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EXPONENTIAL TYPE VECTORS OF ISOMETRIC GROUP GENERATORS

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By means of exponential type vectors a criterion for operator in a Banach space to generate groups of isometries is established. It is also shown that exponential type vectors of operators with separable spectrum can be completely described by spectral subspaces.

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В терминах векторов экспоненциального типа установлен критерий генерации групп изометрий операторами в банаховом пространстве. Показано, что векторы экспоненциального типа операторов с отделенными спектрами могут быть полностью описаны спектральными подпространствами.

Introduction. As is well known (see, for example, [13]), Nelson's and Stone's classical theorems imply the next proposition: a closed operator iA on a Hilbert space generates the unitary group e^{itA} if and only if the operator A is symmetric and its analytic vectors are dense.

In the present article this proposition is extended to isometric groups on Banach spaces. In this case the exponential type vectors fulfill the role of analytic vectors and conservativity fulfills the role of symmetry.

In addition, it is established that exponential type vectors of isometric group generators can be completely described by the spectral subspaces in the sense of J. Lubič and V. Macajev [10]. This fact is proved not only for generators of such groups, but for a more general class of closed operators with spectrum on contours, whose resolvents satisfy Levinson's condition [6]. Such operators belong to the so-called class a operators with separable spectrum [10]. Let's note, that for the operators with separable spectrum the fact of belonging of vectors from spectral spaces to set it of all exponential type vectors was earlier established by V. Gorbachuk and M. Gorbachuk [3].

For the class of operators with meromorphic resolvents the relation between root subspaces and exponential type vectors was established by G. Radzievskii [11]. Independently a similar result was established in the paper [8].

At last we shall note that normed subspaces of exponential type vectors of an unbounded operator and the scale of such subspaces were introduced by Ja. Radyno [12]. He also

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proved density of exponential type vectors for generators of uniformly bounded and strongly continuous one-parametric groups.

1. Vectors of exponential type. Let $(\mathcal{B}, \|\,.\,\|)$ be a Banach space over the field of complex numbers $\mathbb{C}$ and $A\colon \mathcal{D}(A) \subset \mathcal{B} \to \mathcal{B}$ be a closed unbounded linear operator with dense domain $\mathcal{D}(A)$. Next, $\{A^k : k \in \{2,3,\dots\}\}$ are powers of A with domains $\mathcal{D}(A^k) := \left\{x \in \mathcal{D}(A^{k-1}) : Ax \in \mathcal{D}(A^{k-1})\right\}$, $\mathcal{D}(A^\infty) := \bigcap_{k\geq 1}\mathcal{D}(A^k)$ and $A^0 = I$ is the identity operator on $\mathcal{B}$. Algebra $L(\mathcal{B})$ of bounded operators on space $\mathcal{B}$ is equipped with the uniform operator topology. As usual, $\rho(A) \ni \lambda \to (\lambda - A)^{-1}$, $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda - A)^{-1} \in L(\mathcal{B})\}$, $\sigma(A) := \mathbb{C} \setminus \rho(A)$ are the resolvent, resolvent set and the spectrum of the operator A, respectively.

Definition 1. An element $x \in \mathcal{D}(A^\infty)$ is called the *vector of exponential type* τ of the operator A if $\overline{\lim\limits_{k\to\infty}} \left\|A^k x\right\|^{1/k} = \tau < \infty$. The set of all vectors of exponential type τ will be denoted by $\mathcal{E}^\tau(A)$. Next, $\mathcal{E}(A) := \bigcup\limits_{\tau>0} \mathcal{E}^\tau(A)$ denotes all exponential type vectors of the operator A.

Lemma 1. *The following statements are equivalent:*

(a) $x \in \mathcal{E}^\tau(A)$;

(b) *the function* $G(\lambda, x, A) = \sum\limits_{k=0}^{\infty} \dfrac{\|A^k x\|}{k!}\lambda^k$ *of complex variable* λ *is entire and has exponential type* τ, *i.e.*, $\tau = \overline{\lim\limits_{r\to\infty}}\, r^{-1}\ln M(r)$, *where* $M(r) = \max\limits_{|\lambda|=r} |G(\lambda, x, A)|$;

(c) *the radius of convergence of the power series* $g(\lambda, x, A) = \sum\limits_{k=0}^{\infty} \dfrac{\|A^k x\|}{\lambda^{k+1}}$ *is equal to the number* τ.

Proof. Equivalence of conditions (a), (b), (c) follows from the well-known relations for entire functions of exponential type [2, Theorem 1.1.1]. □

Making use of Lemma 1, the space $\mathcal{E}(A)$ can be characterized as the set of vectors $x \in \mathcal{B}$ for which the series $g(\nu, x, A)$ is convergent for same number $\nu > 0$. In other words,

$$\mathcal{E}(A) = \bigcup_{\nu>0} \ell_1^\nu(A), \qquad \ell_1^\nu(A) := \left\{x \in \mathcal{D}(A^\infty) : g(\nu, x, A) < \infty\right\}$$

where for each number $\nu > 0$ the linear subspace $\ell_1^\nu(A)$ is equipped to the norm $\|x\|_\nu = \nu\, g(\nu, x, A)$. Later we shall need certain simple properties of the spaces $\ell_1^\nu(A)$. In a somewhat different form they were derived in [7].

Lemma 2. *The following propositions are valid:*

(a) $\ell_1^\nu(A)$ *is a Banach space*;

(b) *the inclusion* $\ell_1^\nu(A) \subset \mathcal{B}$ *is continuous and* $\|x\| \leq \|x\|_\nu$ *for all* $x \in \ell_1^\nu(A)$;

(c) *the inclusion* $\ell_1^\nu(A) \subset \ell_1^\mu(A)$ *is continuous and* $\|x\|_\mu \leq \|x\|_\nu$ *for all* $x \in \ell_1^\nu(A)$, $0 < \nu < \mu$;

(d) *the space* $\ell_1^\nu(A)$ *is* A*-invariant and the operator norm of the restriction* $A_\nu := A_{\,|\,\ell_1^\nu(A)}$ *satisfies the inequality* $\|A_\nu\|_\nu \leq \nu$, *where* $\|A_\nu\|_\nu := \sup\limits_{0\neq x\in\ell_1^\nu(A)} \|A_\nu x\|_\nu / \|x\|_\nu$;

(e) *for each* $\lambda \in \rho(A)$ *the inequality* $\|(\lambda - A_\nu)^{-1}\|_\nu \le \|(\lambda - A)^{-1}\|$ *holds and the spectra of the operator* A *and of its restriction* A_ν *satisfy the inclusion* $\sigma(A_\nu) \subset \sigma(A)$.

Proof. (a) The obvious inequality

$$\|A^k x\| \le \nu^k \|x\|_\nu, \qquad \forall\, x \in \ell_1^\nu(A), \qquad k \in \{0, 1, 2, \dots\} \tag{1}$$

is valid. Thus, if $\{x_n\}$ is a Cauchy sequence in $\ell_1^\nu(A)$, then $\{A^k x_n\}$ is a Cauchy sequence in $\mathcal{B}$ for each fixed $k \ge 0$. From closedness of the operator A and completeness of the space $\mathcal{B}$ applying the induction over all k we obtain that there exists a vector $x \in \mathcal{B}$ such that

$$\lim_{n\to\infty} \|A^k x_n - A^k x\| = 0, \qquad k \in \{0, 1, 2, \dots.\} \tag{2}$$

By the assumption for any $\varepsilon > 0$ there exists a number $n(\varepsilon)$ such that $\|x_n - x_m\|_\nu < \varepsilon$ for all $n, m \ge n(\varepsilon)$, whence

$$(\forall n \ge n(\varepsilon)) : \|x_n\|_\nu \le \|x_{n(\varepsilon)}\|_\nu + \|x_n - x_{n(\varepsilon)}\|_\nu \le \|x_{n(\varepsilon)}\|_\nu + \varepsilon. \tag{3}$$

Using (3) and (2) for each $N \in \{0, 1, 2, \dots\}$ we have

$$\sum_{k=0}^{N} \frac{\|A^k x\|}{\nu^k} = \lim_{n\to\infty} \sum_{k=0}^{N} \frac{\|A^k x_n\|}{\nu^k} \le \|x_{n(\varepsilon)}\|_\nu + \varepsilon$$

and

$$\sum_{k=0}^{N} \frac{\|A^k (x_n - x)\|}{\nu^k} = \lim_{m\to\infty} \sum_{k=0}^{N} \frac{\|A^k (x_n - x_m)\|}{\nu^k} \le \varepsilon.$$

Hence $x \in \ell_1^\nu(A)$ and $\|x - x_n\|_\nu \le \varepsilon$ for any $n \ge n(\varepsilon)$. The completeness of $\ell_1^\nu(A)$ is proved.

(b) follows immediately from inequality (1) at $k = 0$.

(c) In fact, for each $x \in \ell_1^\nu(A)$ and $\nu < \mu$ the inequality $\|x\|_\mu = \mu\, g(\mu, x, A) \le \nu\, g(\nu, x, A) = \|x\|_\nu$ is valid.

(d) In fact, we have the inequality $\|\, Ax\|_\nu = \nu \sum_{k=0}^{\infty} \|\, A^{k+1} x\| / \nu^{k+1} \le \nu\, \|x\|_\nu$ for each $\nu > 0$ and $x \in \ell_1^\nu(A)$.

(e) In fact, we have $\|(\lambda - A)^{-1} x\|_\nu = \sum_{k=0}^{\infty} \|(\lambda - A)^{-1} A^k x\| / \nu^k \le \|(\lambda - A)^{-1}\|\, \|x\|_\nu$ for each $\lambda \in \rho(A)$ and $x \in \ell_1^\nu(A)$. Taking into account invariance of the spaces $\ell_1^\nu(A)$ with respect to the operator A, we obtain the relation $(\lambda - A)^{-1}\,|_{\,\ell_1^\nu(A)} = (\lambda - A_\nu)^{-1}$. This means that the restriction of the resolvent of A to $\ell_1^\nu(A)$ is the resolvent for the restriction A_ν of the operator A. Therefore $\lambda \in \rho(A_\nu)$. □

2. Operators with separable spectrum on contours. It is assumed that the spectrum $\sigma(A)$ of the operator A is nonempty and lies on the simple open smooth contour Γ of the complex plane $\mathbb{C}$. In particular, contour Γ can be the real axis $\mathbb{R}$.

Let us now consider an arbitrary closed sector Δ on contour Γ and denote by $\mathcal{B}_\Delta$ a closed subspace in $\mathcal{B}$ with the next properties:

(I) the operator A is everywhere defined and bounded on $\mathcal{B}_\Delta$;

(II) the subspace $\mathcal{B}_\Delta$ is A-invariant;

(III) the spectrum $\sigma(A_\Delta)$ of the restriction A_Δ of the operator A onto $\mathcal{B}_\Delta$ satisfies the relation $\sigma(A_\Delta) \subset \Delta$ and $\sigma(A_\Delta) \cap \operatorname{int}\Delta = \sigma(A) \cap \operatorname{int}\Delta$, where int denote the interior of a set;

(IV) if the operator A is everywhere defined and bounded on some closed invariant subspace $\mathcal{M} \subset \mathcal{B}$ and for its restriction $A_{\mathcal{M}}$ onto $\mathcal{M}$ the inclusion $\sigma(A_{\mathcal{M}}) \subset \Delta$ holds, then $\mathcal{M} \subset \mathcal{B}_\Delta$.

Definition 2. [10] A subspace $\mathcal{B}_\Delta$ satisfying conditions (I)–(IV) is called the *spectral subspace* of the operator A.

Lemma 3. *For each closed sector $\Delta \subset \Gamma$ such that $\Delta \subset \{\lambda \in \mathbb{C} : |\lambda| < \nu\}$ the inclusion $\mathcal{B}_\Delta \subset \ell_1^\nu(A)$ is valid.*

Proof. Consider the restriction A_Δ of the operator A onto the subspace $\mathcal{B}_\Delta$. From Definition 2 and the assumptions of the lemma the inclusion $\sigma(A_\Delta) \subset \{\lambda \in \mathbb{C} : |\lambda| < \nu\}$ follows. Therefore, the spectral radius $r_\Delta := \overline{\lim}_{k\to\infty} \|A_\Delta^k\|^{1/k}$ of A_Δ satisfies the inequality $r_\Delta < \nu$. This means that for each $x \in \mathcal{B}_\Delta$ the series $\sum_{k=0}^\infty \|A^k x\|/\nu^k = \sum_{k=0}^\infty \|A_\Delta^k x\|/\nu^k$ is convergent. Since the expression on the left side above is the norm of the space $\ell_1^\nu(A)$, we have $x \in \ell_1^\nu(A)$. □

Theorem 1. *Let for each point $\xi \in \Gamma$ the following conditions be valid:*

(i) there exists a nondecreasing function $M_\xi(\delta)$ satisfying for sufficiently small $\varepsilon > 0$ the Levinson's condition
$$\int_0^\varepsilon \ln\ln M_\xi(\delta)\, d\delta < \infty;$$

(ii) there exists a neighborhood $O(\xi)$ such that for each $\lambda \in O(\xi)\setminus\Gamma$ the resolvent of the operator A satisfies the inequality
$$\|(\lambda - A)^{-1}\| \le M_\xi\big[\delta(\lambda,\Gamma)\big],$$
where $\delta(\lambda,\Gamma)$ is the distance from the point λ to Γ.

Then for each closed sector $\Delta \subset \Gamma$ there is the spectral subspace $\mathcal{B}_\Delta$ and the equality
$$\mathcal{E}(A) = \bigcup_{\Delta\subset\Gamma} \mathcal{B}_\Delta$$
is realized.

Proof. From Lemma 3 it directly follows the inclusion $\bigcup_{\Delta\subset\Gamma} \mathcal{B}_\Delta \subset \mathcal{E}(A)$. We shall proof the inverse inclusion. By Lemma 2(e) for each $\nu > 0$ the set $\sigma(A_\nu)$ is bounded and belong to contour Γ. Therefore there exists a sector Δ such that the inclusions $\sigma(A_\nu) \subset \operatorname{int}\Delta$ is valid. Later we shall use the reasoning and notations from the proof of Lubich-Matsaev's theorem ([10], Theorem 5). The contour C_Δ^- is formed by conjugating without self-intersection of two smooth tangent contours, which start from different endpoints of the sector Δ. We assume that the contour C_Δ^- belongs to the intersection of the circle $\{\lambda \in \mathbb{C} : |\lambda| < \nu\}$ with the resolvent set $\rho(A_\nu)$ and is oriented outside. At the endpoints of the sector Δ the contour C_Δ^- has the spinode inside. Elsewhere the contour C_Δ^- is smooth.

From Levinson's condition [6] it follows that there exists a function $\Phi_\Delta^-(\lambda)$ with the properties: $\Phi_\Delta^-(\lambda)$ is analytic and bounded outside the contour C_Δ^-; $\Phi_\Delta^-(\lambda) \ne 0$ for all finite λ outside the countour C_Δ^-; $\Phi_\Delta^-(\infty) = 0$; for each endpoint ξ of the sector Δ, $\Phi_\Delta^-(\lambda)$ has asymptotics of the form $O\Big(\dfrac{1}{M_\xi[\delta(\lambda,\Gamma)]}\Big)$ as $\lambda \to \xi$ along Δ. Then ([10], Theorem 5), the equality $\mathcal{B}_\Delta = \operatorname{Ker}\Phi_\Delta^-(A)$, where $\Phi_\Delta^-(A) := \dfrac{1}{2\pi i}\displaystyle\oint_{C_\Delta^-} \Phi_\Delta^-(\lambda)(\lambda - A)^{-1}\, d\lambda$, is valid. As it is easily seen from the estimate of the resolvent, the integral $\Phi_\Delta^-(A)$ exists in the operator norm of the space $\mathcal{B}$.

Since $\rho(A) \subset \rho(A_\nu)$, the restriction of the resolvent $(\lambda - A_\nu)^{-1}$ onto the invariant subspace $\ell_1^\nu(A)$ is a continuous function on the contour C_Δ^- in the operator norm on the space $\ell_1^\nu(A)$. From the condition (ii) the inequality $\|(\lambda - A_\nu)^{-1}\| \leq M_\xi[\delta(\lambda, \Gamma)]$ follows. Thus the integral $\Phi_\Delta^-(A_\nu) := \dfrac{1}{2\pi i}\oint_{C_\Delta^-} \Phi_\Delta^-(\lambda)(\lambda - A_\nu)^{-1}\, d\lambda$ exists in the operator norm of the space $\ell_1^\nu(A)$. The spectrum $\sigma(A_\nu) \subset \operatorname{int}\Delta$ is contained inside the contour C_Δ^-, so the resolvent $(\lambda - A_\nu)^{-1}$ is bounded and analytic in the operator norm of the space $\ell_1^\nu(A)$ outside C_Δ^-. The integrand is bounded analytic function outside C_Δ^- and vanishes at infinity. By Cauchy's theorem, $\Phi_\Delta^-(A_\nu) = 0$. Thus, $\Phi_\Delta^-(A)x = \Phi_\Delta^-(A_\nu)x = 0$ for all $x \in \ell_1^\nu(A)$, that is $\ell_1^\nu(A) \subset \operatorname{Ker}\Phi_\Delta^-(A) = \mathcal{B}_\Delta$. Bearing in mind that number ν is unrestricted we have $\mathcal{E}(A) \subset \bigcup_{\Delta \subset \Gamma} \mathcal{B}_\Delta$ □

Let us now consider in the Banach algebra $L(\mathcal{B})$ the one-parametric strongly continuous group $\mathbb{R} \ni t \to e^{itA} \in L(\mathcal{B})$ with generator $T = iA$. Following [10], we shall call the generator A *nonquasianalytic* if it satisfies the condition $\displaystyle\int_{-\infty}^{\infty} \frac{\ln\|e^{itA}\|}{1+t^2}\, dt < \infty$. The spectrum of a nonquasianalytic operator lies on the real axis. Moreover, for a nonquasianalytic operator A the function $M(\delta) = \sup_{|\operatorname{Im}\lambda| \geq \delta} \|(\lambda - A)^{-1}\|$ is proved ([10], Theorem 3) to satisfy the conditions of Theorem 1 for each points on the real axis.

As usual, a one-parametric group $\mathbb{R} \ni t \to e^{itA} \in L(\mathcal{B})$ is called the *isometric group* if $\|e^{itA}x\| = \|x\|$ for all $x \in \mathcal{B}$ and $t \in \mathbb{R}$. It is obvious that every isometric group is generated by a operator iA, where A is nonquasianalytic. Thus, from Theorem 1 and [10] it follows

Remark 1. If an operator A satisfies one of the following conditions:

(i) A is nonquasianalytic;

(ii) $T = iA$ is a generator of an isometric group,

then equality of Theorem 1 is valid.

3. Conservative operators. We denote by $\mathcal{B}'$ the dual space of linear continuous functionals on $\mathcal{B}$. In accordance with Hahn-Banach's theorem for an arbitrary vector $x \in \mathcal{B}$ there exists a functional $e_x \in \mathcal{B}'$ such that $\|x\| = \langle x, e_x\rangle$ and $\|e_x\| = 1$. For this reason the functional of $\varphi_x = \|x\|\, e_x$ satisfies the condition

$$\langle x, \varphi_x\rangle = \|x\|^2 = \|\varphi_x\|^2. \tag{4}$$

We denote the set of functionals $\varphi_x \in \mathcal{B}'$ which satisfy relation (4) by $\mathcal{B}'_x$.

Definition 3. A linear operator $T\colon \mathcal{D}(T) \to \mathcal{B}$ with dense domain $\mathcal{D}(T) \subset \mathcal{B}$ is called *conservative* if for each $x \in \mathcal{D}(T)$ there exists a functional $\varphi_x \in \mathcal{B}'_x$ which satisfies the relation

$$\operatorname{Re}\langle Tx, \varphi_x\rangle = 0. \tag{5}$$

It is not difficult to verify that conservativity of T is equivalent to the inequality $|\operatorname{Re}\lambda|\, \|x\| \leq \|(\lambda - T)x\|$ for all $x \in \mathcal{D}(T)$ and all $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re}\lambda \neq 0$.

Theorem 2. *A closed operator $T = iA$ generates an isometric group on space $\mathcal{B}$ if and only if it satisfies the next conditions*:

(i) $\overline{\mathcal{E}(A)} = \mathcal{B}$ *(closure by the norm of $\mathcal{B}$)*;

(ii) the operator T on $\mathcal{B}$ is conservative.

Proof. Sufficiency. Let $\Delta \subset \{\lambda \in \mathbb{C} : |\lambda| < \nu\} \cap \Gamma$ for some $\nu > 0$. We consider the restriction $(\lambda - A_\Delta)$ of the operator $(\lambda - A)$ to the $\mathcal{B}_\Delta$. From Lemma 3 it follows that on the Banach space $\mathcal{B}_\Delta$ the bounded inverse operator $(\lambda - A_\Delta)^{-1}$ exists for each λ such that $|\lambda| \geq \nu$. Therefore $\mathcal{B}_\Delta = (\lambda - A_\nu)(\mathcal{B}_\Delta)$ for all $|\lambda| \geq \nu$.

Let $\mathcal{B}_\Delta^\perp$ be the annihilator of the subspace $\mathcal{B}_\Delta$ in the dual space $\mathcal{B}'$. We shall show the conservativity of the operator $T_\Delta = iA_\Delta$ on $\mathcal{B}_\Delta$. Let $x \in \mathcal{B}_\Delta$. From the conservativity of T it follows that there exists a functional $\varphi_x \in \mathcal{B}'_x$ satisfying conditions (4) and (5). Putting $e_x = \varphi_x/\|x\|$, we have $\|e_x\| = 1$. Therefore, the relation $\langle x, \widehat{e}_x\rangle = \|x\|$ for the coset $\widehat{e}_x \in \mathcal{B}'/\mathcal{B}_\Delta^\perp$ of functional e_x holds. Hence $\|\widehat{e}_x\| = 1$ in the relative norm of a factor-space $\mathcal{B}'/\mathcal{B}_\Delta^\perp$. Therefore, the functional $\widehat{\varphi}_x = \|x\|\,\widehat{e}_x$ satisfies conditions (4) and (5) relative the duality $\langle \mathcal{B}\,,\, \mathcal{B}'/\mathcal{B}_\Delta^\perp\rangle$. Namely

$$\langle x, \widehat{\varphi}_x\rangle = \langle x, \widehat{e}_x\rangle\,\|x\| = \|x\|^2 = \|\widehat{\varphi}_x\|^2, \qquad \mathrm{Re}\langle T_\Delta x, \widehat{\varphi}_x\rangle = \mathrm{Re}\langle Tx, \varphi_x\rangle = 0.$$

This establishes the conservativity of the operator T_Δ on the space $\mathcal{B}_\Delta$. Now, it follows that

$$\nu\,\|x\|^2 = \nu\langle x, \widehat{\varphi}_x\rangle \pm \mathrm{Re}\langle T_\Delta x, \widehat{\varphi}_x\rangle = \mathrm{Re}\big\langle (\nu \pm T_\Delta)x, \widehat{\varphi}_x\big\rangle \leq \|(\nu \pm T_\Delta)x\|\,\|\widehat{\varphi}_x\|.$$

Therefore, $\|(\nu \pm T_\Delta)^{-1}\| \leq \nu^{-1}$. Let $\mathrm{Re}\,\mu = \mu$ and $|\nu - \mu| < \nu$. Then $\mu > 0$ and $\|(\nu-\mu)(\nu \pm T_\Delta)^{-1}\| < 1$, so that $(\mu \pm T_\Delta)(\mathcal{B}_\Delta) = \big[I - (\nu-\mu)(\nu \pm T_\Delta)^{-1}\big](\nu \pm T_\Delta)(\mathcal{B}_\Delta) = \mathcal{B}_\Delta$. Applying the previous argument, we get

$$\|(\mu \pm T_\Delta)^{-1}\| \leq \mu^{-1}\,. \tag{6}$$

Repeating this process shows that inequality (6) is fulfilled for all $\mu > 0$. From Hille-Yosida's theorem ([4]) we get that the operators $\pm T_\Delta$ on $\mathcal{B}_\Delta$ generate the semigroups $0 \leq t \to e^{\pm itA_\Delta}$ such that $\|e^{\pm itA_\Delta}\| \leq 1$. Thus, the group $\mathbb{R} \ni t \to e^{itA_\Delta}$ generated by the bounded operator T_Δ is isometric on $\mathcal{B}_\Delta$. Therefore the collection of the groups $\{e^{itA_\Delta}\}_{\Delta\subset\Gamma}$ defines the unique isometric group $U(t)$ on the union $\bigcup_{\Delta\subset\Gamma}\mathcal{B}_\Delta$. That is, we have $\|U(t)x\| = \|x\|$ for each $x \in \bigcup\limits_{\Delta\subset\Gamma}\mathcal{B}_\Delta$. In accordance with Theorem 1 we obtain $\overline{\bigcup_{\Delta\subset\Gamma}\mathcal{B}_\Delta} = \mathcal{B}$. This implies the equality $\|U(t)x\| = \|x\|$ for each $x \in \mathcal{B}$.

We have $\lim\limits_{t\to 0} t^{-1}[U(t)x - x] = Tx$ for each $x \in \mathcal{E}(A)$. From uniqueness of the generator of a strongly continuous group it follows that T is a generator of $U(t)$ on $\mathcal{B}$, i.e., $U(t) = e^{itA}$.

Necessity. Let $T = iA$ be a generator of the isometric group $\mathbb{R} \ni t \to e^{itA}$. Then the domain $\mathcal{D}(A)$ is dense in $\mathcal{B}$. The operators $\pm T$ generate the isometric semigroups $0 \leq t \to e^{\pm itA}$ on $\mathcal{B}$. The functions $0 \leq t \to \langle e^{\pm itA}x, \varphi_x\rangle$ are differentiable for each $x \in \mathcal{D}(A)$ and $\varphi_x \in \mathcal{B}'_x$. Moreover,

$$\frac{d}{dt}\,\mathrm{Re}\langle e^{\pm itA}x, \varphi_x\rangle_{|t=0} = \pm\,\mathrm{Re}\langle Tx, \varphi_x\rangle. \tag{7}$$

Since $\big|\langle e^{\pm itA}x, \varphi_x\rangle\big| \leq \|e^{\pm itA}x\|\,\|\varphi_x\| = \|x\|\,\|\varphi_x\| = \langle x, \varphi_x\rangle$, then for each $t > 0$ we have $t^{-1}\,\mathrm{Re}\langle e^{\pm itA}x - x, \varphi_x\rangle \leq 0$ and $\pm\,\mathrm{Re}\langle Tx, \varphi_x\rangle \leq 0$ by virtue of equality (7). Consequently, $\mathrm{Re}\langle Tx, \varphi_x\rangle = 0$ for each $x \in \mathcal{D}(A)$ and $\varphi_x \in \mathcal{B}'_x$. Hence the operator T is conservative on $\mathcal{B}$ and condition (ii) is established.

To the isometric group e^{itA} Radyno's theorem [12] on density of exponential type vectors of bounded strongly continuous groups generators can be applied, whence the condition (i) follows. Correctness of condition (i) also follows from Theorem 1 and Lubič-Macajev's theorem [10] on density of the spectral subspaces of nonquasianalytic group generators. But the following argument is more elementary.

Let a function $f(t) \in L_1(\mathbb{R})$ be the restriction onto the real axis of an entire function of exponential type $\tau > 0$ such that $\int_{-\infty}^{\infty} f(t)\, dt = 1$ and put $P_\alpha \equiv \alpha \int_{-\infty}^{\infty} f(\alpha t) e^{itA} dt$ for all $\alpha > 0$. There exists a constant $c > 0$ independent of k such that Bernstein's inequality $\int_{-\infty}^{\infty} |f^{(k)}(t)|\, dt \le c\tau^k$ for all $k \in \{0, 1, \dots\}$ holds [1]. From the known ([5]) relation $\lim_{|t|\to\infty} |f^{(k)}(t)| = 0$ through integration by parts, we receive

$$(iA)^k P_k x = \alpha \int_{-\infty}^{\infty} f(\alpha t)(iA)^k e^{itA} x\, dt = \alpha^k \int_{-\infty}^{\infty} f^{(k)}(t)\, e^{it/\alpha\, A} x\, dt, \qquad x \in \mathcal{B}. \tag{8}$$

Using (8) and Bernstein's inequality, we obtain $\|A^k P_\alpha x\| \le c\|x\|(\tau\alpha)^k$ for all $x \in \mathcal{B}$. Hence, $\|A^k P_\alpha x\|^{1/k} \le (c\|x\|)^{1/k} \tau\alpha$ and $\overline{\lim}_{k\to\infty} \|A^k P_\alpha x\|^{1/k} \le \tau\alpha$. Consequently, for an arbitrary $x \in \mathcal{B}$ we have $P_\alpha x \in \ell_1^\nu(A)$ if $\nu > \tau\alpha$ and the inclusion

$$\bigcup_{\alpha>0} \big\{P_\alpha x :\, x \in \mathcal{B}\big\} \subset \mathcal{E}(A) \tag{9}$$

holds. It is ease to see that for each $\varphi \in \mathcal{B}'$ and $x \in \mathcal{B}$ the integral

$$\langle P_\alpha x,\, \varphi\rangle = \alpha \int_{-\infty}^{\infty} f(\alpha t)\langle e^{itA} x,\, \varphi\rangle\, dt$$

exists and the relation

$$\lim_{\alpha\to\infty} \|P_\alpha x - x\| = 0 \tag{10}$$

is satisfied. In fact, the identity $\langle P_\alpha x - x,\, \varphi\rangle = \alpha \int_{-\infty}^{\infty} f(\alpha t)\langle e^{itA} x - x,\, \varphi\rangle\, dt$ is obvious. Since the function $\mathbb{R} \ni t \to \langle e^{itA} x - x,\, \varphi\rangle$ is continuous at $t = 0$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\max_{|t|\le\delta} \big|\langle e^{itA} x - x,\, \varphi\rangle\big| \le \varepsilon$. Therefore, we obtain

$$\begin{aligned}\big|\langle P_\alpha x - x,\, \varphi\rangle\big| &\le \varepsilon\, \|\varphi\| \int\limits_{|t|\le\delta} |\alpha f(\alpha t)|\, dt + \int\limits_{|t|>\delta} \big|\alpha f(\alpha t)\langle e^{itA} x - x,\, \varphi\rangle\big|\, dt \\ &\le \varepsilon\, \|\varphi\| \int\limits_{|t|\le\delta\alpha} |f(t)|\, dt + 2\|x\|\, \|\varphi\| \int\limits_{|t|>\delta\alpha} |f(t)|\, dt\,.\end{aligned}$$

Thus, it follows that

$$\|P_\alpha x - x\| \le \varepsilon\, \|f(t)\|_{L_1(\mathbb{R})} + 2\|x\| \int\limits_{|t|>\delta\alpha} |f(t)|\, dt.$$

Passing to the limit in the last inequality, we obtain $\lim_{\alpha\to\infty} \|P_\alpha x - x\| \le \varepsilon\, \|f(t)\|_{L_1(\mathbb{R})}$. Since ε is arbitrary, (10) follows. It follows from (10) that the set $\bigcup_{\alpha>0} \big\{P_\alpha x :\, x \in \mathcal{B}\big\}$ is dense in $\mathcal{B}$. By virtue of (9) the set $\mathcal{E}(A)$ is dense in $\mathcal{B}$ too. □

Remark 2. Using reasonings from work [9, Lemma 5.1] it is possible to show the following isometric isomorphism $\mathcal{B} \simeq \ell_1\left(\mathcal{B}_{[-n,n]};\, \mathcal{B}\right)$, where

$$\ell_1\left(\mathcal{B}_{[-n,n]};\, \mathcal{B}\right) := \left\{ x = \sum_{n=1}^{\infty} x_n \in \mathcal{B} :\ x_n \in \mathcal{B}_{[-n,n]};\ \sum_{n=1}^{\infty} \|x_n\| < \infty \right\}$$

is the space with the norm $\|x\|_1 = \inf \sum_{n=1}^{\infty} \|x_n\|$ (inf is taken over all representations of x in the form $x = \sum x_n$). Using this isometric isomorphism the group $U(t)$ can be submitted through groups $e^{itA_{[-n,n]}}$ as follows $U(t)x = \sum_{n=1}^{\infty} e^{itA_{[-n,n]}} x_n$.

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