference of the progression.

О. В. Савастру. *Несимметрический делитель функции в арифметической прогрессии* // Математичні Студії. – 2002. – Т.18, №2. – С.115–124.

Для функции $\tau(1,1,2;n)$, определяющей количество различных представлений n в виде $n = n_1 n_2 n_3^2$, построена асимптотическая формула ее сумматорной функции в арифметической прогрессии, которая нетривиальна для $q \ll x^{\frac{14}{27}}$, где q — разность прогрессии.

We define the non-symmetric divisor function

$$\tau(a,b,c;n) = \sum_{\substack{n_i \in \mathbb{N} \\ n_1^a n_2^b n_3^c = n}} 1,$$

where a, b, c are positive integers.

For $a = b = c = 1$ we obtain the classical divisor function $\tau_3(n)$, investigated by D. R. Heath-Brown in [1].

The function $\tau(a,b,c;n)$ appears in different problems of statistical theory of numbers, in studying of distribution of integer points with special properties in the 3-dimensional space. Moreover, its summatory function can be found in problems on number of non-isomorphic abelian groups of orders not exceeding X. In this paper we investigate the summatory function

$$D(1,1,2;X,q,a) = \sum_{\substack{n \le X \\ n \equiv a(q)}} \tau(1,1,2;n),$$

where $a \in \mathbb{Z}$, $0 \le a < q$, $q \in \mathbb{N}$ is the difference of arithmetic progression.

Notation: The lower case letters other that x, X, y and z denote integers, the letter p normally denotes a prime number. We write $\|x\|$ for the distance of the real number x from

2000 *Mathematics Subject Classification:* 11N37, 11N64.

the nearest integer, $[x]$ for the integer part of x; the greatest common divisor of a and b is denoted by (a,b). The Vinogradov and Landau symbols $X \ll Y$, $X = \mathrm{O}(Y)$ mean $X \le cY$ with constant c that may depend on ε, $\varphi(n)$ is Euler's function, χ_{q0} denotes a main character mod q. We will use the function

$$e_q(x) = \exp(2\pi i x/q), \quad e_q(\bar{a}b) = e_q(db),$$

where $ad \equiv 1(\mathrm{mod} q)$.

We shall prove the following asymptotic formula.

Theorem. *Let a, q be natural numbers, $0 \le a < q$. Then for $X \to \infty$ we have*

$$D\left(1,1,2;X,q,a\right) = M\left(1,1,2;X,q,a\right) + \mathrm{O}\left(X^{\frac{55}{69}+\varepsilon} q^{-\frac{14}{23}}\right) + \mathrm{O}\left(X^{\frac{63}{80}+\varepsilon} q^{-\frac{693}{1120}}\right)$$

uniformly in a, q, X, where $\delta = (a,q)$,

$$M\left(1,1,2;X,q,a\right) = \frac{X}{\varphi\left(q/\delta\right)} \mathrm{Res}\left\{\left(\sum_{\substack{m=1\\(m,q)=\delta}}^{\infty} \tau\left(1,1,2;m\right) m^{-s}\right) \frac{X^{s-1}}{s}, s=1\right\},$$

ε is any positive number.

The strategy for the proof of Theorem is as follows. We assume that $(a,q)=1$ and introduce the sum $D\left(1,1,2;X,q,a\right)$ as a triple sum. We shall give different estimates according to the ranges of the three variables and make the preparatory transformations needed for each of these estimates. Moreover, we shall make use of auxiliary functions such as $L\left(a,b,c;q\right)$ and $S(k,t_1,t_2,\rho,\sigma;q)$, when the variables r, s, t^2 are typically of size less than $q^{1/2}$. The estimates are then compared and various choice of parameters are made, producing the error term, which appears in the statement of Theorem. In each estimate there is a main term which is not calculated explicitly. We merely know that it is independent of a. But then we show that this is sufficient to determine the leading term explicitly. Finally, we show how to remove the condition $(a,q)=1$.

So, we rewrite $D\left(1,1,2;X,q,a\right)$ as

$$D(1,1,2;X,q,a) = \#\{(u,v,w) \in \mathbb{N}^3;\ \ uvw^2 \le X,\ \ uvw^2 \equiv a(q)\}. \tag{1}$$

At first we must remove the condition $uvw^2 \le X$. Let $qX^{-1} \le \delta \le 1$, $\zeta = 1+\delta$. We split the ranges for u, v, w into intervals $u \in A = (U, \zeta U]$, $v \in B = (V, \zeta V]$, $w \in C = (W, \zeta W]$, where U, V, W run over powers of ζ and $U, V, W \gg 1$, $UVW^2 \le X$. Let

$$N(U,V,W) = \#\{(u,v,w), u \in A, v \in B, w \in C, uvw^2 \equiv a(q)\}. \tag{2}$$

Then

$$\begin{aligned} D(1,1,2;X,q,a) &= \sum_{UVW^2 \le X\zeta^{-4}} N(U,V,W) = \\ &= \sum_{UVW^2 \le X} N(U,V,W) + \mathrm{O}(X^{1+\varepsilon}\delta q^{-1}). \end{aligned} \tag{3}$$

Now we make several preliminary transformations:

$$N(U,V,W)=\sum_{\substack{\alpha,\beta,\gamma=1\\ \alpha\beta\gamma^2\equiv a(q)}}^{q}\#\{(u,v,w);u\in A,v\in B,w\in C,q\,|\alpha-u,\beta-v,\gamma-w\}=$$
$$=q^{-3}\sum_{r,s,t=1}^{q}\Bigg(\sum_{\substack{\alpha,\beta,\gamma=1\\ \alpha\beta\gamma^2\equiv a(q)}}^{q}e_q(r\alpha+s\beta+t\gamma)\Bigg)F_q(r,s,t); \tag{4}$$

where

$$\Big|F_q(r,s,t)\Big|=\left|\left(\sum_{u\in A}e_q(-ru)\right)\left(\sum_{v\in B}e_q(-sv)\right)\left(\sum_{w\in C}e_q(-tw)\right)\right|\ll$$
$$\min\left(I,\left\|\frac{r}{q}\right\|^{-1}\right)\ll\min\left(J,\left\|\frac{s}{q}\right\|^{-1}\right)\min\left(M,\left\|\frac{t}{q}\right\|^{-1}\right),$$
$$I=\#\{u\in A\},\ J=\#\{v\in B\},\ M=\#\{w\in C\}.$$

$$N(U,V,W)=q^{-2}\sum_{w\in C}\sum_{r,s=1}^{q}\sum_{\substack{\alpha,\beta=1\\ \alpha\beta w^2\equiv a(q)}}^{q}e_q(r\alpha+s\beta)F_q(r,s), \tag{5}$$

where $\Big|F_q(r,s)\Big|\ll\min\left(I,\left\|\frac{r}{q}\right\|^{-1}\right)\times\min\left(J,\left\|\frac{s}{q}\right\|^{-1}\right)$;

$$N(U,V,W)=q^{-2}\sum_{u\in A}\sum_{s,t=1}^{q}\Bigg(\sum_{\substack{\beta,\gamma=1\\ u\beta\gamma^2\equiv a(q)}}^{q}e_q(s\beta+t\gamma)\Bigg)F_q(s,t), \tag{6}$$

where $\Big|F_q(s,t)\Big|\ll\min\left(J,\left\|\frac{s}{q}\right\|^{-1}\right)\times\min\left(M,\left\|\frac{t}{q}\right\|^{-1}\right)$.

$$N(U,V,W)=q^{-1}\sum_{u\in A}\sum_{w\in C}\sum_{s=1}^{q}e_q(as\overline{uw^2})F_q(s), \tag{7}$$

where $\Big|F_q(s)\Big|=\Big|\sum_{v\in B}e_q(-sv)\Big|\ll\min\left(J,\left\|\frac{s}{q}\right\|^{-1}\right)$;

$$N(U,V,W)=q^{-1}\sum_{u\in A}\sum_{v\in B}\sum_{t=1}^{q}e_q(at\overline{uv})F_q(t), \tag{8}$$

where

$$\Big|F_q\,(t)\,\Big|=\left|\sum_{w\in C}e_q(-tw^2)\right|\ll\left|\frac{M}{q}\sum_{w(\mathrm{mod}q)}e_q(-tw^2)\right|\ll\frac{M}{q^{1/2}}\,(t,q)^{1/2}\log q.$$

Now we define a sum:

$$L(a,b,c;q)=\sum_{\substack{x,y,z=1\\ xy^2z\equiv1(q)}}^{q}e_q(ax+by+cz). \tag{9}$$

We can show that this sum has several properties.

Lemma 1. *For $a, b, c \in \mathbb{Z}$, $q \in \mathbb{N}$ we have*

(i) $L(a,b,c;q_1q_2) = L(a,q_2^2b,c;q_1)L(a,q_1^2b,c;q_2)$, *where* $(q_1,q_2)=1$.

(ii) $L(p^r a, p^r b, p^r c; p^\alpha) = p^{2t}L(a,b,c;p^{\alpha-t})$, $t=\min\{\alpha, r\}$.

(iii) $$L(a,b,c;p) = \begin{cases} p-1 & \text{if } p|a,b,c; \\ 1-p & \text{if } p|a,b;\ p\not|c \text{ or } p|a,c;\ p\not|b \text{ or } p|b,c;\ p\not|a; \\ 1 & \text{if } p|c;\ p\not|a,b \text{ or } p|a;\ p\not|b,c; \\ p\left(\frac{-ac}{p}\right)+1 & \text{if } p|b;\ p\not|a,c. \end{cases}$$

(iv) $L(a,b,c;p^\alpha)=0$ *if* $(a,b,c,p)=1$ *and* $p|\ abc$.

(v) $|L(a,b,c;q)| \le c_1 q(a,b,c,q)\cdot \tau_3(q)$.

Proof. Properties (i)–(iii) are trivial.
We prove (iv). Let $x = x_0 + p^{\alpha-1}x_1$, $\bar x = \bar x_0 - p^{\alpha-1}\bar x_0^2 x_1$; $y = y_0+p^{\alpha-1}y_1$, $\bar y = \bar y_0 - p^{\alpha-1}\bar y_0^2 y_1$; where $(x_0y_0,p)=1$, $x_0\bar x_0 \equiv 1 (\operatorname{mod} p^\alpha)$, $y_0\bar y_0 \equiv 1(\operatorname{mod} p^\alpha)$; $x_0,y_0\in\{1,\dots,p^{\alpha-1}\}$; $x_1,y_1\in \{1,\dots,p\}$.

Now we have

$$cL(a,b,c;p^\alpha) = \sum_{x_0,y_0\,(\operatorname{mod} p^{\alpha-1})}\ \sum_{x_1,y_1 (\operatorname{mod} p)} e_{p^\alpha}(a(x_0+p^{\alpha-1}x_1) + b(y_0+p^{\alpha-1}y_1) +$$
$$+c(\bar y_0^2 - 2p^{\alpha-1}\bar y_0^3 y_1)(\bar x_0 - p^{\alpha-1}\bar x_0^2 x_1)) = \sum_{x_0,y_0 (\operatorname{mod} p^{\alpha-1})} e_{p^\alpha}(ax_0+by_0+c\bar y_0^2\bar x_0)\times$$
$$\times \sum_{x_1 (\operatorname{mod} p)} e_p(x_1(a - c\bar x_0^2\bar y_0^2)) \sum_{y_1 (\operatorname{mod} p)} e_p(y_1(b-2c\bar y_0^3\bar x_0)),$$

however

$$\sum_{x (\operatorname{mod} p)} e_p(ax) = 0 \quad \text{if } (a,p)=1,$$

therefore we need in consideration of all cases of divisibility a, b, c on p and this leads to (iv). Finally, the estimate of the exponential sum $L(a,b,c;q)$ can be obtained analogically as in Smith [3]. □

Remark. Let $(q,ab)=1$, then $L(a,b,c;q)$ can be transformed to the form $L(1,1,ab^2c;q)$.
Indeed, when $x,y,z (\operatorname{mod}\ q)$ and $(ab,q)=1$ we have ax, by, $\frac{z}{ab^2}(\operatorname{mod}\ q)$ and then

$$L(a,b,c;q) = \sum_{\substack{x,y,z (\operatorname{mod} q),\\ xy^2z\equiv 1 (\operatorname{mod} q)}} e_q(ax+by+cz) =$$
$$= \sum_{\substack{x',y',z (\operatorname{mod} q),\\ x'(y')^2\frac{z}{ab^2}\equiv 1 (\operatorname{mod} q)}} e_q(x'+y'+cz) = \sum_{\substack{x',y',z' (\operatorname{mod} q),\\ x'(y')^2 z'\equiv 1 (\operatorname{mod} q)}} e_q(x'+y'+acb^2z').$$

We denote

$$S = S(k,t_1,t_2,\rho,\sigma;q) = \sum_{j=1}^{q}{}^{*} e_q(kj)L(\rho,\sigma,jt_1;q)\cdot \overline{L(\rho,\sigma,jt_2;q)}. \tag{10}$$

This sum has the following product formula.

Lemma 2. $S(k,t_1,t_2,\rho,\sigma;q_1q_2)=S(q_2^3k,t_1,t_2,\rho,\sigma;q_1)S(q_1^3k,t_1,t_2,\rho,\sigma;q_2)$, *here* $(q_1,q_2)=1$.

This proposition allows us to restrict attention to prime power modulus ($q=p^\alpha$). From (10) and property (iv) of $L(a,b,c;q)$ we can see that $S=0$ when $\alpha\geq 2$, unless $(\rho,p^\alpha)=(\sigma,p^\alpha)=(t_1,p^\alpha)=(t_2,p^\alpha)=p^r$. When $p^r|k$ we have $S=p^{5r}S(k',t_1',t_2',\rho',\sigma',p^{\alpha-r})$, where $k=p^rk'$, $t_1=p^rt_1'$, $t_2=p^rt_2'$, $\rho=p^r\rho'$, $\sigma=p^r\sigma'$.

In another case ($p^r\not|k$) we infer $S=0$.

Lemma 3. *Let* $S=S(k,t_1,t_2,\rho,\sigma;p)$. *Then*

$$|S|\leq c_3p^{5/2}\Big\{(k,t_1-t_2,p)\cdot(t_1,p)\cdot(t_2,p)\cdot(\rho,p)\cdot(\sigma,p)\Big\}^{1/2},$$

c_3 *is an absolute constant.*

Proof. We need in consideration of all cases of divisibility t_1,t_2,ρ,σ,k on p. When $p\not|kt_1t_2\rho\sigma$ we have

$$S=\sum_{j=1}^{p}e_p(kj)L(\rho,\sigma,jt_1;p)\overline{L(\rho,\sigma,jt_2;p)}-|L(\rho,\sigma,0;p)|^2=$$
$$=\sum_{x,y,X,Y=1}^{p}{}^{*}e_p(\rho x+\sigma y-\rho X-\sigma Y)\sum_{j=1}^{p}e_p\left(j\left(k+t_1\overline{xy^2}-t_2\overline{XY^2}\right)\right)+\mathrm{O}(1)=$$
$$=pS_1+\mathrm{O}(1),$$

where

$$S_1=\sum_{\substack{x,y,X,Y=1\\ p|k+t_1\overline{xy^2}-t_2\overline{XY^2}}}^{p}e_p(\rho x+\sigma y-\rho X-\sigma Y)=\sum_{\substack{x,y,X,Y=1\\ \frac{\alpha}{xy^2}+\frac{\beta}{XY^2}=1}}^{p}e_p(x+y+X+Y),$$

where $\alpha=-t_1\rho\sigma\bar{k}$ $\beta=t_2\rho\sigma\bar{k}$.

An analogous sum has been considered by Birch and Bombieri in [2], who proved that $|S_1|\leq c_2p^{3/2}$ for $p\not|\alpha\beta$, where c_2 is an absolute constant. □

Now we can combine the results and prove the lemma.

Lemma 4. *Let* $p\not|t_1t_2$, $\alpha\geq 2$, $S=S(k,t_1,t_2,1,1;p^\alpha)$, $g=\left[\frac{\alpha}{2}\right]$. *Then* $S=0$ *unless* $(k,p^g)|t_1-t_2$, *in which case*

$$|S|\leq 12p^{\frac{5\alpha}{2}}(k,p^g),\quad (\alpha \textit{ is even});$$

$$|S|\leq c_4p^{\frac{5\alpha}{2}}\cdot(k,p^g)\cdot(k,p)^{\frac{1}{2}},\quad (\alpha \textit{ is odd}).$$

Proof. By the definition of S we have

$$S=\sum_{j=1}^{p^\alpha}e_{p^\alpha}(kj)\cdot L(1,1,jt_1;p^\alpha)\cdot\overline{L(1,1,jt_2;p^\alpha)}=$$
$$=\sum_{j=1}^{p^\alpha}\sum_{\substack{m=1\\ m=(m_1,m_2,m_3,m_4)}}^{p^\alpha}e_{p^\alpha}(kj+m_1+m_2+jt_1m_1m_2^2-m_3-m_4-jt_2\overline{m_3m_4^2})=$$
$$=p^\alpha\sum_{m=1}^{p^\alpha}{}^{(1)}e_{p^\alpha}(m_1+m_2-m_3-m_4),$$

where $\sum^{(1)}$ denotes the conditions

$$(m_i, p) = 1 \text{ and } k + t_1\overline{m_1 m_2^2} \equiv t_2\overline{m_3 m_4^2}(p^\alpha) \text{ or } km_1m_3m_2^2m_4^2 + t_1m_3m_4^2 \equiv t_2m_1m_2^2(p^\alpha).$$

Hence

$$S = \sum_{j=1}^{p^\alpha} \sum_{m=1}^{p^\alpha}{}^{*} e_{p^\alpha}\left(F(j,m)\right), \tag{11}$$

where $F(j,m) = m_1 + m_2 - m_3 - m_4 + j(km_1m_2^2m_3m_4^2 + t_1m_3m_4^2 - t_2m_1m_2^2)$.

Now our proposition follows from Heath-Brown's Lemma ([1], Lemma 2). □

Lemma (D. R. Heath-Brown) *Let $F[x] \in Z[x_1, \dots, x_n]$ and let A be a set of residue classes for $x(\mathrm{mod}\ p)$. Write*

$$S = \sum_{\substack{x=1 \\ x\in A}}^{p^f}{}^{*} e_{p^f}(F(x))$$

and

$$B = \{x(\mathrm{mod}\ p^g);\ x \in A, p^g | \nabla F(x)\}.$$

Then

$$|S| \le p^{nf/2} \# B \qquad (f = 2g \ge 2),$$

$$|S| \le p^{nf/2} \sum_{x\in B} p^{(n-r(x))/2} \qquad (f = 2g+1 \ge 3),$$

where ∇F is the gradient of F, $r(x)$ is the rank$(\mathrm{mod}\ p)$ of the quadratic form Q_x whose matrix is

$$\left(\frac{1}{2}\cdot\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{i,j=1}^{n}.$$

(If $p = 2$ we define $r(x) = 0$.)

Lemma 5. *Let $k, t_1, t_2, \rho, \sigma, q$ be natural numbers. Then*

$$|S(k, t_1, t_2, \rho, \sigma; q)| \le$$

$$\le q^{5/2+\varepsilon}\left\{(k^2, (t_1-t_2)^2, q)(t_1, q)(t_2, q)(\rho, q)(\sigma, q)\right\}^{1/2}((t_1-t_2)^3, q)^{1/6}$$

where ε is any positive number.

Our next step is to deduce the principal estimate for $N(U, V, W)$ when U, V, W^2 is of order $X^{1/3}$.

Lemma 6. *There exists a function $N_1(U,V,W)$, independent of a, such that*

$$N(U,V,W)-N_1(U,V,W)\ll X^{4\varepsilon}\left\{q^{17/24}M^{1/2}+q^{3/8}M^{1/2}(I^{1/2}+J^{1/2})\right\}$$

for any $\varepsilon>0$.

Proof. Our starting point is (4). Let

$$N_1=N_1(U,V,W)=q^{-3}\sum_{\substack{r,s,t=1\\ r,s \text{ or } t=q}}^{q}L(r,t,sa;q)F_q(r,s,t).$$

We shift the ranges for r,s,t so that each variable runs over $(-\frac{q}{2};\frac{q}{2}]$ and divide these intervals into subsets $R<|r|\le 2R$, $S<|s|\le 2S$, $T<|t|\le 2T$, where $1\ll R,S,T\ll q$. Since $(a,q)=1$, N_1 will be independent of a. We then transform $L(r,t,as;q)$ into a sum of the form $L(1,1,art^2s;q)$ as follows. Let $r=\rho r'$, $t=\sigma t'$ where ρ and σ are the products of the powers of those primes that divide q, $(r',q)=(t',q)=1$. On replacing r' by r, t' by t we find that

$$N-N_1\ll q^{-3+2\varepsilon}F(R\,|\rho|\,;q)F(T\,|\sigma|\,;q)\sum\nolimits_1, \tag{12}$$

where

$$\sum\nolimits_1=\sideset{}{^*}\sum_{R<r\le 2R}\ \sideset{}{^*}\sum_{T<t\le 2T}\left|\sum_{S<|s|\le 2S}L(\rho,\sigma,art^2s;q)F_q(s)\right|,$$

$$F(R\,|\rho|\,;q)=\min(I,q(R\,|\rho|)^{-1}),\ \ F(T\,|\sigma|\,;q)=\min(M,q(T\,|\sigma|)^{-1}).$$

We write $h=rt^2$, $H=RT^2$. We can show that

$$N-N_1\ll H^{1/2}q^{-3}X^{3\varepsilon}F(R\,|\rho|\,;q)F(T\,|\sigma|\,;q)F(S;q)\sum\nolimits_2^{1/2}, \tag{13}$$

where $\sum_2=\sum\limits_{S<|s_1|,|s_2|\le 2S}|\sum(s_1,s_2)|$, $\sum(s_1,s_2)=\sideset{}{^*}\sum\limits_{H<h\le 8H}L(hs_1)\overline{L(hs_2)}$,
$L(hs_i)=L(\rho,\sigma,ahs_i;q)$,

$$\sum_{s_1\ne s_2}\left|\sum(s_1,s_2)\right|\ll q^{5/2+2\varepsilon}(\rho,q)^{1/2}(\sigma,q)^{1/2}S^2(Hq^{-5/6}+1), \tag{14}$$

$$\sum\left|\sum(s,s)\right|\ll q^{2+\varepsilon}HS(\rho,q)(\sigma,q). \tag{15}$$

We now insert these bounds (14) and (15) into (13), using the estimates

$$S\cdot F(S,q)\ll q,\ \ R\cdot(\rho,q)^{1/4}F(R\,|\rho|\,,q)\ll q,$$
$$T^2\cdot F(T\,|\sigma|\,,q)\cdot(\sigma,q)^{1/4}\ll q^{7/8}M^{1/2}.$$

This leads to Lemma 6. □

We also shall need a bound for $N\,(U,V,W)$ that will be efficient when one of U,V,W^2 is “small”.

Lemma 7. *For suitable function $N_1 = N_1(U,V,W)$, independent of a, one has*

$$N - N_1 \ll q^{1/2+\varepsilon} M$$

for any $\varepsilon > 0$.

Proof. If W^2 is "small", then we shall return to (5) and write

$$N_1 = q^{-2} \sum_{w\in C} \sum_{\substack{s,r=1 \\ s \text{ or } r=q}}^{q} \Bigg(\sum_{\substack{\alpha,\beta=1 \\ \alpha\beta w^2 \equiv a(q)}}^{q} e_q(r\alpha + s\beta) \Bigg) F_q(r,s),$$

which is independent of a.

If U is "small" then we need (6) and

$$N_1 = q^{-2} \sum_{u\in A} \sum_{\substack{s,t=1 \\ s \text{ or } r=q}}^{q} \Bigg(\sum_{\substack{\beta,\gamma=1 \\ u\beta\gamma^2 \equiv a(q)}}^{q} e_q(s\beta + t\gamma) F_q(s,t) \Bigg),$$

which is independent of a too. □

Moreover we shall obtain the estimate of $N(U,V,W)$ when one of U,V,W^2 is "large" and the others are "small". We shall also consider different cases.

Lemma 8. *(V is "large".) There exists $N_1 = N_1(U,V,W)$, independent of a, such that*

$$N - N_1 \ll X^{\varepsilon} I^{3/4} M^{1/2} [J+U]^{1/4} (1 + W^4 q^{-1} + W^{5/2} q^{-1/2})^{1/4}$$

for any $\varepsilon > 0$.

Proof. We take $N_1 = q^{-1} \sum_{u\in A} \sum_{w\in C} F_q(0)$, where $F_q(s) = \sum_{v\in B} e_q(-sv)$ and then from (7) we infer our Lemma. □

Lemma 8′ *(W^2 is "large".) There exists $N_1 = N_1(U,V,W)$, independent of a, such that*

$$N - N_1 \ll X^{\varepsilon} I^{3/4} M J^{1/2} U^{1/4} \left[1 + V^2 q^{-1} + V^{3/2} q^{-1/2}\right]^{1/4}$$

for any $\varepsilon > 0$.

By Lemmas 6–8′ we infer from formula (3)

$$D(1,1,2;X,q,a) = M^*(1,1,2;X,q) + \mathrm{O}(X^{\frac{55}{69}+\varepsilon} q^{-\frac{14}{23}}) + \mathrm{O}(X^{\frac{63}{80}+\varepsilon} q^{-\frac{693}{1120}}) \qquad (16)$$

for $q \ll X^{\frac{14}{27}}$, where $M^*(1,1,2;X,q)$ does not depend on a. When $X^{-\frac{742}{777}} \le q \le X^{\frac{14}{27}}$ the last error term vanishes.

The main term is not calculated explicitly, but on one hand,

$$\sum_{a=1}^{q}{}^{*} D(1,1,2;X,q,a) = \varphi(q) M^*(1,1,2;X,q) + \mathrm{O}(X^{\frac{55}{69}+\varepsilon} q^{\frac{9}{23}}) + \mathrm{O}(X^{\frac{63}{80}+\varepsilon} q^{\frac{427}{1120}}) \ , \qquad (17)$$

on the other hand, by Perron's formula,

$$\sum_{a=1}^{q}{}^{*}D(1,1,2;X,q,a)=\varphi(q)M(1,1,2;X,q)+\mathrm{O}(X^{\frac{1}{2}}(qX)^{\varepsilon}), \tag{18}$$

where

$$M(1,1,2;X,q)=\frac{X}{\varphi(q)}\mathrm{Res}(s^{-1}\cdot L^2(s,\chi_{q0})\cdot L(2s,\chi_{q0})\cdot X^{s-1},s=1), \tag{19}$$

Comparison (17) with (18) then produces Theorem for $(a,q)=1$. For going over the case $(a,q)>1$ we introduce the function

$$F(n,\delta)=\sum_{\alpha_1\alpha_2\alpha_3^2\beta=n}\mu(\alpha_1)\mu(\alpha_2)\mu(\alpha_3)\tau(1,1,2;\beta\delta).$$

We can prove that this function is multiplicative in the sense that

$$F(n_1n_2,\delta_1\delta_2)=F(n_1,\delta_1)F(n_2,\delta_2),\quad (n_1\delta_1,n_2\delta_2)=1.$$

Indeed,

$$F(n_1,\delta_1)F(n_2,\delta_2)=$$

$$=\sum_{\alpha_1\alpha_2\alpha_3^2\beta=n_1}\mu(\alpha_1)\mu(\alpha_2)\mu(\alpha_3)\tau(1,1,2;\beta\delta_1)\sum_{\alpha_1'\alpha_2'(\alpha_3')^2\beta'=n_2}\mu(\alpha_1')\mu(\alpha_2')\mu(\alpha_3')\tau(1,1,2;\beta'\delta_2)=$$
$$=\sum_{\alpha_1\alpha_1'\alpha_2\alpha_2'(\alpha_3\alpha_3')^2\beta\beta'=n_1n_2}\mu(\alpha_1\alpha_1')\mu(\alpha_2\alpha_2')\mu(\alpha_3\alpha_3')\tau(1,1,2;\beta\beta'\delta_1\delta_2)=F(n_1n_2,\delta_1\delta_2).$$

An easy calculation shows that $F(p^e,1)=0$ for $e\geq 1$ and $F(p^e,p^f)=F(p^{e+f},1)=0$ for $e\geq 3$, $f>0$; and consequently $F(n,\delta)=0$ if $n\not|\delta^2$.

We note that if $(a,q)=1$ then

$$\sum_{\substack{n\,|\delta^2\\(n,q)=1}}F(n,\delta)D(1,1,2;Xn^{-1},q,a\bar{n})=\sum_{\substack{m\leq X\\ m\equiv a(q)}}\sum_{\substack{n|m\\(n,q)=1}}F(n,\delta)\tau\left(1,1,2;\frac{m}{n}\right). \tag{20}$$

The condition $(n,q)=1$ is redundant, because $m\equiv a(q)$ and $n|m$.

We can prove the identity

$$\sum_{n|m}F(n,\delta)\tau(1,1,2;\frac{m}{n})=\tau(1,1,2;m\delta). \tag{21}$$

From (20) and (21) we now have

$$D(1,1,2;X\delta,q\delta,a\delta)=\sum_{\substack{n|\delta^2\\(n,q)=1}}F(n,\delta)D(1,1,2;Xn^{-1},q,a\bar{n}). \tag{22}$$

Taking new values for X,q,a, from (20) and (22) we now infer

$$D(1,1,2;X,q,a)=\sum_{n|\delta^2,(n,q)=1}F(n,\delta)M(1,1,2;X(\delta n)^{-1},q\delta^{-1})+\mathrm{O}(X^{\frac{55}{69}+\varepsilon}q^{-\frac{14}{23}+4\varepsilon'})+$$
$$+\mathrm{O}(\mathrm{X}^{\frac{63}{80}+\varepsilon}\mathrm{q}^{-\frac{693}{1120}+4\varepsilon'}),$$

where the main term

$$M(1,1,2;X,q,a)=\frac{X}{\delta\varphi(\frac{q}{\delta})}\mathrm{Res}\left\{s^{-1}\left(\frac{X}{\delta}\right)^{s-1};\left(\sum_{\substack{m=1\\(m,q_1)=1}}^{\infty}\tau(1,1,2;m\delta)m^{-s}\right);s=1\right\}.$$

This completes the proof of Theorem.

REFERENCES

1. Heath-Brown D.R. *The divisor function $d_3(n)$ in arithmetic progressions* // Acta Arith. XLVII (1986), P. 29-56.

2. Birch B.J. and Bombieri E. *Appendix: On some exponential sums* // Annals of Math. 121(1985), P. 345-350.

3. R.A. Smith, *On n-dimensional Kloosterman sums* // J. Number Theory 11 (1979), p. 324-343.

Department of Computer Algebra and Discrete Mathematics
Institute of Mathematics, Economics and Mechanics
Odessa National University
Dvoryanskaya Str. 2
65000 Odessa
Ukraine

Received 31.10.2001
Revised 24.11.2002