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KALEIDOSCOPIC GRAPHS

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Let $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ be a connected graph with a set of vertices $\mathcal{V}$ and a set of edges E, $B(x,1) = \{y \in \mathcal{V} : (x,y) \in E\} \cup \{x\}$, $x \in \mathcal{V}$. A graph $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ is called kaleidoscopic if there exist a natural number $s > 1$ and a $(s+1)$-coloring of $\mathcal{V}$ such that $|B(x,1)| = s+1$ and $B(x,1)$ contains the vertices of all colors for every $x \in \mathcal{V}$. We present two methods for construction of kaleidoscopic graphs based on the Cayley graphs of the groups.

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Пусть $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ — связный граф с множеством вершин $\mathcal{V}$ и множеством ребер E, $B(x,1) = \{y \in \mathcal{V} : (x,y) \in E\} \cup \{x\}$, $x \in \mathcal{V}$. Граф $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ называется калейдоскопическим, если существует натуральное число $s > 1$ и $(s+1)$-раскрашивание множества $\mathcal{V}$, такие, что для каждой вершины x множество $B(x,1)$ имеет мощность $s+1$ и содержит вершины всех цветов раскраски графа G_r Предложены два метода построения калейдоскопических графов, основанные на графах Кэли групп.

A problem under consideration arises from the following combinatorial concept of resolvability. Let X be a set and let $\mathcal{F}$ be a family of subsets of X. A subset A of X is called *$\mathcal{F}$-dense* if $A \cap F \neq \varnothing$ for every $F \in \mathcal{F}$. Given a cardinal α, we say that X is *α-resolvable* with respect to $\mathcal{F}$ if X can be partitioned into α $\mathcal{F}$-dense subsets. Put $\Delta(\mathcal{F}) = \min\{|F| : F \in \mathcal{F}\}$ and say that X is *maximally resolvable* with respect to $\mathcal{F}$ if X is $\Delta(\mathcal{F})$-resolvable with respect to $\mathcal{F}$. For resolvability of topological spaces and groups see the surveys [1,2,3]. Let $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ be a connected graph with the set of vertices $\mathcal{V}$ and the set of edges E. We suppose that Gr has no loops and multiple edges. Given any $x, y \in \mathcal{V}$, denote by $d(x,y)$ the length of the shortest path between x and y. For $x \in \mathcal{V}$ and nonnegative integer m, put $B(x,m) = \{y \in \mathcal{V} : d(x,y) \leq m\}$. For a natural number $s \geq 2$, a connected graph $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ is called *homogeneous of degree* s if $|B(x,1)| = s+1$ for each $x \in \mathcal{V}$. By [4, Theorems 1,6], for every connected graph $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ and every natural number r, $|\mathcal{V}| \geq r$, the set $\mathcal{V}$ is r-resolvable with respect to the family $\{B(x, r-1) : x \in \mathcal{V}\}$. In particular, if $|\mathcal{V}| \geq 2$, then $\mathcal{V}$ is 2-resolvable with respect to the family $\{B(x,1) : x \in \mathcal{V}\}$. On the other hand [4, Example 2], for every natural number m, there exists a finite connected graph $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ such that $|B(x,1)| \geq m$ for every $x \in \mathcal{V}$, but $\mathcal{V}$ is not 3-resolvable with respect to the family $\{B(x,1) : x \in \mathcal{V}\}$. In this paper we introduce and investigate a special class of graphs $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ such that the set $\mathcal{V}$ is maximally resolvable with respect to the family of unit balls.

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Definition 1. A homogeneous graph $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ of degree s is called *kaleidoscopic* if there exists a $(s+1)$-coloring χ of $\mathcal{V}$ such that $|\chi(B(x,1))| = s+1$ for every $x \in \mathcal{V}$. In this case χ is called a kaleidoscopic coloring. Equivalently, a $(s+1)$-coloring χ of $\mathcal{V}$ is kaleidoscopic if there are no distinct monochrome vertices in each unit ball of the graph.

We begin with elementary properties and examples of kaleidoscopic graphs.

Lemma 1. *Let* $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ *be a finite kaleidoscopic graph,* $|\nu| = n$*, of degree* s *and let*

$$\chi \colon \mathcal{V} \longrightarrow \{0, 1, \dots, s\}$$

be a kaleidoscopic coloring. Then the following statements holds

(i) $(s+1)|n$;

(ii) $|\chi^{-1}(0)| = |\chi^{-1}(1)| = \dots = |\chi^{-1}(s)|$.

Proof. If $i \in \{0, 1, \dots, s\}$, then $\{B(x,1) : x \in \chi^{-1}(i)\}$ is a partition of $\mathcal{V}$. Since

$$|B(x,1) \cap \chi^{-1}(i)| = 1,$$

$(s+1)|\chi^{-1}(i)| = |\mathcal{V}|$. □

Example 1. Let $\mathrm{Gr_n} = (\mathcal{V}_\mathrm{n}, \mathrm{E_n})$ be a cyclicgraph with $n > 2$ vertices. Suppose that $\mathrm{Gr_n}$ is kaleidoscopic. Since $\mathrm{Gr_n}$ is a homogeneous graph of degree 2, we have $3|n$ by Lemma 1(i). On the other hand, if $3|n$, then any periodical 3-coloring of $\mathcal{V}$ is kaleidoscopic.

Example 2. Among five Plato polyhedran (considered as graphs) the only kaleidoscopic graphs are simplex, cube and icosahedron. Every 4-coloring of the simplex is kaleidoscopic. Fix any vertex x of cube and any 4-coloring of $B(x,1)$. Then, for every $y \in B(x,1)$, color the symmetric to y (with respect to the center of cube) point y' in the color of vertex y. The same coloring is kaleidoscopic also for the icosahedron. Since the octahedron has 6 vertices and degree $s = 4$, it is not kaleidoscopic by Lemma 1(i). At last, take a pentagon which is a facet of dodecahedron. Note that dodecahedron is a homogeneous graph of degree 3. Under every 4-coloring of the set of vertices of dodecahedron there exist two monochrome vertices on the facet. Hence, some unit ball with the centre in the vertex of this facet has two distinct monochrome points.

The following two lemmas will help us to analyze the structure of some kaleidoscopic graphs.

Lemma 2. *Let* $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ *be a finite homogeneous graph of degree* s*,* $|\mathcal{V}| = 2m$ *and let* $X \subseteq \mathcal{V}$*. Suppose that there exist a partition* $\mathcal{V} = \mathcal{V}(0) \cup \mathcal{V}(1)$*,* $|\mathcal{V}(0)| = |\mathcal{V}(1)| = m$ *and natural numbers* p*,* q *such that the following conditions hold:*

(i) $\bigcup\{B(x,1) : x \in X\} = \mathcal{V}$;

(ii) $(s+1)|X| = 2m$;

(iii) $s+1 = p+q, \quad p > q$;

(iv) if $x \in \mathcal{V}(i), \quad i \in \{0,1\}$ *then* $|B(x,1) \cap \mathcal{V}(i)| = p, \quad |B(x,1) \cap (\mathcal{V} \setminus \mathcal{V}(i))| = q$.

Then $|X \cap \mathcal{V}(0)| = |X \cap \mathcal{V}(1)|$.

Proof. Put $k_0 = |X \cap \mathcal{V}(0)|$, $k_1 = |X \cap \mathcal{V}(1)|$. By (i) and (iv), the following inequalities hold

$$pk_0 + qk_1 \geq m, qk_0 + pk_1 \geq m.$$

Adding these inequalities, we get $p|X| + q|X| \geq 2m$. By (ii) and (iii), $(p+q)|X| = 2m$. Hence, k_0, k_1 satisfy the following system of linear equations

$$px_0 + qx_1 = m, qx_0 + px_1 = m.$$

Since $p > q$, this system has only one solution. On the other hand, the system has an evident solution $x_0 = x_1 = \frac{m}{p+q}$. □

Lemma 3. *Let* $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ *be a finite homogeneous graph of degree* s, $|\mathcal{V}| = nm$, n, m *are natural numbers* ≥ 2 *and let* $X \subseteq \mathcal{V}$. *Suppose that there exist a partition*

$$\mathcal{V} = \mathcal{V}(0) \cup \mathcal{V}(1) \cup \ldots \cup \mathcal{V}(n-1), \quad |\mathcal{V}(0)| = |\mathcal{V}(1)| = \ldots = |\mathcal{V}(n-1)| = m$$

and natural numbers p, q *such that the following conditions are satisfied:*

(i) $\cup\{B(x,1) : x \in X\} = \mathcal{V}$;

(ii) $(s+1)|X| = nm$;

(iii) $s+1 = p+2q, \quad p > 2q$;

(iv) if $x \in \mathcal{V}(i)$, $i \in \{0, 1, \ldots, n-1\}$ *then*

$$B(x,1) \subseteq \mathcal{V}((i-1) \bmod n) \cup \mathcal{V}(i \bmod n) \cup \mathcal{V}((i+1) \bmod n),$$

$$|B(x,1) \cap \mathcal{V}(i \bmod n)| = p, \quad |B(x,1) \cap \mathcal{V}((i-1) \bmod n| = |B(x,1) \cap \mathcal{V}((i+1) \bmod n)| = q.$$

Then $|X \cap \mathcal{V}(0)| = |X \cap \mathcal{V}(1)| = \ldots = |X \cap \mathcal{V}(n-1)|$.

Proof. Put $k_i = |X \cap \mathcal{V}(i)|$, $i \in \{0, 1, \ldots, n-1\}$. By (i) and (iv), the following inequalities hold

$$pk_0 + qk_{n-1} + qk_1 \geq m,$$
$$pk_1 + qk_0 + qk_2 \geq m.$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$pk_{n-2} + qk_{n-3} + qk_{n-1} \geq m,$$
$$pk_{n-1} + qk_{n-2} + qk_0 \geq m.$$

Adding all these inequalities, we get $p|X| + 2q|X| \geq nm$. By (ii) and (iii), $(p+2q)|X| = nm$. It follows that the numbers $k_0, k_1, \ldots, k_{n-1}$ satisfy the following system of linear equations

$$\begin{pmatrix} p & q & 0 & 0 & . & . & . & 0 & 0 & q \\ q & p & q & 0 & . & . & . & 0 & 0 & 0 \\ 0 & q & p & q & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & q & p & q \\ q & 0 & 0 & 0 & . & . & . & 0 & q & p \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \ldots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} m \\ m \\ m \\ \ldots \\ m \\ m \end{pmatrix}.$$

Denote by $\triangle$ the determinant of the matrix of this system and put $f(x) = p + qx + qx^{n-1}$. It is a standard fact that $\triangle = f(\varepsilon_0)\ f(\varepsilon_1) \cdots f(\varepsilon_{n-1})$, where $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$ are complex roots of the equation $z^n = 1$. Since $p > 2q$ then $f(\varepsilon_i) \neq 0$ for each $i \in \{0, 1, \ldots, n-1\}$. Hence, this system has only a unique solution. On the other hand, this system has the evident solution $x_0 = x_1 = \ldots = x_{n-1} = \frac{m}{p+2q}$. □

In order to apply Lemmas 2, 3 we remind definition of the Cayley graph of a group. Given any subset A of a group G, denote by $\langle A\rangle$ the smallest subgroup of G containing A. Let G be a group with the identity e, $S\subseteq G$, $e\in S$, $S=S^{-1}$ and $G=\langle S\rangle$. The *Cayley graph* of a group G determined by the system of generators S is a graph $\mathrm{Cay(G,S)}=(\mathrm{G,E})$, where $x,y\in E$ if and only if $x\neq y$ and $x^{-1}y\in S$. If S is finite and $|S|-1=s$, $s\geq 2$, then $\mathrm{Cay(G,S)}$ is a homogeneous graph of degree s.

Example 3. Let $G=\langle a\rangle\times\langle b\rangle$, $\langle a\rangle\simeq\mathbb{Z}_2$, $\langle b\rangle\simeq\mathbb{Z}_m$, $m>2$, $S=\{a,b,b^{-1},e\}$. Geometrically, the graph $\mathrm{Cay(G,S)}$ is a prism with $2m$ vertices. Suppose that $\mathrm{Cay(G,S)}$ is kaleidoscopic and show that $4|m$. In what follows (Example 6) we shall prove that $\mathrm{Cay(G,S)}$ is kaleidoscopic provided that $4|m$. To apply Lemma 2, put $s=3$, $p=3$, $q=1$, $\mathcal{V}=G$, $\langle a\rangle=\{0,1\}$, $\mathcal{V}(0)=\{(0,x):\ x\in\ \langle b\rangle\}$, $\mathcal{V}(1)=\{(1,x): x\in\langle b\rangle\}$. Fix a kaleidoscopic coloring $\chi\colon\mathcal{V}\longrightarrow\{0,1,2,3\}$ and put $X_i=\chi^{-1}(i)$, $i\in\{0,1,2,3\}$. By Lemma 2, $|X_i\cap\mathcal{V}(0)|=|X_i\cap\mathcal{V}(1)|$ for every $i\in\{0,1,2,3\}$. By Lemma 1(ii), $|X_i|=m/2$. Since $\mathcal{V}(0)=(X_0\cap\mathcal{V}(0))\cup(X_1\cap\mathcal{V}(0))\cup(X_2\cap\mathcal{V}(0))\cup(X_3\cap\mathcal{V}(0))$, we have $4|m$.

Example 4. Let $G=\langle a\rangle\times\langle b\rangle$, $\langle a\rangle\simeq\mathbb{Z}_n$, $\langle b\rangle\simeq\mathbb{Z}_m$, $n,m>2$, $S=\{a,a^{-1},b,b^{-1},e\}$. Suppose that $\mathrm{Cay(G,S)}$ is kaleidoscopic and show that $5|m$, $5|n$. To apply Lemma 3, put $s=4$, $p=3$, $q=1$, $\mathcal{V}=G$, $\langle a\rangle=\{0,1,\dots,n-1\}$, $\mathcal{V}(i)=\{(i,x):x\in\langle b\rangle\}$, $i\in\{0,1,\dots,n-1\}$. Fix a kaleidoscopic coloring $\chi:\mathcal{V}\longrightarrow\{0,1,2,3,4\}$ and put $X_j=\chi^{-1}(j)$, $j\in\{0,1,2,3,4\}$. By Lemma 3,

$$|X_j\cap\mathcal{V}(0)|=|X_j\cap\mathcal{V}(1)|=\ldots=|X_j\cap\mathcal{V}(n-1)|$$

for every $j\in\{0,1,2,3,4\}$. By Lemma 1(ii), $|X_j|=nm/5$, $j\in\{0,1,2,3,4\}$. Since $\mathcal{V}(0)=\cup_{j=0}^{4}(X_j\cap\mathcal{V}(0))$, we have $5|m$. Replacing n and m, we obtain $5|n$.

Our first method of construction of kaleidoscopic graphs is based on the next definition.

Definition 2. Let G be a group with the identity e and let $X,Y\subseteq G$. We say that (X,Y) is a *kaleidoscopic pair* of G provided that X is finite and the following conditions hold

(i) $e\in X$, $X=X^{-1}$ and $G=\langle X\rangle$;

(ii) $e\in Y$ and $G=XY$;

(iii) $x^{-1}XXx\cap YY^{-1}=\{e\}$ for every $x\in X$.

By (i) and (iii), $XX\cap YY^{-1}=\{e\}$. In view of (ii) this observation implies that, for every element $g\in G$, there exist $x\in X$, $y\in Y$ such that $g=xy$ and this representation is unique. For every kaleidoscopic pair (X,Y) of a group G define the *standard coloring* $\chi\colon G\longrightarrow X$ by the following rule. For every $x\in X$, put $\chi(x)=x$. Take any element $g\in G$ and pick $x\in X$, $y\in Y$ with $g=xy$. Put $\chi(g)=x$. Show that a standard coloring is kaleidoscopic. Assume that $g_1,g_2,g\in G$, $g_1,g_2\in B(g,1)$ and $\chi(g_1)=\chi(g_2)$. Choose $x_1,x_2\in X$ and $y_1,y_2\in Y$ such that $g_1=x_1y_1$, $g_2=x_2g_2$. Since $\chi(g_1)=x_1$, $\chi(g_2)=x_2$ and $\chi(g_1)=\chi(g_2)$, we have $x_1=x_2$. Since $g_1,g_2\in B(g,1)$, there exist $z_1,z_2\in X$ such that $g_1=z_1g$, $g_2=z_2g$. Thus, $x_1y_1=z_1g$, $x_1y_2=z_2g$ and $z_1^{-1}x_1y_1=z_2^{-1}x_1y_2$. It follows that $x_1^{-1}z_2z_1^{-1}x_1=y_2y_1^{-1}$. By Definition 2(iii), $x_1^{-1}z_2z_1^{-1}x_1=e$, so $z_1=z_2$ and $g_1=g_2$. Thus, we have proved the following statement.

Theorem 1. *If (X,Y) is a kaleidoscopic pair of a group G, then* $\mathrm{Cay(G,X)}$ *is a kaleidoscopic graph.*

Definition 3. A kaleidoscopic pair (X,Y) of a group G is called a *Hamming pair* provided that Y is a subgroup of G. In this case a kaleidoscopic graph $\mathrm{Cay(G,X)}$ is called a *Hamming graph.*

A motivation of this definition is given in Example 5.

Theorem 2. *Let (X,Y) be a kaleidoscopic pair of a group G and let χ be the standard coloring of G. Then the following statements are equivalent:*

(i) (X,Y) is a Hamming pair;

(ii) if $g_1, g_2 \in G,\ x \in X$ and $\chi(g_1) = \chi(g_2)$, then $\chi(xg_1) = \chi(xg_2)$.

Proof. $(i) \Rightarrow (ii)$. Choose $y_1, y_2 \in Y$ and $a \in X$ such that $g_1 = ay_1$, $g_2 = ay_2$. Pick $z_1, z_2 \in Y$ and $b_1, b_2 \in X$ such that $xg_1 = b_1z_1$, $xg_2 = b_2z_2$. Then $z_1 = b_1^{-1}xay_1$, $z_2 = b_2^{-1}xay_2$. Since Y is a subgroup of G, $b_1^{-1}xa \in Y$, $b_2^{-1}xa \in Y$, so $b_1^{-1}b_2 \in Y$. By Definition 2(iii) $b_1 = b_2$. Hence, $\chi(xg_1) = \chi(xg_2)$.

$(ii) \Rightarrow (i)$. Let $y_1, y_2 \in Y$. Then $\chi(y_1) = \chi(y_2) = \chi(e) = e$. By (ii), $\chi(y_1y_2) = \chi(y_1e) = e$, $\chi(y_1^{-1}y_1) = \chi(e) = \chi(y_1^{-1}e) = \chi(y_1^{-1})$. Hence, $y_1y_2 \in Y$, $y_1^{-1} \in Y$. □

Example 5. Let $G = \langle a_1\rangle\times\langle a_2\rangle\times\ldots\times\langle a_n\rangle$, $\langle a_i\rangle \simeq \mathbb{Z}_2, i \in \{1,2,\ldots,n\}$, $S = \{e, a_1, a_2, \ldots, a_n\}$. Suppose that a cube $\mathrm{Cay(G,S)}$ is a kaleidoscopic graph. By Lemma 1(i), $n+1|2^n$. On the other hand, if $n+1|2^n$, then there exists a subgroup H of G such that $\{Sh : h \in H\}$ is a partition of G [5, Section 87]. The subgroup H is called a Hamming code. Clearly, (S,H) is a Hamming pair, so $\mathrm{Cay(G,S)}$ is a Hamming graph.

Example 6. Let $G = \langle a\rangle \times \langle b\rangle$, $\langle a\rangle \simeq \mathbb{Z}_2$, $\langle b\rangle \simeq \mathbb{Z}_m$, $m > 2$. Suppose that $4|m$ and show that $\mathrm{Cay(G,S)}$ is a Hamming graph, where $S = \{e, a, b, b^{-1}\}$. Put $H = \langle ab^2\rangle$. Then $H = \langle b^4\rangle \cup \{a^kb^{2k} : k \in \{1,3,\ldots,m-1\}\}$. Hence, the pair (S,H) satisfies Definition 3 with $X = S, Y = H$.

Example 7. Let $G = \langle a\rangle\times\langle b\rangle$, $\langle a\rangle \simeq \mathbb{Z}_5$, $\langle b\rangle \backsimeq \mathbb{Z}_5$, $S = \{e, a, a^{-1}, b, b^{-1}\}$. Show that $\mathrm{Cay(G,S)}$ is a Hamming graph. Put $H = \langle a^3b\rangle$. Then $H = \{a^3b, ab^2, a^4b^3, a^2b^4\}$ and $S^{-1}S \cap H = \{e\}$, $SH = G$.

Example 8. Let $G = \langle a\rangle \times \langle b\rangle$, $\langle a\rangle \simeq \mathbb{Z}$, $\langle b\rangle \backsimeq \mathbb{Z}$, $S = \{e, a, a^{-1}, b, b^{-1}\}$. Show that $\mathrm{Cay(G,S)}$ is a Hamming graph. Note that

$$SS = \{e, a, a^{-1}, b, b^{-1}, a^2, b^2, a^{-2}, b^{-2}, ab, a^{-1}b^{-1}, a^{-1}b, ab^{-1}\}$$

and put $H = \langle a^2b^2, a^{-2}b^2\rangle$. A routine verification shows that $SH = G$ and $SS \cap H = \{e\}$. Note that this graph can be interpreted as a square mosaic on the plane. It is easy to find kaleidoscopic colorings for triangle and hexagonal mosaics on the plane. Now we describe another way of construction of kaleidoscopic graphs from the Cayley graphs $\mathrm{Cay(G,S)}$ of groups under some restrains on S.

Step 1. Let G be a group with the identity e and let S be a finite subset of G such that $e \notin S$, $S = S^{-1}$ and $G = \langle S\rangle$. Suppose that $|S| = r(r-1)$ for some natural number $r > 1$ and S has no elements of order 2. Write $S = S_1 \cup S_2 \cup \cdots \cup S_r$ so that $|S_i| = r-1$, $i \in \{1,2,\ldots,r\}$ and $|S_i^{-1} \cap S_j| = 1$ for all distinct indexes $i,j \in \{1,2,\ldots,r\}$.

Step 2. We may suppose that G is linearly ordered, so every edge of $\mathrm{Cay(G,S)}$ is a pair (x,y), $x < y$. Let N be a cardinality of the set of edges of $\mathrm{Cay(G,S)}$. Take a set W of cardinality $2N$, $W \cap G = \varnothing$ and construct an auxiliary graph $\mathrm{Gr} = (\mathrm{G \cup W, E})$. For every edge (x,y) of $\mathrm{Cay(G,S)}$, take distinct elements $z,t \in W$ and replace the edge (x,y) by the path x,z,t,y. List the edges (x,z), (z,t), (t,y) to E. We suppose that if (x,y) and (x',y') are distinct edges of $\mathrm{Cay(G,S)}$ and (x,y) (x',y') are replaced by x,z,t,y and x',z',t',y', respectively, then $\{z,t\} \cap \{z',t'\} = \varnothing$.

Step 3. Define a coloring $\chi\colon W\cup G\longrightarrow\{0,1,2,\dots,r\}$ by the following rule. Put $\chi(y)=0$ for every $g\in G$. Let $x,y\in G$ and (x,y) is be edge of Cay(G, S), so $x^{-1}y=s$, $s\in S$. Suppose that (x,y) is replaced by the path x,z,t,y. Choose $i,j\in\{1,2,\dots,r\}$ such that $s\in S_i$, $s^{-1}\in S_j$. Put $\chi(z)=i$, $\chi(t)=j$.

Step 4. For every $x\in G$ and every $i\in\{1,2,\dots,r\}$ there are exactly $r-1$ vertices $z_1,\dots,z_{r-1}\in W$ with $(x,z_1),\dots,(x,z_{r-1})\in E$ and $\chi(z_1)=\dots=\chi(z_{r-1})=i$. Stick together these vertices and identify the edges $(x,z_1),\dots,(x,z_{r-1})$. After this factorization we obtain kaleidoscopic graphs.

We finish this paper with constructions of free kaleidoscopic graphs, semigroups and groups.

Definition 4. Let $\mathrm{Gr}_1=(\mathcal{V}_1,\mathrm{E}_1)$, $\mathrm{Gr}_2=(\mathcal{V}_2,\mathrm{E}_2)$ be kaleidoscopic graphs of degree $s>1$ and let $\chi_1\colon\mathcal{V}_1\longrightarrow\{0,1,\dots,s\}$, $\chi_2\colon\mathcal{V}_2\longrightarrow\{0,1,\dots,s\}$ be kaleidoscopic colorings. A mapping f from $\mathcal{V}_1$ onto $\mathcal{V}_2$ is called a *kaleidoscopic homomorphism* if the following conditions hold

(i) $\chi_1(x)=\chi_2(f(x))$ for every $x\in\mathcal{V}_1$;

(ii) if $(x,y)\in E_1$, then $(f(x),f(y))\in E_2$.

Definition 5. A tree $Tr=(\mathcal{V},E)$ of degree $s>1$ with a fixed kaleidoscopic coloring $\chi\colon\mathcal{V}\longrightarrow\{0,1,\dots,s\}$ is called a *free kaleidoscopic tree* of degree s.

Theorem 3. *Let $Tr=(\mathcal{V},E)$ be a free kaleidoscopic tree of degree s with a kaleidoscopic coloring $\chi\colon\mathcal{V}\longrightarrow\{0,1,\dots,s\}$. Let $\mathrm{Gr}_1=(\mathcal{V}_1,\mathrm{E}_1)$ be any kaleidoscopic graph of degree s with a kaleidoscopic coloring $\chi_1\colon\mathcal{V}_1\longrightarrow\{0,1,\dots,s\}$. Then there exists a kaleidoscopic homomorphism $f\colon\mathcal{V}\longrightarrow\mathcal{V}_1$.*

Proof. Take any vertices $x\in\mathcal{V}$, $y\in\mathcal{V}_1$ with $\chi(x)=\chi_1(y)$ and put $f(x)=y$. For every nonnegative integer m, denote $S_m(x)=\{z\in\mathcal{V}:d(x,z)=m\}$. Suppose that we have extended f onto $S_0(x)\cup\ldots\cup S_m(x)$. Take any vertex $z\in S_{m+1}(x)$ and choose $z'\in S_m(x)$ such that $d(z,z')=1$. Pick $t\in B(f(z'),1)$ such that $\chi(z')=\chi_1(t)$ and put $f(z)=t$. By the construction, f is a kaleidoscopic homomorphism. □

Definition 6. Let $s>1$ be a natural number and let $X=\{0,1,\dots,s\}$. A *free kaleidoscopic semigroup* $KS(X)$ is a semigroup in the alphabet X determined by the relations of the form $xx=x$, $xyx=x$ for all $x,y\in X$. We identify $KS(X)$ with the set of all nonempty words in the alphabet X without factors xx, xyx, $x,y\in X$.

Show that $KS(X)$ acts transitively on the set of vertices of every kaleidoscopic graph $\mathrm{Gr}=(\mathcal{V},\mathrm{E})$ of degree s with kaleidoscopic coloring $\chi\colon\mathcal{V}\longrightarrow\{0,1,\dots,s\}$. Take any $x\in X$ and $v\in\mathcal{V}$. Choose $u\in B(v,1)$ such that $\chi(u)=x$ and put $x(v)=u$. Then extend this action inductively from the set of letters X onto $KS(X)$ by the following rule. If $w\in KS(X)$, $w=xw_1$, $w_1\in KS(X)$ and $v\in\mathcal{V}$, put $w(v)=x(w_1(v))$. Note that the colors of the vertices of the shortest path between any two vertices $v_1,v_2\in\mathcal{V}$ determine the word $w\in KS(X)$ such that $w(v_1)=v_2$, so this action is transitive.

Definition 7. For every $x\in X$, the subset $KG(X,x)$ of all words from $KS(X)$ with the first letter x and the last letter x is a subgroup of $KS(X)$ with the identity x. To obtain the inverse element to the word $W\in KG(X,x)$ write the letter of W in the inverse order. Let us call $KG(X,x)$ a *kaleidoscopic group.*

The following statements obtained by routine words arguments explain the algebraic structures of free kaleidoscopic groups and semigroups.

1. $KG(X,x)$ is a free group with the set of free generators $\{xyzx : y,z \in X, y < z, y \neq x, z \neq x\}$.
2. The only idempotents of semigroup $KG(X)$ are the words $x, xy, x, y \in X$.
3. Put $L(x) = \{yx : y \in X\}$, $R(X) = \{xy : y \in X\}$. Then $KS(X)$ is isomorphic to the sandwich product $L(x) \times KG(X,x) \times R(X)$ with the multiplication

$$(l_1, g_1, r_1)(l_2, g_2, r_2) = (l_1, g_1\lambda(r_1, l_2)g_2, r_2),$$

where $\lambda(r_1, l_2) = r_1 l_2$.

The final remark concerns homogeneous graphs $\mathrm{Gr} = (\mathcal{V}, \mathrm{E})$ of infinite degree λ. By standard diagonal process it is easy to find a coloring $\chi\colon \mathcal{V} \longrightarrow \lambda$ of ν such that each ball of radius 1 has the vertices of all colours. Hence, Gr is kaleidoscopic. This is why we consider only kaleidoscopic graphs of finite degree.

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