УДК 517.548

A. L. Golberg

ON GEOMETRIC AND ANALYTIC DEFINITIONS OF QUASICONFORMALITY

A. L. Golberg. *On geometric and analytic definitions of quasiconformality*, Matematychni Studii, **18** (2002) 29–34.

The relations between two definitions of quasiconformality of homeomorphisms are considered. A new geometric condition ensuring a boundedness of distortions and therefore quasiconformality is established.

А. Л. Гольберг. *О геометрическом и аналитическом определениях квазиконформности* // Математичні Студії. – 2002. – Т.18, №1. – С.29–34.

Рассмотрены соотношения между двумя определениями квазикоформности гомеоморфизмов. Установлено новое геометрическое условие гарантирующее ограниченность искажения, а значит квазиконформность.

There are various (in general, not equivalent) definitions of quasiconformality, which involve different features of mappings. The mostly accepted are the definitions suggested by Reshetnyak [2] and Väisälä [3]. In the present paper, we investigate the relations between these definitions and obtain a new geometric criterion for quasiconformality of homeomorphisms.

Let G and G^* be bounded domains in $\mathbb{R}^n$, $n \geq 2$, and let a mapping $f\colon G \to G^*$ be differentiable at a point $x \in G$. This means that there exists a linear mapping $f'(x)\colon \mathbb{R}^n \to \mathbb{R}^n$, called the (strong) derivative of the mapping f at x, such that

$$f(x+h) = f(x) + f'(x)h + \omega(x,h)|h|, \quad \omega(x,h) \to 0 \ as\ h \to 0.$$

Denote

$$J(x,f) = \det f'(x), \quad l(x,f) = \min_{|h|=1} |f'(x)h|, \quad L(x,f) = \max_{|h|=1} |f'(x)h|.$$

The quantities

$$H_I(x,f) = \frac{|J(x,f)|}{l^n(x,f)}, \quad H_O(x,f) = \frac{L^n(x,f)}{|J(x,f)|}, \quad H(x,f) = \frac{L(x,f)}{l(x,f)}$$

are called, respectively, the inner, outer and linear dilatations of f at x. If $n = 2$, then $H_I(x,f) = H_O(x,f) = H(x,f)$. In the general case, we have the relations:

$$H(x,f) \leq \min(H_I(x,f), H_O(x,f)) \leq H^{n/2}(x,f) \leq \max(H_I(x,f), H_O(x,f)) \leq H^{n-1}(x,f).$$

2000 *Mathematics Subject Classification:* 30C65.

Definition 1 [3]. A homeomorphism $f\colon G \to G^*$ is called *quasiconformal* if:

(a) f is ACL (absolutely continuous on lines);

(b) f is differentiable almost everywhere in G;

(c) for almost all $x \in G$ there exists a number q $(1 \le q < \infty)$ such that

$$H_I(x, f) \le q, \quad H_O(x, f) \le q.$$

The quantity $q_V = \inf q$ is called the *quasiconformality coefficient* of the mapping f in G. Here the infimum is taken over all admissible q.

The geometric approach to investigation of properties of quasiconformal mappings is going back to the classical work of Menshov [1]. His method is based on the appropriate change of the radii of the normal neighborhood bases.

Let x be a point in $\mathbb{R}^n$. Assume that some closed neighborhood $\mathcal{G}_t(x)$ of x is defined for any $t \in (0, 1]$. Following [2], we say that a set of the neighborhoods $\mathcal{G}_t(x)$ of the point x constitutes a *normal system*, if there exists a continuous function $v\colon \mathbb{R}^n \to \mathbb{R}$ such that $v(x) = 0$, $v(y) > 0$ for any $y \ne x$. Here $\mathcal{G}_t(x) = \{y \in \mathbb{R}^n\colon v(y) \le t\}$ for any $t \in (0, 1]$. Let $\Gamma_t(x) = \{y \in \mathbb{R}^n : \; v(y) = t\}$ denote the boundary of $\mathcal{G}_t(x)$.

The function v is called the *generating function* for a given normal system $\{\mathcal{G}_t(x)\}$.

Denote

$$r(x, t) = \inf_{y \in \Gamma_t(x)} |y - x|, \quad \mathcal{R}(x, t) = \sup_{y \in \Gamma_t(x)} |y - x|.$$

These values $r(x, t)$ and $\mathcal{R}(x, t)$ are equal, respectively, to the minimal and the maximal radii of the neighborhood $\mathcal{G}_t(x)$. The limit

$$\Delta(x) = \limsup_{t \to 0} \frac{\mathcal{R}(x, t)}{r(x, t)}$$

is called the *regularity parameter* of the family $\{\mathcal{G}_t(x), 0 < t \le 1\}$. Any such system $\{\mathcal{G}_t(x)\}$ is called the *regular normal system*, provided $\Delta(x) < \infty$.

Let $f\colon G \to G^*$ be a homeomorphism and let $\{\mathcal{G}_t(x)\}$ be a normal system of neighborhoods of $x \in G$. One can introduce similar to above the minimal and the maximal radii for the image of $\mathcal{G}_t(x)$ by

$$r^*(x, t) = \inf_{y \in \Gamma_t(x)} |f(y) - f(x)|, \quad \mathcal{R}^*(x, t) = \sup_{y \in \Gamma_t(x)} |f(y) - f(x)|.$$

Definition 2 [2]. A mapping f is called quasiconformal at point $x \in G$ if there exists a normal regular system $\{\mathcal{G}_t(x)\}$ of neighborhoods of x such that

$$\limsup_{t \to 0} \frac{\mathcal{R}^*(x, t)}{r^*(x, t)} \frac{\mathcal{R}(x, t)}{r(x, t)} = q(x, f) < \infty.$$

This quantity is called the *coefficient of quasiconformality* of f at x in the sense of Reshetnyak.

Similarly, a homeomorphism $f\colon G \to G^*$ is called quasiconformal in G, if f is quasiconformal at every point $x \in G$ and there exists a constant q, $1 \le q < \infty$, such that $q(x, f) \le q$ for all $x \in G$.

The next example shows that in the case $n \geq 3$ the above two definitions of quasiconformality are not equivalent.

Let $G = \{x = (x_1, \dots, x_n) : 0 < x_k < 1, k \in \{1, \dots, n-1\}, q^{-1/p} < x_n < 1\}$, $q > 1$, $0 < p < 1$ and

$$g(x) = \left(x_1, \dots, x_{n-1}, \frac{x_n^{1-p}}{1-p} \right).$$

The mapping g is a homeomorphism, which is differentiable in G, and $G^* = g(G) = \{y = (y_1, \dots, y_n) : 0 < y_k < 1, k \in \{1, \dots, n-1\}, q^{(p-1)/p}/(1-p) < y_n < 1/(1-p)\}$. It is easy to see that

$$l(x,g) = 1,\ L(x,g) = |J(x,g)| = H_I(x,g) = H(x,g) = x_n^{-p},\ H_O(x,g) = x_n^{(1-n)p},$$

and that the inequality $q(x,f) \leq q$ is equalent to the condition $H(x,f) \leq q$. Hence, g is a quasiconformal mapping in the sense of Definition 2, and its coefficient of quasiconformality equals q. On the other hand, the mapping g is quasiconformal also in the sense of Definition 1, but in this sense, g has the coefficient of quasiconformality $q^{n-1} \geq q$.

Let us now define quasiconformal homeomorphisms in somewhat other way. We set

$$K_I(x, \{\mathcal{G}_t(x)\}) = \limsup_{t \to 0} \frac{mf(B(x, \mathcal{R}(x,t)))}{\Omega r^{*^n}(x,t)}, \tag{1}$$

and

$$K_O(x, \{\mathcal{G}_t(x)\}) = \limsup_{t \to 0} \frac{\Omega \mathcal{R}^{*^n}(x,t)}{mf(B(x, r(x,t)))}, \tag{2}$$

where $B(x,h) = \{y \in \mathbb{R}^n : |y - x| < h\}$ is n-dimensional ball in $\mathbb{R}^n$ and mA denotes the n-dimensional Lebesgue measure of the set A. Let $\Omega = mB(0,1)$.

Definition 3. A homeomorphism $f\colon G \to G^*$ is called quasiconformal in the domain G, if for almost all $x \in G$ there exist the normal regular systems $\{\mathcal{G}_t(x)\} \subset G$ of neighborhoods of x, which satisfy

$$K_I(x, \{\mathcal{G}_t(x)\}) \leq K(\{\mathcal{G}_t\}) < \infty, \quad K_O(x, \{\mathcal{G}_t(x)\}) \leq K(\{\mathcal{G}_t\}) < \infty. \tag{3}$$

We set $K = \inf K(\{\mathcal{G}_t\})$, where the infinum is taken over all such systems of neighborhoods. This value will be called the *quasiconformality coefficient* of f in G.

Theorem. *Definitions 1 and 3 are equivalent.*

Proof. 1) $\Rightarrow$ 3). Let f be a homeomorphism satisfying the conditions of Definition 1. We show that f satisfies Definition 3, with $K \leq q_V$.

Let $a \in G$ be a point of differentiability of the mapping f and suppose that $|J(a,f)| \neq 0$. The image of the unit ball under the linear mapping f' is an ellipsoid $E(f)$. Let $l_1, l_2, \dots, l_n$ be the semi-axes of $E(f)$, $l_1 \geq l_2 \geq \dots \geq l_n > 0$. We assume that $f(a) = a = 0$, and $f'(0)$ is given by

$$f'(0)e_1 = l_1 e_1,\ \dots,\ f'(0)e_n = l_n e_n,$$

where e_ν denotes the νth unit basis vector.

Consider the normal system of neighborhoods

$$\{\mathcal{G}_t^0(0), 0 < t \le 1\} = \left\{ x \in \mathbb{R}^n : \left(\frac{x_1}{1+tl_1}\right)^2 + \cdots + \left(\frac{x_n}{1+tl_n}\right)^2 \le \mu^2 t^2 \right\} \tag{4}$$

of the origin and choose $\mu > 0$ so that $\mathcal{G}_t^0(0) \subset G$ for any $t \in (0,1]$. Then $\mathcal{R}(0,t) = \mu t(1+tl_1)$, $r(0,t) = \mu t(1+tl_n)$.

Since the mapping f is differentiable at $a = 0$, there exists ε, $0 < \varepsilon < l_n$, such that $|f(x) - f'(0)x| \le \varepsilon \mu t$ for $x \in \mathcal{G}_t^0(0)$. Fix any such ε. From the last inequality we obtain

$$\mathcal{R}^*(0,t) \le \mu t l_1 + \mu t^2 l_1^2 + \varepsilon \mu t,$$

and

$$r^*(0,t) \ge \mu t l_n + \mu t^2 l_n^2 - \varepsilon \mu t.$$

Thus

$$\frac{mf(B(0,\mathcal{R}(0,t)))}{\Omega \mathcal{R}^n(0,t)} \left(\frac{\mathcal{R}(0,t)}{r^*(0,t)}\right)^n \le \frac{mf(B(0,\mathcal{R}(0,t)))}{\Omega \mathcal{R}^n(0,t)} \left(\frac{\mu t + \mu t^2 l_1}{\mu t \lambda_n + \mu t^2 l_n^2 - \varepsilon \mu t}\right)^n,$$

$$\frac{\Omega r^n(0,t)}{mf(B(0,r(0,t)))} \left(\frac{\mathcal{R}^*(0,t)}{r(0,t)}\right)^n \le \frac{\Omega r^n(0,t)}{mf(B(0,r(0,t)))} \left(\frac{\mu t l_1 + \mu t^2 \lambda_1^2 + \varepsilon \mu t}{\mu t + \mu t^2 l_n}\right)^n.$$

But

$$\limsup_{t\to 0} \frac{mf(B(0,h))}{\Omega h^n} = |J(0,f)|;$$

thus one obtains, letting $\varepsilon \to 0$ and then $t \to 0$, the relations

$$\limsup_{t\to 0} \frac{mf(B(0,\mathcal{R}(0,t)))}{\Omega {r^*}^n(0,t)} \le \frac{|J(0,f)|}{l_n^n} = H_I(0,f),$$

$$\limsup_{t\to 0} \frac{\Omega {\mathcal{R}^*}^n(0,t)}{mf(B(0,r(0,t)))} \le \frac{l_1^n}{|J(0,f)|} = H_O(0,f).$$

It follows that there exists the normal system (4) of neighborhoods at the point $a \in G$ for which

$$K_I(a, \{\mathcal{G}_t^0(a)\}) \le q_V, \quad K_O(a, \{\mathcal{G}_t^0(a)\}) \le q_V.$$

Since the mapping f is differentiable almost everywhere in G, and the point a is arbitrary, the above relations imply $K(\{\mathcal{G}_t^0\}) \le q_V$, and, hence, $K \le q_V$.

3) $\Rightarrow$ 1). We show that the conditions (a)–(c) of Definition 1 follow from Definition 3 and that $q_V \le K$. To this end, we need the following statements.

Lemma 1. *Let $f\colon G \to G^*$ be a homeomorphism. Then almost all points $x \in G$ admit the normal regular systems $\{\mathcal{G}_t(x)\} \subset G$ of neighborhoods such that under inequalities (3) the mapping f is ACL-mapping and differentiable almost everywhere in G.*

The proof of this lemma is similar to that of Lemmas 8.2–8.3 and Theorem 8.1 from [2] by using the inequality

$$\limsup_{t\to 0}\frac{\mathcal{R}^*(x,t)}{r^*(x,t)}\le K^{2/n}. \tag{5}$$

instead of the corresponding Reshetnyak's inequality. The inequality (5) is a consequence of relations (1)–(3).

Lemma 2. *Let $f\colon G\to G^*$ be a homeomorphism. Suppose that almost all points $x\in G$ have the normal regular systems $\{\mathcal{G}_t(x)\}\subset G$ of neighborhoods of x satisfying (3). Then there exists a number q, $1\le q<\infty$, which does not depend on x and such that for almost all $x\in G$, we have inequalities*

$$H_I(x,f)\le q,\quad H_O(x,f)\le q.$$

Proof. Let $x_0\in G$ be a point at which the mapping f is differentiable and $|J(x_0,f)|\ne 0$. Without loss of generality, one can assume that $f(x_o)=x_0=0$ and

$$f'(0)e_1=l_1e_1,\ \dots,\ f'(0)e_n=l_ne_n,$$

where

$$l_1\ge l_2\ge\cdots\ge l_n>0.$$

This can be achieved by suitable choice of the coordinate system.

Let

$$r(t)=r(0,t),\ \mathcal{R}(t)=\mathcal{R}(0,t),\ r^*(t)=r^*(0,t),\ \mathcal{R}^*(t)=\mathcal{R}^*(0,t),\ \Gamma_t=\Gamma_t(0).$$

The differentiability of f at 0 yields

$$f(x)=f'(0)x+\omega(x)|x|,$$

where $\omega(x)\to 0$ as $x\to 0$.

Suppose that $a(t)>0$ and $b(t)>0$ are such that $a(t)e_1\in\Gamma_t$, $b(t)e_n\in\Gamma_t$. Then

$$r^*(t)\le|f(b(t)e_n)|,\quad \mathcal{R}^*(t)\ge|f(a(t)e_1)|.$$

It follows that

$$\frac{mf(B(0,\mathcal{R}(t)))}{\Omega r^{*^n}(t)}\ge\frac{mf(B(0,\mathcal{R}(t)))}{\Omega\mathcal{R}^n(t)}\left(\frac{b(t)}{|f(b(t)e_n)|}\right)^n, \tag{6}$$

$$\frac{\Omega\mathcal{R}^{*^n}(t)}{mf(B(0,r(t)))}\ge\frac{\Omega r^n(t)}{mf(B(0,r(t)))}\left(\frac{|f(a(t)e_1)|}{a(t)}\right)^n, \tag{7}$$

and in view of differentiability of f at 0,

$$f(a(t)e_1)=l_1a(t)e_1+\omega_1(a(t)e_1)a(t),\quad f(b(t)e_n)=l_nb(t)e_n+\omega_2(b(t)e_n)b(t),$$

where $\omega \to 0$ as $t \to 0$. Hence

$$\frac{|f(a(t)e_1)|}{a(t)} \to l_1, \quad \frac{|f(b(t)e_n)|}{b(t)} \to l_n,$$

as $t \to 0$. Letting in (5)–(6) $t \to 0$, we get

$$H_I(0, f) \leq K_I(0, \{\mathcal{G}_t(0)\}), \quad H_O(0, f) \leq K_O(0, \{\mathcal{G}_t(0)\}).$$

Since $x_0 \in G$ was an arbitrary point and f is differentiable almost everywhere in G, it follows that for almost all $x \in G$,

$$H_I(x, f) \leq K_I(x, \{\mathcal{G}_t(x)\}) \leq K_I(\{\mathcal{G}_t\}), \quad H_O(x, f) \leq K_O(x, \{\mathcal{G}_t(x)\}) \leq K_O(\{\mathcal{G}_t\}).$$

Letting

$$q = \max\{\sup K_I(x, \{\mathcal{G}_t(x)\}), \sup K_O(x, \{\mathcal{G}_t(x)\})\},$$

where each supremum is taken over those $x \in G$, at which the inequalities (3) hold, one obtains

$$H_I(x, f) \leq q \leq K(\{\mathcal{G}_t\}) < \infty, \quad H_O(x, f) \leq q \leq K(\{\mathcal{G}_t\}) < \infty, \tag{8}$$

also almost everywhere in G. This completes the proof of Lemma 2. □

The last inequalities (8) show that $q_V \leq K(\{\mathcal{G}_t\})$ which implies $q_V \leq K$. Now applying Lemmas 1–2 completes the proof of Theorem. □

REFERENCES

1. Menchoff D. E. *Sur une généralisation d'un théorème de M. H. Bohr*, Mat. sbor. **2** (1937), 339–356.
2. Reshetnyak Yu. G. *Spacial Mappings with Bounded Distortion*, Amer. Math. Soc., 1989.
3. Väisälä J. *Lectures on n-dimensional Quasiconformal Mappings*, Berlin, Springer-Verlag, 1971.

Department of Mathematics and Statistics
Bar-Ilan University, 52900 Ramat-Gan, Israel

Received 6.03.2002