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A SPECTRAL STIFF PROBLEM FOR AN ELASTICITY SYSTEM

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We study spectral properties of the stiff problem for an elasticity system with the Dirichlet boundary condition. The stiff part is located strictly inside the domain. The ratio of rigidities of soft and stiff parts is associated with a small parameter. We construct and justify the complete asymptotic expansions of prime eigenvalues and the corresponding eigenvectors with a fixed number. These expansions describe low frequency eigenvibration of the system.

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В работе изучаются спектральные свойства жесткой задачи для системы уравнений теории упругости с краевыми условиями Дирихле, когда "жесткая" область является внутренней подобластью системы. Отношение жесткостей менее и более жесткой частей системы характеризует малый параметр. Построены и обоснованы полные асимптотические разложения по малому параметру простых собственных значений и соответствующих собственных векторов задачи, описывающие низкочастотные формы собственных колебаний системы.

The stiff problem is a boundary value problem whose differential operator has different orders of the coefficients in different subdomains. In the elasticity theory such problems model systems with perturbed stiffness of an elastic medium. The ratio of rigidities of less stiff part of the system and of more one is associated with a small parameter. The study of stiff problems was originated in [1]. J.- L. Lions, who introduced the term *stiff problem*, has developed asymptotic techniques for investigation of such problems.

Since the 1980's the theory of nonhomogeneous media attracts much attention of researchers. The most popular topics are the averaging of differential operators, problems with concentrated masses, stiff problems, problems with singular perturbed domains, combinations of such problems, etc. The extensive yet not comprehensive bibliography of this area is given in [1]–[6]. G. P. Panasenko [7, 8] has initiated studies of the spectral stiff problems. Perturbation of coefficients which describe the stiffness and the density of systems occurs also in spectral problems with singular perturbed domains (see [6] and the references there). Various research methods of the spectral stiff problems have been developed [1, 5], [9]–[15].

It turns out that there are two kinds of the eigenvibrations of the systems with singular perturbed stiffness. The *low frequency* vibrations correspond to eigenvalues with a fixed number. Such eigenvalues are infinitesimal as the small parameter tends to zero. The

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energy of low frequency vibrations is located on a less stiff part of the system and they vanish on a more one. Other kind is so-called *high frequency* vibrations. They are supported by eigenfunctions corresponding to some sequences of eigenvalues, with different numbers, converging to nonzero limits.

The phenomenon of high frequency vibrations in stiff problems was first mentioned in [5]. The asymptotic analysis of these vibrations is more complicated in comparison with the low frequency case. The research on high frequency oscillations has been attempted, in particular, in [13]. The complete asymptotic analysis, based on quasiclassical WKB-method, was done in [14] for the one-dimensional case. In the multidimensional case this approach faces with considerable difficulties. In any case, the asymptotic analysis should start with investigation of low frequency eigenvibrations, and this is the subject of our paper.

We construct and justify the complete asymptotic expansions of prime eigenvalues and the corresponding eigenvectors of a spectral stiff problem for an elasticity system with the Dirichlet boundary condition. The stiffer region is located strictly inside the domain.

Spectral stiff problems with such allocation of the stiff region have been considered in [5], [7]–[9], [15]. In [5] the complete asymptotic expansions of solutions of spectral problems for the second-order elliptic equation in $\mathbb{R}^3$ were constructed. In these problems together with stiffness also the density was perturbed. As a result, the orders of the expansion coefficients are balanced, and formation of problems for this coefficients is simpler in comparison to one in the case of the stiffness perturbation only. Moreover, the eigenvalues are not infinitesimal as small parameter tends to zero. In [7]–[9] the expansions of the eigenvalues of elliptic equations in $\mathbb{R}^n$ $(n \geqslant 1)$ were constructed. However the problems for principal terms obtained in [7]–[9] are underspecified. This causes using auxiliary boundary value problems and so-called quasicharacteristic equation. The latter is an equation for principal terms of eigenvalues. It is rather intricate and can be effectively analyzed only in a few cases. In [15] the complete asymptotic expansions of eigenvalues and eigenvectors for a one-dimensional fourth-order operator were constructed and justified. A recurrent sequence of completely specified problems for the expansion coefficients was obtained.

In this paper we apply the approach [15] for construction of formal asymptotics. To obtain completely specified problems for coefficients of the asymptotics, in addition to relations for this coefficients, we use the compatibility condition for problems for the next order terms in the rigid part. In the soft part, the principal terms of eigenelements are solutions of some spectral problem containing a spectral parameter both in the equation and the boundary condition. In the stiff part, the principal terms of the eigenfunctions belong to the kernel of the homogeneous Neumann problem These terms correspond to motions of the rigid inclusion treated as a perfectly rigid body.

We employ methods of theory of self-adjoint compact operators in Hilbert spaces. Justification of asymptotic expansions is based on a lemma on almost proper elements of self-adjoint operators in [16]. The results of this paper were announced in [17].

1. Statement of the problem

Let Ω_0, Ω be bounded connected domains in $\mathbb{R}^n$ $(n \geqslant 3)$ with smooth boundaries $\partial\Omega_0$ and $\partial\Omega$. Let $\overline{\Omega}_0 \subset \Omega$. Set $\Omega_1 = \Omega \setminus \overline{\Omega}_0$. We consider the following spectral Dirichlet problem for the linear elasticity system:

$$\mathcal{L}^\varepsilon(x,\partial_x)u_\varepsilon(x)+\lambda_\varepsilon\rho(x)u_\varepsilon(x)=0, \qquad x\in\Omega_0\cup\Omega_1; \tag{1.1}$$

$$u_\varepsilon(x)=0, \qquad x\in\partial\Omega; \tag{1.2}$$

$$[u_\varepsilon(x)]_{\partial\Omega_0}=0, \qquad [\sigma^\varepsilon(x,\partial_x)u_\varepsilon(x)]_{\partial\Omega_0}=0. \tag{1.3}$$

Here $\mathcal{L}^\varepsilon(x,\partial_x)=\frac{\partial}{\partial x_j}\big(A^{\varepsilon\, jl}(x)\frac{\partial}{\partial x_l}\big)$ is the elasticity operator, $u_\varepsilon=(u_1^\varepsilon,\dots,u_n^\varepsilon)$ is the displacement vector, $\sigma^\varepsilon(x,\partial_x)=\nu_j(x)A^{\varepsilon\, jl}(x)\frac{\partial}{\partial x_l}$ is the tension operator, ν is the outward normal on $\partial\Omega_0$, $\rho>0$ is a smooth bounded function on Ω, λ_ε is a spectral parameter. We will denote by $[f]_{\partial\Omega_0}$ the jump of f while crossing $\partial\Omega_0$. It is customary to assume that expressions containing the same index twice should be summed over that index from 1 to n. By $A^{\varepsilon\, jl}$ we denote $n\times n$-matrices

$$A^{\varepsilon\, jl}(x)=\begin{cases} A_0^{jl}(x), & x\in\Omega_0;\\ \varepsilon A_1^{jl}(x), & x\in\Omega_1, \end{cases} \qquad j,l\in\{1,\dots,n,\}$$

where $\varepsilon>0$ is a small real parameter. Elements $a_{m\,ik}^{\;jl}$ $(i,k\in\{1,\dots,n\})$ of A_m^{jl} are smooth bounded functions on Ω_m such that the conditions

$$a_{m\,ik}^{\;jl}=a_{m\,jk}^{\;il}=a_{m\,ki}^{\;lj};$$

$$\varkappa_1\eta_{ij}\eta_{ij}\leqslant a_{m\,ik}^{\;jl}(x)\eta_{ij}\eta_{kl}\leqslant\varkappa_2\eta_{ij}\eta_{ij}, \quad \varkappa_1,\varkappa_2=\text{const}>0, \quad x\in\Omega_m, \quad m\in\{0,1\}$$

hold for an arbitrary real symmetric matrix $\{\eta_{ij}\}$. It is easy to check that elements $a^{\varepsilon\, jl}_{\;ik}$ $(i,k\in\{1,\dots,n\})$ of $A^{\varepsilon\, jl}$ satisfy

$$\varepsilon\varkappa_1\eta_{ij}\eta_{ij}\leqslant a^{\varepsilon\, jl}_{\;ik}(x)\eta_{ij}\eta_{kl}\leqslant\varkappa_2\eta_{ij}\eta_{ij}, \qquad x\in\Omega_0\cup\Omega_1, \quad \varepsilon\in(0;\,1). \tag{1.4}$$

The left-hand side inequality is the condition of strong ellipticity for $\mathcal{L}^\varepsilon$.

For column-vectors $u=(u_1,u_2,\dots,u_n)^\top$, $v=(v_1,v_2,\dots,v_n)^\top$ and matrices $A=\{a_{ij}\}$, $B=\{b_{ij}\}$ we introduce notation $u\cdot v=u_iv_i$, $|u|=(u\cdot u)^{1/2}$, $(A,B)=a_{ij}\,b_{ij}$, $|A|=(A,A)^{1/2}$. Here "$\top$" denotes the transposition symbol. If vector components or matrix elements belong to a Hilbert space $\mathfrak{H}$ with the scalar product $(\cdot\,,\cdot)_{\mathfrak{H}}$, we denote

$$(u,v)_{\mathfrak{H}}=(u_i,v_i)_{\mathfrak{H}}, \qquad \|u\|_{\mathfrak{H}}=(u,u)_{\mathfrak{H}}^{1/2};$$

$$(A,B)_{\mathfrak{H}}=(A_{ij},B_{ij})_{\mathfrak{H}}, \qquad \|A\|_{\mathfrak{H}}=(A,A)_{\mathfrak{H}}^{1/2},$$

and write $u,\ v\in\mathfrak{H}$ or $A,\ B\in\mathfrak{H}$ instead of $u,\ v\in\mathfrak{H}^n$ or $A,\ B\in\mathfrak{H}^{n^2}$. Later on we use the term *function* instead of *vector-function*. For a function u, let ∇u denote the matrix $\big\{\frac{\partial u_i}{\partial x_j}\big\}$.

Solutions of the problem (1.1)–(1.3) are understood in the sense of the corresponding integral identity

$$\int\limits_\Omega\big(\mathfrak{A}^\varepsilon\nabla u_\varepsilon,\nabla v\big)\,dx=\lambda_\varepsilon\int\limits_\Omega\rho u_\varepsilon\cdot v\,dx \tag{1.5}$$

in the Sobolev space $H_0^1(\Omega)$. Here we denote by $\mathfrak{A}^\varepsilon$ the linear transformation in the space of $n\times n$-matrices that for each $x\in\Omega_0\cup\Omega_1$ takes a matrix $\xi=\{\xi_{ij}\}$ to the matrix

$\mathfrak{A}^\varepsilon\xi = \{a^{\varepsilon\, jl}_{ik}(x)\xi_{ij}\}$. The identity (1.5) can be derived from (1.1)–(1.3) by multiplying both sides (1.1) by $v \in C_0^\infty(\Omega)$ and using the *first Betti formula*

$$\int\limits_G \mathcal{L}u \cdot v\, dx = \int\limits_{\partial G} \sigma u \cdot v\, dx - \int\limits_G \big(\mathfrak{A}\nabla u, \nabla v\big)\, dx, \quad u, v \in C^2(G), \tag{1.6}$$

where $\mathcal{L}$ is an elasticity operator, $G \subset \mathbb{R}^n$ is a domain with piecewise smooth boundary ∂G. We here used the fact that $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$.

Let $\mathfrak{a}^\varepsilon(u,v)$ and $\mathfrak{b}(u,v)$ be bilinear forms in $H_0^1(\Omega)$ and in the weighted Lebesgue space $L_2(\rho,\Omega)$ given by the integrals occurring in (1.5); then (1.5) takes the form

$$\mathfrak{a}^\varepsilon(u_\varepsilon, v) = \lambda_\varepsilon \mathfrak{b}(u_\varepsilon, v), \qquad u_\varepsilon, v \in H_0^1(\Omega). \tag{1.7}$$

The next two lemata yield the information about spectral properties of the problem (1.1)–(1.3).

Lemma 1.1. *For any* $\varepsilon > 0$ *the spectrum of the problem* (1.1)–(1.3) *consists of a sequence of eigenvalues*

$$0 < \lambda_1^\varepsilon \leqslant \lambda_2^\varepsilon \leqslant \cdots \leqslant \lambda_s^\varepsilon \leqslant \cdots \xrightarrow[s\to+\infty]{} +\infty,$$

where multiplicity is taken into account in the indexing. The multiplicity of each λ_s^ε *is finite, and the corresponding eigenvectors* $\{u_\varepsilon^s(x)\}_{s=1}^\infty$ *can be chosen as an orthonormal basis in* $L_2(\rho,\Omega)$.

Proof. The proof is based on reduction of (1.1)–(1.3) to an operator equation. For this purpose we introduce in $H_0^1(\Omega)$ the equivalent scalar product $(\cdot,\,\cdot)_\varepsilon$ associated with $\mathfrak{a}^\varepsilon$, i.e., defined by the left-hand side of (1.5). In view of

$$C_1\varepsilon\, \|u\|^2_{H_0^1(\Omega)} \leqslant \mathfrak{a}^\varepsilon(u,u) \leqslant C_2\, \|u\|^2_{H_0^1(\Omega)}, \qquad u \in H_0^1(\Omega) \tag{1.8}$$

and the symmetry of $\mathfrak{a}^\varepsilon$ the equivalence holds. (1.8) follows from (1.4), boundedness of $\mathfrak{A}^\varepsilon$, the Korn inequality $\|\nabla u\|^2_{L_2(\Omega)} \leqslant C\, \|e(u)\|^2_{L_2(\Omega)}$, $u \in H_0^1(\Omega)$ [3, p.17], and the Friedrichs inequality $\|u\|_{L_2(\Omega)} \leqslant C\, \|\nabla u\|_{L_2(\Omega)}$, $u \in H_0^1(\Omega)$ [3, p.10], where $e(u) = \{e_{ij}\} = \big\{\frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big)\big\}$ is a deformation tensor. By $\|\cdot\|_\varepsilon$ denote the norm (the energetic norm) associated with $(\cdot,\,\cdot)_\varepsilon$. We use the same designation $H_0^1(\Omega)$ for the space with this norm.

Fix $u \in H_0^1(\Omega)$; then the map $v \to \mathfrak{b}(u,v)$ is a linear bounded functional in $H_0^1(\Omega)$. By the Riesz representation theorem, there exists a unique $w \in H_0^1(\Omega)$ such that $\mathfrak{b}(u,v) = \mathfrak{a}^\varepsilon(w,v)$. By that the operator $u \overset{A^\varepsilon}{\longmapsto} w$ is defined in $H_0^1(\Omega)$. The identity (1.7) and the relation $\mathfrak{b}(u_\varepsilon, v) = \mathfrak{a}^\varepsilon(A^\varepsilon u_\varepsilon, v)$ lead to the spectral problem

$$A^\varepsilon u_\varepsilon = \lambda_\varepsilon^{-1} u_\varepsilon, \qquad u_\varepsilon \in H_0^1(\Omega) \tag{1.9}$$

which is equivalent to (1.1)–(1.3). From boundedness, symmetry and positiveness of $\mathfrak{a}^\varepsilon$ it follows that A^ε is bounded, self-adjoint and positive. In addition, since the embedding $H_0^1(\Omega) \subset L_2(\rho,\Omega)$ is compact, it follows that A^ε is also a compact operator. Then the assertion of the lemma is the immediate consequence of the theory of compact operators. $\square$

Below we use notation

$$\mathcal{L}_m(x,\partial_x) = \frac{\partial}{\partial x_j}\Big(A_m^{jl}(x)\frac{\partial}{\partial x_l}\Big), \qquad \sigma_m(x,\partial_x) = \nu_j(x)A_m^{jl}(x)\frac{\partial}{\partial x_l},$$

$$\mathfrak{a}_m(u,v) = \int\limits_{\Omega_m}\big(\mathfrak{A}_m\nabla u,\nabla v\big)\,dx, \qquad \mathfrak{b}_m(u,v) = \int\limits_{\Omega_m}\rho u\cdot v\,dx, \qquad u,v\in H_0^1(\Omega)$$

for $m = 0,1$. Here $\mathfrak{A}_m$ is the linear transformation taking for each $x\in\Omega_m$ $n\times n$-matrix $\xi = \{\xi_{ij}\}$ to the matrix $\mathfrak{A}_m\xi = \{a_{m\,ik}^{\,jl}(x)\xi_{ij}\}$. Note that $\mathfrak{a}^\varepsilon = \mathfrak{a}_0 + \varepsilon\mathfrak{a}_1$. By γ_Σ denote the trace operator on a hypersurface $\Sigma\subset\mathbb{R}^n$. We shall often write $f|_\Sigma$ instead of $\gamma_\Sigma f$.

Lemma 1.2. *The eigenvalues of the problem* (1.1)–(1.3) *are continuous functions of* ε. *For each eigenvalue* λ_s^ε *the estimates*

$$C_1\varepsilon \leqslant \lambda_s^\varepsilon \leqslant C_2(s)\varepsilon, \quad s\in\{1,2,\dots\}, \quad \varepsilon\in(0;\,1), \tag{1.10}$$

are valid. Here C_1, C_2 *do not depend on* ε, *and* C_1 *does not depend yet on* s.

Proof. The assertions of the lemma can be obtained from the minimax property

$$\lambda_s^\varepsilon = \sup_{\substack{P\subset L_2(\rho,\Omega)\\ \dim P = s-1}}\ \inf_{\substack{u\in H_0^1(\Omega)\\ u\in P^\perp,\,u\neq 0}} \frac{\mathfrak{a}^\varepsilon(u,\,u)}{\mathfrak{b}(u,\,u)}, \qquad s\in\{1,2,\dots\} \tag{1.11}$$

of the eigenvalues λ_s^ε of the problem (1.1)–(1.3). Here $P^\perp$ is the orthogonal complement of a linear $(s-1)$-dimensional subspace P in $L_2(\rho,\Omega)$. Note that the supremum is attained at the linear span of eigenvectors corresponding to the eigenvalues $\lambda_1^\varepsilon,\dots,\lambda_{s-1}^\varepsilon$. From (1.11) it follows immediately that all eigenvalues are continuously depending on ε.

For λ_1^ε, starting from (1.11), we have

$$\lambda_1^\varepsilon = \inf_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\mathfrak{a}_0(u,\,u)+\varepsilon\mathfrak{a}_1(u,\,u)}{\mathfrak{b}(u,\,u)} \geqslant \varepsilon \inf_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\mathfrak{a}_0(u,\,u)+\mathfrak{a}_1(u,\,u)}{\mathfrak{b}(u,\,u)} = \varepsilon\lambda_1^1,$$

where λ_1^1 is λ_1^ε at $\varepsilon = 1$. Hence $\lambda_1^\varepsilon \geqslant \lambda_1^1\varepsilon$, and since λ_1^ε is the smallest eigenvalue of (1.1)–(1.3), then $\lambda_s^\varepsilon \geqslant \lambda_1^1\varepsilon$ for all $s\in\{1,2,\dots\}$. The left-hand side of (1.10), with $C_1 = \lambda_1^1$, is proved.

Let us prove the right-hand side of (1.10). For this purpose let us compare each λ_s^ε with the corresponding eigenvalue ν_s of the problem

$$\begin{aligned} \mathcal{L}_1(x,\partial_x)u(x) + \nu\rho(x)u(x) = 0, &\quad x\in\Omega_1;\\ u(x) = 0, &\quad x\in\partial\Omega_1. \end{aligned} \tag{1.12}$$

By $\overset{\circ}{H}{}_0^1(\Omega)$ denote the space of extensions of all $u\in H_0^1(\Omega_1)$ to the entire domain Ω by zero on the set $\Omega\setminus\Omega_1$. It is obvious that $\overset{\circ}{H}{}_0^1(\Omega)\subset H_0^1(\Omega)$. Let Q be the linear span of eigenvectors corresponding to $\lambda_1^\varepsilon,\dots,\lambda_{s-1}^\varepsilon$. Evidently, $Q\subset L_2(\rho,\Omega)$, $\dim Q = s-1$. The restrictions of all $u\in Q$ to Ω_1 form the subspace $\widetilde{Q}\subset L_2(\rho,\Omega_1)$ of dimension $\dim\widetilde{Q} = m\leqslant s-1$. Then, taking into account that the supremum occurring in (1.11) is attained at Q, we have

$$\begin{aligned}\lambda_s^\varepsilon = \inf_{\substack{u\in H_0^1(\Omega)\\ u\in Q^\perp,\, u\neq 0}} \frac{\mathfrak{a}^\varepsilon(u,\,u)}{\mathfrak{b}(u,\,u)} \leqslant \varepsilon \inf_{\substack{u\in \overset{\circ}{H}{}_0^1(\Omega)\\ u\in Q^\perp,\, u\neq 0}} \frac{\mathfrak{a}_1(u,\,u)}{\mathfrak{b}_1(u,\,u)} = \varepsilon \inf_{\substack{u\in H_0^1(\Omega_1)\\ u\in \widetilde{Q}^\perp,\, u\neq 0}} \frac{\mathfrak{a}_1(u,\,u)}{\mathfrak{b}_1(u,\,u)} \leqslant \\ \leqslant \sup_{\substack{P\subset L_2(\rho,\Omega_1)\\ \dim P=\dim \widetilde{Q}=m\leqslant s-1}} \inf_{\substack{u\in H_0^1(\Omega_1)\\ u\in P^\perp,\, u\neq 0}} \frac{\mathfrak{a}_1(u,\,u)}{\mathfrak{b}_1(u,\,u)} = \varepsilon\nu_{m+1} \leqslant \varepsilon\nu_s.\end{aligned} \tag{1.13}$$

Here the first inequality is obtained by changing from $H_0^1(\Omega)$ to $\overset{\circ}{H}{}_0^1(\Omega) \subset H_0^1(\Omega)$; therefore the infimum may be increase only; we use also the fact that $\mathfrak{a}^\varepsilon = \varepsilon\mathfrak{a}_1$, $\mathfrak{b} = \mathfrak{b}_1$ on $\overset{\circ}{H}{}_0^1(\Omega)$. The second equality holds because the restrictions of all $u \in \overset{\circ}{H}{}_0^1(\Omega) \cap Q^\perp$ to Ω_1 form exactly the set $H_0^1(\Omega_1) \cap \widetilde{Q}^\perp$ while the functional $\mathfrak{a}_1/\mathfrak{b}_1$ depends only on values of u at points of Ω_1. The second inequality of the chain (1.13) is the result of the passage from the fixed subspace $\widetilde{Q} \subset L_2(\rho, \Omega_1)$ of dimension m to taking of the supremum over all subspaces $P \subset L_2(\rho, \Omega_1)$ of the same dimension. In view of the minimax property this supremum is equal to the eigenvalue ν_{m+1} of (1.12). Obviously $\nu_{m+1} \leqslant \nu_s$ because $m + 1 \leqslant s$ while all eigenvalues increase in ascending order of their indices.

Thus from (1.13) we see that $\lambda_s^\varepsilon \leqslant \nu_s\varepsilon$ what is the right-hand side of (1.10) with $C(s) = \nu_s$. The lemma is proved. □

It is our purpose to construct and justify the complete asymptotic expansions in the parameter ε of eigenvalues λ_ε and the corresponding eigenvectors u_ε of (1.1)–(1.3). By Lemma 1.2, all eigenvalues are infinitesimal as $\varepsilon \to 0$, i.e., the asymptotic expansions, constructed below, describe the low frequency eigenvibrations.

2. Formal asymptotics

2.1. Principal terms. Taking into account the structure of the coefficients of $\mathcal{L}^\varepsilon$ and notation $u_\varepsilon|_{\Omega_0} = v_\varepsilon$, $u_\varepsilon|_{\Omega_1} = w_\varepsilon$, we rewrite (1.1)–(1.3) in the form

$$\mathcal{L}_0(x, \partial_x)v_\varepsilon(x) + \lambda_\varepsilon\rho(x)v_\varepsilon(x) = 0, \qquad x \in \Omega_0; \tag{2.1}$$

$$\varepsilon\mathcal{L}_1(x, \partial_x)w_\varepsilon(x) + \lambda_\varepsilon\rho(x)w_\varepsilon(x) = 0, \qquad x \in \Omega_1; \tag{2.2}$$

$$w_\varepsilon(x) = 0, \qquad x0 \in \partial\Omega; \tag{2.3}$$

$$v_\varepsilon(x) = w_\varepsilon(x), \qquad x \in \partial\Omega_0; \tag{2.4}$$

$$\sigma_0(x, \partial_x)v_\varepsilon(x) = \varepsilon\sigma_1(x, \partial_x)w_\varepsilon(x), \qquad x \in \partial\Omega_0. \tag{2.5}$$

We seek formal asymptotic expansions of eigenelements of the form

$$\lambda_\varepsilon \sim \varepsilon(\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots); \tag{2.6}$$

$$v_\varepsilon(x) \sim v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \cdots, \quad x \in \Omega_0; \tag{2.7}$$

$$w_\varepsilon(x) \sim w_0(x) + \varepsilon w_1(x) + \varepsilon^2 w_2(x) + \cdots, x \in \Omega_1. \tag{2.8}$$

Substituting (2.6)–(2.8) into (2.1)–(2.5) and arranging the coefficients by the same powers of ε, we obtain the recurrence relations

$$\mathcal{L}_0(x, \partial_x)v_0(x) = 0, \qquad x \in \Omega_0; \tag{2.9}$$

$$\mathcal{L}_0(x, \partial_x) v_{k+1}(x) + \rho(x) \sum_{i=0}^{k} \lambda_i v_{k-i}(x) = 0, \qquad x \in \Omega_0; \tag{2.10}$$

$$\mathcal{L}_1(x, \partial_x) w_k(x) + \rho(x) \sum_{i=0}^{k} \lambda_i w_{k-i}(x) = 0, \qquad x \in \Omega_1; \tag{2.11}$$

$$w_k(x) = 0, \qquad x \in \partial\Omega; \tag{2.12}$$

$$v_k(x) = w_k(x), \qquad x \in \partial\Omega_0; \tag{2.13}$$

$$\sigma_0(x, \partial_x) v_0(x) = 0, \qquad x \in \partial\Omega_0; \tag{2.14}$$

$$\sigma_0(x, \partial_x) v_{k+1}(x) = \sigma_1(x, \partial_x) w_k(x), \qquad x \in \partial\Omega_0, \tag{2.15}$$

where $k \in \{0, 1, 2, \dots\}$.

Collecting (2.9) and (2.14), we have the problem

$$\mathcal{L}_0(x, \partial_x) v_0(x) = 0, \qquad x \in \Omega_0; \tag{2.16}$$

$$\sigma_0(x, \partial_x) v_0(x) = 0, \qquad x \in \partial\Omega_0 \tag{2.17}$$

for determination of v_0. It is the Neumann problem for the linear homogeneous elasticity system. Its kernel is nontrivial and consists of functions $a + Ax$, where $a \in \mathbb{R}^n$, A is a skew-symmetric $(n \times n)$-matrix with constant real elements. In the elasticity theory, such functions are called *rigid displacements*. Following [3], we denote the set of ones by $\mathfrak{R}$. Evidently, $\mathfrak{R}$ is a linear space of dimension $r = \frac{1}{2}n(n+1)$. In terms of bilinear forms it, what has been said above, means that $\mathfrak{a}_0(u, v) = 0 \ \forall v \in H^1(\Omega_0)$ iff $u \in \mathfrak{R}$. It will cause no confusion if we use the same letter $\mathfrak{R}$ to designate the space of restrictions of rigid displacements to a domain or to a hypersurface in $\mathbb{R}^n$.

Let us give an other description of the rigid displacements. Let Ψ be an $n \times r$-matrix whose columns are basis elements of $\mathfrak{R}$; then $\mathfrak{R} = \{\Psi\alpha \mid \alpha \in \mathbb{R}^r\}$. Specifically, we can take $\Psi = (E, \Psi_1, \Psi_2, \dots, \Psi_{n-1})$, where E is the identity matrix of order n; Ψ_p $(p = 1, 2, \dots, n-1)$ is an $n \times (n-p)$-matrix of the form

$$\Psi_p = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ x_{p+1} & x_{p+2} & \cdots & x_n \\ -x_p & 0 & \cdots & 0 \\ 0 & -x_p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -x_p \end{pmatrix}.$$

Thus all solutions of (2.16)–(2.17) are of the form $\Psi\alpha$, $\alpha \in \mathbb{R}^r$. Let us remark that they are C^∞-functions. However the relation $v_0 = w_0$ on $\partial\Omega_0$ (see (2.13) for $k = 0$) means that v_0 is not an arbitrary rigid displacement. Therefore we can write

$$v_0 = \Psi\alpha_0, \tag{2.18}$$

where the components of $\alpha_0 \in \mathbb{R}^r$ subject to the further determination.

Taking $k = 0$ in (2.11)–(2.13), we obtain the relations

$$\mathcal{L}_1(x, \partial_x) w_0(x) + \lambda_0 \rho(x) w_0(x) = 0, \qquad x \in \Omega_1; \tag{2.19}$$

$$w_0(x) = 0, \qquad x \in \partial\Omega; \tag{2.20}$$

$$w_0(x) = v_0(x), \qquad x \in \partial\Omega_0, \tag{2.21}$$

for w_0. (2.18) shows that (2.21) can be written as

$$w_0 = \Psi(x)\alpha_0, \qquad x \in \partial\Omega_0. \tag{2.22}$$

Now, collecting (2.10) and (2.15) in the case $k = 0$, we derive the equalities

$$\mathcal{L}_0(x, \partial_x) v_1(x) + \lambda_0 \rho(x) v_0 = 0, \qquad x \in \Omega_0; \tag{2.23}$$

$$\sigma_0(x, \partial_x) v_1(x) = \sigma_1(x, \partial_x) w_0(x), \qquad x \in \partial\Omega_0 \tag{2.24}$$

for determination of v_1. The kernel of the corresponding to (2.23)–(2.24) homogeneous problem (2.16)–(2.17) is nontrivial and coincides with $\mathfrak{R}$. By the third Fredholm theorem (see [18, Th.3, P.98]) the solution of (2.23)–(2.24) exists under the compatibility condition; it is defined uniquely up to an additive kernel term, i.e. up to a rigid displacement.

To obtain the above-mentioned condition we apply the Betti formula (1.6) to (2.23) for each $v = \Psi e_k$, ($k \in \{1, \dots r\}$), where $\{\Psi e_k\}_{k=1}^r$ is the basis of $\mathfrak{R}$; $\{e_k\}_{k=1}^r$ is the canonical basis of $\mathbb{R}^r$. Taking into account too (2.18) and (2.24), it is not hard to compute that

$$\int\limits_{\Omega_0} \mathcal{L}_0 v_1 \cdot \Psi e_k \, dx + \lambda_0 \int\limits_{\Omega_0} \rho v_0 \cdot \Psi e_k \, dx =$$

$$= \int\limits_{\partial\Omega_0} \sigma_0 v_1 \cdot \Psi e_k \, dx - \int\limits_{\Omega_0} \big(\mathfrak{A}_0 \nabla v_1, \, \nabla \Psi e_k\big) \, dx + \lambda_0 \int\limits_{\Omega_0} \rho v_0 \cdot \Psi e_k \, dx =$$

$$= \Big\langle \int\limits_{\partial\Omega_0} \Psi^\top (\sigma_1 w_0) \, dx, \; e_k \Big\rangle_{\mathbb{R}^r} + \lambda_0 \Big\langle \int\limits_{\Omega_0} \rho \Psi^\top \Psi \alpha_0 \, dx, \; e_k \Big\rangle_{\mathbb{R}^r} = 0, \quad k \in \{1, \dots, r\}$$

whence it follows that

$$\lambda_0 J \alpha_0 = - \int\limits_{\partial\Omega_0} \Psi^\top (\sigma_1 w_0) \, dx, \tag{2.25}$$

where $J = \int_{\Omega_0} \rho \Psi^\top \Psi \, dx$ is the symmetric positive-definite $r \times r$-matrix (the Gramian matrix of the functions $\{\Psi e_k\}_{k=1}^r$), $\langle \cdot, \, \cdot \rangle_{\mathbb{R}^r}$ denotes the Euclidean scalar product in $\mathbb{R}^r$. The compatibility condition (2.25) assures the solvability of (2.23)–(2.24).

Henceforth we shall not special stipulate the mention of the Fredholm theorems under consideration of problems with nontrivial kernels. We shall put down compatibility conditions of such problems at once. This conditions can be obtained following to the just now considered scheme.

Eliminating α_0 from (2.22) and (2.25), and combining the resulting relation with (2.19) and (2.20), we obtain the spectral problem

$$\mathcal{L}_1(x, \partial_x) w_0(x) + \lambda_0 \rho(x) w_0(x) = 0, \qquad x \in \Omega_1; \tag{2.26}$$

$$w_0(x) = 0, \qquad x \in \partial\Omega; \tag{2.27}$$

$$\Psi(x) J^{-1} \int\limits_{\partial\Omega_0} \Psi^\top (\sigma_1 w_0) \, dx + \lambda_0 w_0(x) = 0, \qquad x \in \partial\Omega_0 \tag{2.28}$$

containing a spectral parameter λ_0 both in the equation (2.26) and the boundary condition (2.28). We now concentrate on this problem, deferring construction of the asymptotics for a while.

We shall establish spectral properties of (2.26)–(2.28) by reduction of one to an equivalent operator equation in appropriate Hilbert space. For this purpose let us consider the subspace

$$\mathcal{H} = \{u \in H^1(\Omega_1) \mid \gamma_{\partial\Omega} u = 0, \quad \gamma_{\partial\Omega_0} u \in \mathfrak{R}\}$$

of $H^1(\Omega_1)$. For any $u \in \mathcal{H}$ it is obviously that $u = \Psi\alpha$ on $\partial\Omega_0$, $\alpha \in \mathbb{R}^r$. The continuity of trace operators in $H^1(\Omega_1)$ and finite dimensionality of $\mathfrak{R}$ imply closureness of $\mathcal{H}$. Let $\tau\colon \mathcal{H} \to \mathbb{R}^r$ be the map $u \overset{\tau}{\mapsto} \alpha$. Its kernel is $H_0^1(\Omega_0)$. Since the codimension of one is finite, equals to $r = \frac{1}{2}n(n+1)$, it is follows that τ is continuous in $\mathcal{H}$, i.e.,

$$\tau(u) \leqslant C\,\|u\|_{\mathcal{H}}, \qquad u \in \mathcal{H}, \tag{2.29}$$

where C does not depend on u.

Let us introduce in $\mathcal{H}$ the scalar product associated with $\mathfrak{a}_1$. It is equivalent to the standard scalar product $(\cdot\,,\cdot)_{H^1(\Omega_1)}$. Reason similar to that in the proof of Lemma 1.1 shows that this equivalence is a consequence of strong ellipticity of $\mathcal{L}_1$, the Korn and Friedrichs inequalities (they are valid for $u \in \mathcal{H}$ too). The corresponding norm (the energetic norm) $\mathfrak{a}_1^{1/2}(\cdot\,,\cdot)$ we denote by $\|\cdot\|_{\mathcal{H}}$.

In the variational formulation of (2.26)–(2.28) we deal with the integral identity

$$\int\limits_{\Omega_1} (\mathfrak{A}_1 \nabla w_0, \nabla v)\, dx = \lambda_0 \Big(\int\limits_{\Omega_1} \rho w_0 \cdot v\, dx + \big\langle J\tau(w_0),\, \tau(v)\big\rangle_{\mathbb{R}^r} \Big), \qquad w_0, v \in \mathcal{H}. \tag{2.30}$$

It can be obtained by applying of the Betti formula to (2.26)–(2.28). We use also that $u = \Psi\tau(u)$ on $\partial\Omega_0$ for all $u \in \mathcal{H}$.

Lemma 2.1. *The spectrum of the problem* (2.26)–(2.28) *consists of a sequence of eigenvalues*

$$0 < \lambda_0^1 \leqslant \lambda_0^2 \leqslant \cdots \leqslant \lambda_0^s \leqslant \cdots \xrightarrow[s\to+\infty]{} +\infty,$$

where multiplicity is taken into account in indexing. The multiplicity of each λ_0^s is finite while the corresponding eigenvectors $\{w_0^s(x)\}_{s=1}^\infty$ can be chosen as an orthonormal basis in $L_2(\rho, \Omega_1)$.

Proof. Let $\mathfrak{b}_\tau$ be the bilinear form

$$\mathfrak{b}_\tau(w, v) = \int_{\Omega_1} \rho w \cdot v\, dx + \big\langle J\tau(w), \tau(v)\big\rangle_{\mathbb{R}^r}$$

in $\mathcal{H}$. Then (2.30) takes the form

$$(w_0, v)_{\mathcal{H}} = \lambda_0 \mathfrak{b}_\tau(w_0, v), \quad w_0, v \in \mathcal{H}. \tag{2.31}$$

Since $\mathfrak{b}_\tau$ is positive, symmetric and bounded in $\mathcal{H}$ (see (2.29)), then by the Riesz representation theorem there exists a unique $f \in \mathcal{H}$ such that $\mathfrak{b}_\tau(w, v) = (f, v)_{\mathcal{H}}$. Denote by B the map that takes w to f. Then (2.31) takes the form $(w_0, v)_{\mathcal{H}} = \lambda_0 (Bw_0, v)_{\mathcal{H}}$, whence we obtain the operator equation

$$Bw_0 = \lambda_0^{-1} w_0, \qquad w_0 \in \mathcal{H}. \tag{2.32}$$

It is readily seen that B is defined in $\mathcal{H}$ by $\mathfrak{b}_\tau(w, v) = (Bw, v)_{\mathcal{H}}$. Since, as stated above, $\mathfrak{b}_\tau$ is positive, symmetric, bounded in $\mathcal{H}$, and the embedding $\mathcal{H} \subset L_2(\rho, \Omega_1)$ is compact, it follows that B is also a positive, self-adjoint and compact operator in $\mathcal{H}$. Therefore the assertion of the lemma follows from properties of compact operators. □

Let us return to construction of the expansions (2.6)–(2.8). Suppose further that λ_0 is a prime eigenvalue of (2.26)–(2.28), and w_0 is the corresponding eigenvector normalized in $L_2(\rho, \Omega_1)$. By results [19, Part 1, Sec.6] for strongly elliptic systems, $w_0 \in C^\infty(\overline{\Omega}_1)$. Now, solving (2.25), we obtain

$$\alpha_0 = -\lambda_0^{-1} J^{-1} \int\limits_{\partial\Omega_0} \Psi^\top (\sigma_1 w_0)\, dx.$$

Consequently (see (2.18)),

$$v_0 = \Psi \alpha_0 = -\lambda_0^{-1} \Psi J^{-1} \int\limits_{\partial\Omega_0} \Psi^\top (\sigma_1 w_0)\, dx.$$

Thus the principal terms λ_0, w_0, v_0 of the expansions (2.6)–(2.8) are completely determined.

2.2. Next terms of asymptotics Let us seek the next terms of (2.6)–(2.8). With this objection in mind, we return to the problem (2.23)–(2.24) with already determined C^∞-data and ensured compatibility condition (2.25). Its solution, as stated above, is defined up to a rigid displacement

$$v_1 = \tilde{v}_1 + \Psi \alpha_1, \tag{2.33}$$

where $\tilde{v}_1$ is a fixed solution of (2.23)–(2.24), and $\alpha_1 \in \mathbb{R}^r$ is an unknown for now vector. Fix $\tilde{v}_1$ by orthogonality condition $\int_{\Omega_0} \rho \Psi^\top \tilde{v}_1\, dx = 0$. By [19, Part 1, Sec.12], $\tilde{v}_1 \in C^\infty(\overline{\Omega}_0)$.

Setting $k = 1$ in (2.11)–(2.13), we derive the relations

$$\mathcal{L}_1(x, \partial_x) w_1(x) + \lambda_0 \rho(x) w_1(x) = -\lambda_1 \rho(x) w_0(x), \qquad x \in \Omega_1; \tag{2.34}$$

$$w_1(x) = 0, \qquad x \in \partial\Omega; \tag{2.35}$$

$$w_1(x) = v_1(x), \qquad x \in \partial\Omega_0 \tag{2.36}$$

for w_1. According to (2.33) the last equality can be written as

$$w_1(x) = \tilde{v}_1(x) + \Psi(x) \alpha_1, \quad x \in \partial\Omega_0. \tag{2.37}$$

Just as by seeking of principal terms, we write out a problem for the next term of asymptotics in Ω_0, in the present case for v_2. One is (2.10),(2.15) for $k = 1$

$$\mathcal{L}_0(x, \partial_x) v_2(x) + \lambda_0 \rho(x) v_1 + \lambda_1 \rho(x) v_0 = 0, \qquad x \in \Omega_0; \tag{2.38}$$

$$\sigma_0(x, \partial_x) v_2(x) = \sigma_1(x, \partial_x) w_1(x), \qquad x \in \partial\Omega_0. \tag{2.39}$$

Its compatibility condition, on account of (2.18), (2.33) and $\int_{\Omega_0} \rho\Psi^\top \tilde{v}_1\, dx = 0$, is

$$\int\limits_{\partial\Omega_0} \Psi^\top(\sigma_1 w_1)\, dx + \lambda_0 J\alpha_1 + \lambda_1 J\alpha_0 = 0. \tag{2.40}$$

Eliminating α_1 from (2.40) and (2.37), and combining the resulting relation with (2.34) and (2.35), we obtain the problem

$$\mathcal{L}_1(x,\partial_x)w_1(x) + \lambda_0\rho(x)w_1(x) = -\lambda_1\rho(x)w_0(x), \qquad x \in \Omega_1; \tag{2.41}$$

$$w_1(x) = 0, \qquad x \in \partial\Omega; \tag{2.42}$$

$$\Psi(x)J^{-1}\int\limits_{\partial\Omega_0} \Psi^\top(\sigma_1 w_1)\, dx + \lambda_0 w_1(x) = \lambda_0\tilde{v}_1(x) - \lambda_1\Psi(x)\alpha_0, \qquad x \in \partial\Omega_0 \tag{2.43}$$

for determination of λ_1 and w_1. Since the corresponding homogeneous problem is the considered above spectral problem (2.26)–(2.28), the solution of (2.41)–(2.43) exists under the compatibility condition

$$\lambda_1 = -\Big(1 + \big\langle J\alpha_0, \alpha_0\big\rangle_{\mathbb{R}^r}\Big)^{-1}\int\limits_{\partial\Omega_0} \tilde{v}_1 \cdot \sigma_1 w_0\, dx. \tag{2.44}$$

Note that $\big\langle J\alpha_0, \alpha_0\big\rangle_{\mathbb{R}^r} > 0$ because the matrix J is positive defined.

Thus λ_1 is determined by (2.44); next from (2.41)–(2.43) we determine w_1 up to an additive term Cw_0. Since $\partial\Omega_1$ and the right-hand sides in (2.41)–(2.43) are of class C^∞, we have $w_1 \in C^\infty(\overline{\Omega}_1)$ [19, Part 1, Sec.12]. Fix w_1 by $(w_1, w_0)_{L_2(\rho,\Omega_1)} = 0$. Then we determine $\alpha_1 \in \mathbb{R}^r$ from (2.40):

$$\alpha_1 = -\lambda_0^{-1}J^{-1}\int\limits_{\partial\Omega_0} \Psi^\top(\sigma_1 w_1)\, dx - \lambda_0^{-1}\lambda_1\alpha_0.$$

Consequently $v_1 = \tilde{v}_1 + \Psi\alpha_1$, and v_1 is already completely determined. Next, from (2.38)–(2.39), v_2 is determined up to a rigid displacement. Namely $v_2 = \tilde{v}_2 + \Psi\alpha_2$, where $\tilde{v}_2 \in C^\infty(\overline{\Omega}_0)$ is the solution of (2.38)–(2.39) fixed by $\int_{\Omega_0} \rho\Psi^\top\tilde{v}_1\, dx = 0$, and $\alpha_2 \in \mathbb{R}^r$ is a vector with undefined for now components.

Continuing in the same manner, we see that all terms of (2.6)–(2.8) up to an arbitrary order can be determined. Indeed, let us assume, by induction, that the coefficients λ_i, w_i, v_i of (2.6)–(2.8) are determined for $i \in \{0, 1, \ldots, k-1\}$, and $w_i \in C^\infty(\overline{\Omega}_1)$, $v_i \in C^\infty(\overline{\Omega}_0)$; each of w_i satisfies $(w_i, w_0)_{L_2(\rho,\Omega_1)} = \delta_{i0}$ (δ_{i0} is the Kronecker symbol); v_k is found up to a rigid displacement

$$v_k = \tilde{v}_k + \Psi\alpha_k, \qquad \text{where} \quad \int\limits_{\Omega_0} \rho\Psi^\top\tilde{v}_k\, dx = 0, \qquad \tilde{v}_k \in C^\infty(\overline{\Omega}_0). \tag{2.45}$$

Let us show that then λ_k, w_k and α_k can be determined, whence the same is true for v_k; $w_k \in C^\infty(\overline{\Omega}_1)$, $v_k \in C^\infty(\overline{\Omega}_0)$; v_{k+1} can be found up to a rigid displacement; $v_{k+1} \in C^\infty(\overline{\Omega}_0)$.

Combining (2.10) and (2.15), we have the relations

$$\mathcal{L}_0(x,\partial_x)v_{k+1}(x) = -\rho(x)\sum_{i=0}^{k}\lambda_i v_{k-i}, \qquad x \in \Omega_0; \tag{2.46}$$

$$\sigma_0(x,\partial_x)v_{k+1}(x) = \sigma_1(x,\partial_x)w_k(x), \qquad x \in \partial\Omega_0 \tag{2.47}$$

with smooth right-hand members. The compatibility condition for (2.46)–(2.47) is

$$\int\limits_{\partial\Omega_0} \Psi^\top(\sigma_1 w_k)\,dx + \sum_{i=0}^{k} \lambda_i J\alpha_{k-i} = 0 \tag{2.48}$$

(compare (2.40)). In view of (2.45) the boundary condition (2.13) ($w_k = v_k$ on $\partial\Omega_0$) takes the form

$$w_k(x) = \tilde v_k(x) + \Psi(x)\alpha_k, \quad x \in \partial\Omega_0.$$

Let us eliminate α_k from here and (2.48); combination of the resulting relation, (2.11) and (2.12) leads to the problem

$$\mathcal{L}_1(x,\partial_x)w_k(x) + \lambda_0\rho(x)w_n(x) = -\rho(x)\sum_{i=1}^{k}\lambda_i w_{k-i} = 0, \qquad x \in \Omega_1; \tag{2.49}$$

$$w_k(x) = 0, \qquad x \in \partial\Omega; \tag{2.50}$$

$$\Psi(x)J^{-1}\int\limits_{\partial\Omega_0} \Psi^\top(\sigma_1 w_k)\,dx + \lambda_0 w_k(x) = \lambda_0\tilde v_k(x) - \sum_{i=1}^{k}\lambda_i\Psi\alpha_{k-i}, \qquad x \in \partial\Omega_0 \tag{2.51}$$

for determination of λ_k and w_k. Its compatibility condition is

$$\lambda_k = -\big(1 + \langle J\alpha_0, \alpha_0\rangle_{\mathbb{R}^r}\big)^{-1}\Big(\int\limits_{\partial\Omega_0} \tilde v_1\cdot\sigma_1 w_0\,dx + \sum_{i=1}^{k-1}\lambda_i\langle J\alpha_{k-i},\,\alpha_0\rangle_{\mathbb{R}^r}\Big).$$

It gives λ_k while w_k is determined up to a term Cw_0. Fix w_k by $(w_k,\,w_0)_{L_2(\rho,\Omega_1)} = 0$. From (2.48)

$$\alpha_k = -\lambda_0^{-1}\Big(J^{-1}\int\limits_{\partial\Omega_0}\Psi^\top(\sigma_1 w_k)\,dx + \sum_{i=1}^{k}\lambda_i\alpha_{k-i}\Big).$$

From (2.46)–(2.47), we can determine v_{k+1} in the form $v_{k+1} = \tilde v_{k+1} + \Psi\alpha_{k+1}$, where $\tilde v_{k+1}$ is the solution of (2.46)–(2.47) fixed by

$$\int_{\Omega_0}\rho\Psi^\top\tilde v_{k+1}\,dx = 0; \quad \tilde v_{k+1} \in C^\infty(\overline{\Omega}_0);$$

components of $\alpha_{k+1} \in \mathbb{R}^r$ subject to the further determination. Smoothness of the found coefficients follows [19, Part 1, Sec.12].

The induction step in construction of the expansions (2.6)–(2.7) of prime eigenvalues of (1.1)–(1.3) and the corresponding eigenvectors is performed. The procedure of determination of the coefficients of asymptotics is completely described.

2.3. Estimates of remainder terms. Let us see the Nth partial sums

$$\Lambda_N^\varepsilon = \varepsilon\sum_{k=0}^{N}\lambda_k\varepsilon^k, \qquad U_N^\varepsilon(x) = \begin{cases}\sum\limits_{k=0}^{N} v_k(x)\varepsilon^k, & x \in \Omega_0;\\ \sum\limits_{k=0}^{N} w_k(x)\varepsilon^k, & x \in \Omega_1\end{cases} \tag{2.52}$$

of the series (2.6)–(2.8). Substituting their into (1.1)–(1.3), we obtain

$$\mathcal{L}^\varepsilon(x,\partial_x)U_N^\varepsilon(x)+\rho(x)\Lambda_N^\varepsilon U_N^\varepsilon(x)=F_N^\varepsilon(x), \qquad x\in\Omega_0\cup\Omega_1; \tag{2.53}$$
$$U_N^\varepsilon(x)=0, \qquad x\in\partial\Omega; \tag{2.54}$$
$$[U_N^\varepsilon(x)]_{\partial\Omega_0}=0, \quad [\sigma^\varepsilon(x,\partial_x)U_N^\varepsilon(x)]_{\partial\Omega_0}=G_N^\varepsilon(x). \tag{2.55}$$

From (2.12) and (2.13), it is follows that (2.54) and the first condition (2.55) are valid.

Let us estimate the remainder terms F_N^ε, G_N^ε. To compute F_N^ε we use the evident equality

$$\sum_{k=0}^N a_k\sum_{k=0}^N b_k=\sum_{k=0}^N\sum_{i=0}^k a_ib_{k-i}+\sum_{k=1}^N\sum_{i=0}^{N-k}a_{k+i}b_{N-i}.$$

Thus for F_N^ε on Ω_0 we have

$$F_N^\varepsilon=\sum_{k=0}^N(\mathcal{L}_0v_k)\varepsilon^k+\rho\varepsilon\sum_{k=0}^N\lambda_k\varepsilon^k\sum_{k=0}^N v_k\varepsilon^k=\sum_{k=0}^N(\mathcal{L}_0v_k)\varepsilon^k+\rho\varepsilon\sum_{k=0}^N\sum_{i=0}^k\lambda_iv_{k-i}\varepsilon^k+$$
$$+\rho\varepsilon\sum_{k=1}^N\sum_{i=0}^{N-k}\lambda_{k+i}v_{N-i}\varepsilon^{k+N}=\mathcal{L}_0v_0+\sum_{k=1}^N\Big(\mathcal{L}_0v_k+\rho\sum_{i=0}^{k-1}\lambda_iv_{k-1-i}\Big)\varepsilon^k+$$
$$+\varepsilon^{N+1}\rho\Big(\sum_{i=0}^N\lambda_iv_{N-i}+\sum_{k=1}^N\sum_{i=0}^{N-k}\lambda_{k+i}v_{N-i}\varepsilon^k\Big)=\varepsilon^{N+1}\rho\Big(\sum_{i=0}^N\lambda_iv_{N-i}+\sum_{k=1}^N\sum_{i=0}^{N-k}\lambda_{k+i}v_{N-i}\varepsilon^k\Big).$$

The last equality being a consequence of (2.9)–(2.10). It is easy to check that similar computation of F_N^ε on Ω_1, with using (2.11), leads to

$$F_N^\varepsilon=\varepsilon^{N+1}\rho\Big(\sum_{k=1}^N\sum_{i=0}^{N-k}\lambda_{k+i}w_{N-i}\varepsilon^k\Big).$$

Thus we see at once that

$$\|F_N^\varepsilon\|_{L_2(\rho,\Omega)}\leqslant C(N)\varepsilon^{N+1}. \tag{2.56}$$

For G_N^ε on $\partial\Omega_0$ we have

$$G_N^\varepsilon=[\sigma^\varepsilon U_N^\varepsilon]_{\partial\Omega_0}=\sigma_0v_0+\sum_{k=1}^N(\sigma_0v_k-\sigma_1w_{k-1})\varepsilon^k+(\sigma_1w_N)\varepsilon^{N+1}=(\sigma_1w_N)\varepsilon^{N+1},$$

where (2.14) and (2.15) were taken into account. Hence

$$\|G_N^\varepsilon\|_{L_2(\partial\Omega_0)}\leqslant C(N)\varepsilon^{N+1}. \tag{2.57}$$

2.4. Justification of formal asymptotics Justification of asymptotic expansions will be divided into 2 steps. We first shall ascertain that a passage to the limit as $\varepsilon\to0$ in (1.1)–(1.3) leads to the problems (2.16)–(2.17) and (2.26)–(2.28) for determination of the principal terms of the asymptotics. Thereupon we shall show that the partial sums (2.52) approximate vrai eigenelements, and this proximity will be estimated.

So let us first prove the lemma on passage to the limit.

Lemma 2.2. *Let λ_ε be an eigenvalue of the problem* (1.1)–(1.3), *and u_ε the corresponding eigenvector normalized in $H_0^1(\Omega)$. Suppose $\varepsilon^{-1}\lambda_\varepsilon \longrightarrow \lambda_* \neq 0$ as $\varepsilon \to 0$, and $u_{\varepsilon'} \to u_*$ weakly in $H_0^1(\Omega)$ as $\varepsilon' \to 0$, where ε' is a subsequence of ε; then the restriction $u_*|_{\Omega_0}$ belongs to $\mathfrak{R}$ while λ_* and the restriction $u_*|_{\Omega_1}$ are the eigenvalue and the corresponding eigenvector of the problem* (2.26)–(2.28).

Proof. For convenience we shall not write the stroke by ε. Under the assumption of the lemma, the pair $(\lambda_\varepsilon,\, u_\varepsilon)$ satisfies the integral identity (1.5) corresponding to (1.1)–(1.3). In terms of previously introduced bilinear forms $\mathfrak{a}_m$, $\mathfrak{b}$, $(m \in \{0,1\})$, it means that

$$\mathfrak{a}_0(u_\varepsilon,\varphi) + \varepsilon\mathfrak{a}_1(u_\varepsilon,\varphi) = \lambda_\varepsilon\mathfrak{b}(u_\varepsilon,\varphi) \qquad \text{for all} \quad \varphi \in H_0^1(\Omega) \tag{2.58}$$

Since $\lambda_\varepsilon \sim \varepsilon\lambda_*$ as $\varepsilon \to 0$, (2.58) shows that

$$|\mathfrak{a}_0(u_\varepsilon,\varphi)| \leqslant C\varepsilon\, \|\varphi\|_{H_0^1(\Omega)}, \qquad |\mathfrak{a}_1(u_\varepsilon,\varphi)| \leqslant C\, \|\varphi\|_{H_0^1(\Omega)}, \tag{2.59}$$

where constants C do not depend on ε. In particular, $\mathfrak{a}_0(u_\varepsilon,u_\varepsilon) \leqslant C\varepsilon$ (recall that u_ε is normalized in $H_0^1(\Omega)$).

Let us denote $u_*|_{\Omega_0} = v_*$, $u_*|_{\Omega_1} = w_*$. The sequence $\{u_\varepsilon\}$ is weakly convergent in $H^1(\Omega_0)$, which gives $\mathfrak{a}_0(u_\varepsilon,\varphi) \to \mathfrak{a}_0(v_*,\varphi)$, and $\mathfrak{a}_1(u_\varepsilon,\varphi) \to \mathfrak{a}_1(w_*,\varphi)$ as $\varepsilon \to 0$. Taking into account this, we conclude from the first inequality (2.59) that $\mathfrak{a}_0(u_\varepsilon,\varphi) \to \mathfrak{a}_0(v_*,\varphi) = 0$ as $\varepsilon \to 0$ for an arbitrary $\varphi \in H^1(\Omega_0)$, hence that $v_* \in \mathfrak{R}$, i.e., $v_* = \Psi\alpha_*$, $\alpha_* \in \mathbb{R}^r$; then $w_* \in \mathcal{H}$, and $\tau(w_*) = \alpha_*$.

Let ψ be a function in $H_0^1(\Omega)$ such that $\psi|_{\Omega_0} = \psi_0 \in \mathfrak{R}$. Note that $\mathfrak{a}_0(u_\varepsilon,\psi_0) = 0$; $\psi|_{\Omega_1} = \psi_1 \in \mathcal{H}$; $\psi_0 = \Psi\tau(\psi_1)$ on Ω_0. Evidently, for such functions, (2.58) takes the form

$$\mathfrak{a}_1(u_\varepsilon,\psi) = \varepsilon^{-1}\lambda_\varepsilon\mathfrak{b}(u_\varepsilon,\psi), \qquad \psi \in H_0^1(\Omega), \quad \psi|_{\Omega_0} \in \mathfrak{R}. \tag{2.60}$$

Since $\|u_\varepsilon\|_{H_0^1(\Omega)} = 1$, it follows that $\{u_\varepsilon\}$ is strongly convergent in $L_2(\rho,\Omega)$, which implies

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathfrak{b}(u_\varepsilon,\psi) &= \mathfrak{b}_1(w_*,\psi_1) + \mathfrak{b}_0(v_*,\psi_0) = \mathfrak{b}_1(w_*,\psi_1) + \mathfrak{b}_0\big(\Psi\alpha_*, \Psi\tau(\psi_1)\big) = \\ &= \int\limits_{\Omega_1} \rho w_* \cdot \psi_1\, dx + \Big\langle \int\limits_{\Omega_0} \rho\Psi^\top\Psi\tau(w_*)\, dx,\ \tau(\psi_1)\Big\rangle_{\mathbb{R}^r} = \\ &= \int\limits_{\Omega_1} \rho w_* \cdot \psi_1\, dx + \big\langle J\tau(w_*),\, \tau(\psi_1)\big\rangle_{\mathbb{R}^r}.\end{aligned}$$

According to what has been said, and recalling that $\varepsilon^{-1}\lambda_\varepsilon \to \lambda_*$, we see that the passage to the limit as $\varepsilon \to 0$ in (2.60) gives

$$\int\limits_{\Omega_1} (\mathfrak{A}_1\nabla w_*, \nabla\psi_1)\, dx = \lambda_*\Big(\int\limits_{\Omega_1} \rho w_* \cdot \psi_1\, dx + \big\langle J\tau(w_*),\, \tau(\psi_1)\big\rangle_{\mathbb{R}^r}\Big), \qquad w_*, \psi_1 \in \mathcal{H}.$$

It is even the integral identity (2.30) corresponding to (2.26)–(2.28). Let us remark that $w_* \not\equiv 0$. Indeed, taking $\varphi = u_\varepsilon$ in (2.58), using the left-hand side of (1.8) and the normalization of u_ε, we have

$$1 = \|u_\varepsilon\|^2_{H_0^1(\Omega)} \leqslant C\varepsilon^{-1}\mathfrak{a}_\varepsilon(u_\varepsilon,u_\varepsilon) = C\varepsilon^{-1}\lambda_\varepsilon\mathfrak{b}(u_\varepsilon,u_\varepsilon).$$

Letting $\varepsilon \to 0$ yields

$$1 \leqslant C\lambda_* \mathfrak{b}(u_*, u_*) = C\lambda_*\big(\mathfrak{b}_1(w_*, w_*) + \langle J\tau(w_*),\, \tau(w_*)\rangle_{\mathbb{R}^r}\big).$$

Hence if it were true that $w_* \equiv 0$, there would be $1 \leqslant 0$, a contradiction. From this we conclude, that the pair $(\lambda_*,\, w_*)$ consists of the eigenvalue and the corresponding eigenvector of (2.26)–(2.28). The lemma is proved. □

For any $N \in \mathbb{N}$ denote by $\widetilde{U}^\varepsilon_N$ the partial sum U^ε_N (see (2.52)) normalized in $H^1_0(\Omega)$. To prove the next theorem, we need the lemma

Lemma 2.3. [16] *Let $A : \mathfrak{H} \longrightarrow \mathfrak{H}$ be a self-adjoint positive compact operator in a Hilbert space $\mathfrak{H}$. Let $(\mu, v) \in \mathbb{R} \times \mathfrak{H}$ be a vector such that $\|Av - \mu v\|_{\mathfrak{H}} \leqslant \delta$, $\|v\|_{\mathfrak{H}} = 1$, $\delta = \mathrm{const} > 0$. Then for any $d > \delta$ there exists a pair $(\lambda, u) \in \mathbb{R} \times \mathfrak{H}$, where λ is an eigenvalue of A and $\|u\|_{\mathfrak{H}} = 1$, such that*

$$|\lambda - \mu| \leqslant \delta, \qquad \|u - v\|_{\mathfrak{H}} \leqslant 2d^{-1}\delta;$$

here u is a linear combination of eigenvectors of A corresponding to eigenvalues in the interval $[\mu - d,\, \mu + d]$.

Theorem 2.4. *Let λ_0 be a prime eigenvalue of the problem* (2.26)–(2.28). *Then there exist an eigenvalue λ_ε of the problem* (1.1)–(1.3) *and the corresponding eigenvector u_ε, normalized in $H^1_0(\Omega)$, such that the estimates*

$$|\lambda_\varepsilon - \Lambda^\varepsilon_N| \leqslant C_1(N)\varepsilon^{N+2}, \qquad \|u_\varepsilon - \widetilde{U}^\varepsilon_N\|_{H^1_0(\Omega)} \leqslant C_2(N)\varepsilon^{N+1} \tag{2.61}$$

are valid; here C_1, C_2 do not depend on ε.

Proof. The problem (1.1)–(1.3) is equivalent to the operator equation (1.9) in $H^1_0(\Omega)$ with scalar product $(\cdot,\, \cdot)_\varepsilon$. Let us take Λ^ε_N and $\widetilde{U}^\varepsilon_N$ as an almost proper value and vector of A_ε, in the sense of Lemma 2.3. Note that $\widetilde{U}^\varepsilon_N = (C_0 + O(\varepsilon))U^\varepsilon_N$, where C_0 does not depend on ε. Recall that $(A_\varepsilon u,\, \varphi)_\varepsilon = (u,\, \varphi)_{L_2(\rho,\Omega)}$, and $(u,\, \varphi)_\varepsilon = \mathfrak{a}_\varepsilon(u,\, \varphi)$; then on account of just now said and the inequality $(\cdot,\, \cdot)_{H^1_0(\Omega)} \leqslant C_1\varepsilon^{-1}(\cdot,\, \cdot)_\varepsilon$ (see (1.8)), we have

$$\begin{gathered}\big|\big(\Lambda^\varepsilon_N A_\varepsilon \widetilde{U}^\varepsilon_N - \widetilde{U}^\varepsilon_N,\, \varphi\big)_{H^1_0(\Omega)}\big| \leqslant C_2\varepsilon^{-1}\big|\big(\Lambda^\varepsilon_N A_\varepsilon U^\varepsilon_N - U^\varepsilon_N,\, \varphi\big)_\varepsilon\big| = \\ = C_2\varepsilon^{-1}\big|\Lambda^\varepsilon_N\big(A_\varepsilon U^\varepsilon_N,\, \varphi\big)_\varepsilon - \big(U^\varepsilon_N,\, \varphi\big)_\varepsilon\big| = C_3\varepsilon^{-1}\big|\Lambda^\varepsilon_N\big(U^\varepsilon_N,\, \varphi\big)_{L_2(\rho,\Omega)} - \mathfrak{a}^\varepsilon\big(U^\varepsilon_N,\, \varphi\big)\big|\end{gathered} \tag{2.62}$$

for any $\varphi \in H^1_0(\Omega)$. Applying the Betti formula (1.6), with $v = \varphi$, to (2.53), and using (2.54),(2.55), we obtain that

$$\Lambda^\varepsilon_N\big(U^\varepsilon_N,\, \varphi\big)_{L_2(\rho,\Omega)} - \mathfrak{a}^\varepsilon\big(U^\varepsilon_N,\, \varphi\big) = \int\limits_\Omega F^\varepsilon_N \cdot \varphi\, dx - \int\limits_{\partial\Omega_{\Omega_0}} G^\varepsilon_N \cdot \varphi\, dx.$$

According to (2.56),(2.57) and continuity of a trace operator, we conclude that the right-hand side of the last equality is estimated by $C_3(N)\varepsilon^{N+1}\,\|\varphi\|_{H^1_0(\Omega)}$; therefore (2.62) now becomes

$$\big|\big(\Lambda^\varepsilon_N A_\varepsilon \widetilde{U}^\varepsilon_N - \widetilde{U}^\varepsilon_N,\, \varphi\big)_{H^1_0(\Omega)}\big| \leqslant C_4(N)\varepsilon^N\,\|\varphi\|_{H^1_0(\Omega)} \quad \forall \varphi \in H^1_0(\Omega).$$

Hence, taking into account $\Lambda_N^\varepsilon = \lambda_0\varepsilon + O(\varepsilon^2)$, we have

$$\|A_\varepsilon \widetilde{U}_N^\varepsilon - (\Lambda_N^\varepsilon)^{-1}\widetilde{U}_N^\varepsilon\|_{H_0^1(\Omega)} \leqslant C_5(N)\varepsilon^{N-1}.$$

By Lemma 2.3, there exists an eigenvalue λ_ε of (1.1)–(1.3) such that

$$\left|\lambda_\varepsilon^{-1} - (\Lambda_N^\varepsilon)^{-1}\right| \leqslant C_6(N)\varepsilon^{N-1}.$$

Since λ_ε, Λ_N^ε are of order ε, we have

$$\left|\lambda_\varepsilon - \Lambda_N^\varepsilon\right| \leqslant C_7(N)\varepsilon^{N+1}.$$

Applying the last estimate to $\Lambda_{N+1}^\varepsilon$, we obtain

$$\left|\lambda_\varepsilon - \Lambda_N^\varepsilon\right| = \left|\lambda_\varepsilon - \Lambda_{N+1}^\varepsilon + \varepsilon^{N+2}\lambda_{N+1}\right| \leqslant \left|\lambda_\varepsilon - \Lambda_{N+2}^\varepsilon\right| + \varepsilon^{N+2}|\lambda_{N+1}| \leqslant C_8(N)\varepsilon^{N+1}.$$

Hence the first estimation (2.61) is established.

Suppose λ_ε is an eigenvalue of (1.1)–(1.3) such that $\varepsilon^{-1}\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$. By $U_{\lambda_0}(\varepsilon)$ denote a linear span of the eigenvectors corresponding to the all such eigenvalues. Let us take $\tilde{d} > 0$ such that the interval $[\lambda_0^{-1} - \tilde{d};\ \lambda_0^{-1} + \tilde{d}]$ contains no points of the spectrum of operator B (see (2.32)) distinct from λ_0^{-1}. Hence, by sufficiently small values of ε, the interval $[(\Lambda_N^\varepsilon)^{-1} - \varepsilon^{-1}\tilde{d};\ (\Lambda_N^\varepsilon)^{-1} + \varepsilon^{-1}\tilde{d}]$ contains only those eigenvalues λ_ε such that $\varepsilon^{-1}\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$. Set $d = \varepsilon^{-1}\tilde{d}$, where d is the number from Lemma 2.3. By this lemma there exists a vector $u_\varepsilon \in U_{\lambda_0}(\varepsilon)$, $\|u_\varepsilon\|_{H_0^1(\Omega)} = 1$, such that

$$\|u_\varepsilon - \widetilde{U}_N^\varepsilon\|_{H_0^1(\Omega)} \leqslant C_9(N)\varepsilon^N.$$

Using of the last estimate for $\widetilde{U}_{N+1}^\varepsilon$, we have

$$\begin{aligned}\|u_\varepsilon - \widetilde{U}_N^\varepsilon\|_{H_0^1(\Omega)} &\leqslant \|u_\varepsilon - \widetilde{U}_{N+1}^\varepsilon\|_{H_0^1(\Omega)} + \|\widetilde{U}_{N+1}^\varepsilon - \widetilde{U}_N^\varepsilon\|_{H_0^1(\Omega)} \leqslant \\ &\leqslant C_{10}(N)\varepsilon^{N+1} + \|\widetilde{U}_{N+1}^\varepsilon - \widetilde{U}_N^\varepsilon\|_{H_0^1(\Omega)}.\end{aligned}$$

Easy computation shows that the last summand can be estimated by $C_{11}\varepsilon^{N+1}$, where C_{11} does not depend on ε; hence the second estimate (2.61) is valid.

It remains to proof that $\dim U_{\lambda_0}(\varepsilon) = 1$. For this purpose we show that there exists a unit eigenvalue λ_ε of (1.1)–(1.3) satisfying the first estimate (2.61), and such that $\varepsilon^{-1}\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$, We assume the contrary, and let the same is true for two different eigenvalues λ_ε^1, λ_ε^2. Let $u_\varepsilon^{(1)}$, $u_\varepsilon^{(2)}$ be the corresponding normalized in $H_0^1(\Omega)$ eigenvectors. They are orthogonal in $L_2(\rho,\Omega)$, i.e.,

$$\left(u_\varepsilon^{(1)}, u_\varepsilon^{(2)}\right)_{L_2(\rho,\Omega)} = 0, \quad \left\|u_\varepsilon^{(1)}\right\|_{H_0^1(\Omega)} = \left\|u_\varepsilon^{(2)}\right\|_{H_0^1(\Omega)} = 1.$$

Since $\left\|u_\varepsilon^{(1)}\right\|_{H_0^1(\Omega)} = 1$ than there exists a subsequence $\varepsilon' \to 0$ of ε such that $u_{\varepsilon'}^{(1)} \to u^{(1)}$ weakly in $H_0^1(\Omega)$. The same is true for $u_{\varepsilon'}^{(2)}$. Namely, there exists a subsequence $\varepsilon'' \to 0$ of ε' such that $u_{\varepsilon''}^{(2)} \to u^{(2)}$ weakly in $H_0^1(\Omega)$. Hence $u_{\varepsilon''}^{(k)} \to u^{(k)}$ strongly in $L_2(\rho,\Omega)$ as $\varepsilon'' \to 0$ ($k \in \{1,2\}$), and

$$0 = \left(u_{\varepsilon''}^{(1)}, u_{\varepsilon''}^{(2)}\right)_{L_2(\rho,\Omega)} \xrightarrow[\varepsilon''\to 0]{} \left(u^{(1)}, u^{(2)}\right)_{L_2(\rho,\Omega)} = 0. \tag{2.63}$$

By Lemma 2.2 the restrictions $u^{(1)}|_{\Omega_1} = w^{(1)}$, $u^{(2)}|_{\Omega_1} = w^{(2)}$ belong to $\mathcal{H}$, and they are eigenvectors of (2.26)–(2.28) corresponding to the eigenvalue λ_0. Moreover, $w^{(1)} \not\equiv w^{(2)}$, otherwise, since

$$\left(u^{(1)}, u^{(2)}\right)_{L_2(\rho,\Omega)} = \left(w^{(1)}, w^{(2)}\right)_{L_2(\rho,\Omega_1)} + \left\langle J\tau(w^{(1)}), \tau(w^{(2)})\right\rangle_{\mathbb{R}^r},$$

the right-hand equality in (2.63) would not be valid. On the other hand, the fact that $w^{(1)} \not\equiv w^{(2)}$ contradicts the primality of λ_0. The theorem is proved. □

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