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T. O. Banakh

(METRICALLY) QUARTER-STRATIFIABLE SPACES AND THEIR APPLICATIONS IN THE THEORY OF SEPARATELY CONTINUOUS FUNCTIONS

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We introduce and study (metrically) quarter-stratifiable spaces and then apply them to generalize Rudin and Kuratowski-Montgomery theorems about the Baire and Borel complexity of separately continuous functions.

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Вводятся и изучаются (метрически) четверть-стратифицируемые пространства, которые впоследствии применяются для обобщения теорем Рудина и Куратовского-Монтгомери, касающихся борелевой и бэровой классификации раздельно непрерывных функций.

The starting point for writing this paper was the desire to improve the results of V. K. Maslyuchenko et al. [15], [16], [11], [12] who generalized a classical theorem of W. Rudin [19] which states that every separately continuous function $f\colon X\times Y\to\mathbb{R}$ on the product of a metrizable space X and a topological space Y belongs to the first Baire class. It was proven in [15] that the metrizability of X in the Rudin theorem can be weakened to the σ-metrizability and paracompactness of X. A subtle analysis of Rudin's original proof reveals that this theorem is still valid for a much wider class of spaces X. These spaces are of independent interest, so we decided to give them a special name — metrically quarter-stratifiable spaces. (Metrically) quarter-stratifiable spaces are introduced and studied in details in the first three sections of this paper, where we investigate relationships between the class of (metrically) quarter-stratifiable spaces and other known classes of generalized metric spaces. It turns out that each semi-stratifiable space is quarter-stratifiable (this is a reason for the choice of the term "quarter-stratifiable"), while each quarter-stratifiable Hausdorff space has G_δ-diagonal. Because of this, the class of quarter-stratifiable spaces is "orthogonal" to the class of compacta — their intersection contains only metrizable compacta. The class of quarter-stratifiable spaces is quite wide and has many nice inheritance properties. Moreover, every (submetrizable) space with G_δ-diagonal is homeomorphic to a closed subset of a (metrically) quarter-stratifiable T_1-space. The following diagram describes the interplay between the class of (metrically) quarter-stratifiable spaces and other classes of generalized metric spaces in the framework of Hausdorff spaces.

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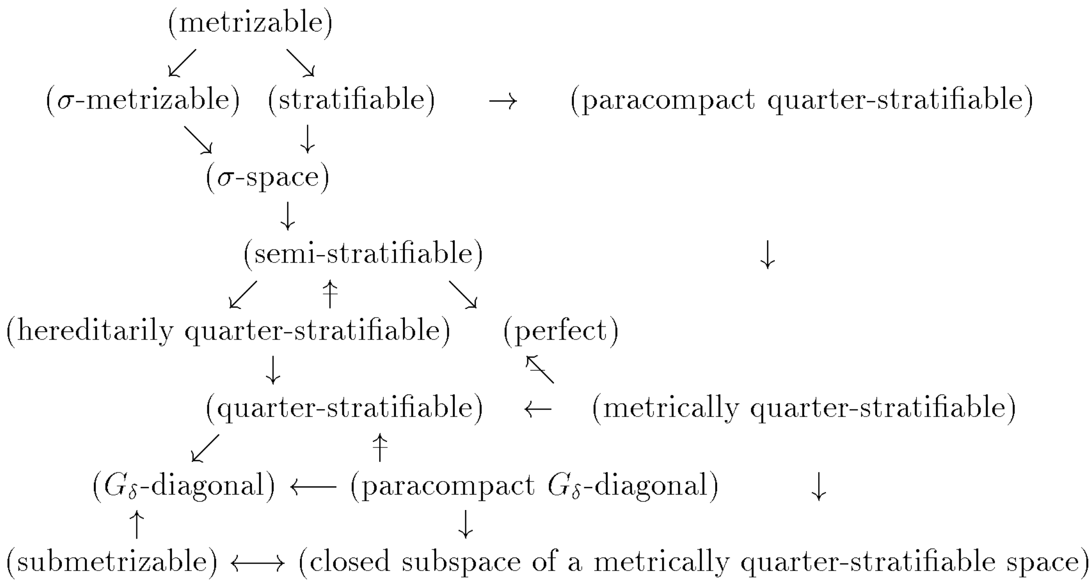

In the last three sections we apply metrically quarter-stratifiable spaces for generalizing classical theorems of Rudin [19] and Kuratowski-Montgomery [13], [14], [17].

1. Quarter-stratifiable spaces

Before introducing quarter-stratifiable spaces we recall some concepts from the theory of generalized metric spaces, see [8]. All topological spaces considered in this paper are T_1-spaces, all maps (unlike to functions) are continuous.

1.1. Definition. A topological space X is

(1) *perfect* if each open set in X is an F_σ-set;

(2) *submetrizable* if X admits a *condensation* (i.e., a bijective continuous map) onto a metrizable space;

(3) *σ-metrizable* if X is covered by a countable collection of closed metrizable subspaces;

(4) a *G_δ-diagonal* if the diagonal is a G_δ-set in the square $X \times X$;

(5) a *σ-space* if X has a σ-discrete network;

(6) *developable* if there exists a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X such that for each $x \in X$ the set $\{\mathcal{S}t(x,\mathcal{U}_n)\}_{n\in\mathbb{N}}$ is a base at x, where $\mathcal{S}t(x,\mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n : x \in U\}$;

(7) a *Moore space* if X is regular and developable;

(8) *semi-stratifiable* (resp. *stratifiable*) if there exists a function G which assigns to each $n \in \mathbb{N}$ and a closed subset $H \subset X$, an open set $G(n,H)$ containing H such that

   (i) $H = \bigcap_{n\in\mathbb{N}} G(n,U)$ (resp. $H = \bigcap_{n\in\mathbb{N}} \overline{G(n,U)}$) and

   (ii) $G(n,K) \supset G(n,H)$ for every closed subset $K \supset H$ and $n \in \mathbb{N}$.

The semi-stratifiable spaces admit the following characterization, see [8, 5.8].

1.2. Theorem. *A topological space (X,τ) is semi-stratifiable if and only if there exists a function $g\colon \mathbb{N}\times X \to \tau$ such that for every $x \in X$*

(1) $\{x\} = \bigcap_{n\in\mathbb{N}} g(n,x)$

(2) $(x \in g(n, x_n),\ n \in \mathbb{N}) \Rightarrow (x_n \to x)$.

Weakening the first condition to $\bigcup_{x \in X} g(n, x) = X$ leads to the definition of a quarter-stratifiable space — the principal concept in this paper.

1.3. Definition. A topological space (X, τ) is called *quarter-stratifiable* if there exists a function $g\colon \mathbb{N} \times X \to \tau$ (called *quarter-stratifying function*) such that

(1) $X = \bigcup_{x \in X} g(n, x)$ for every $n \in \mathbb{N}$;

(2) $(x \in g(n, x_n),\ n \in \mathbb{N}) \Rightarrow (x_n \to x)$.

A space X is *hereditarily quarter-stratifiable* if every subspace of X is quarter-stratifiable.

Since the semi-stratifiability is a hereditary property, we get that each semi-stratifiable space is hereditarily quarter-stratifiable. As we shall see later, the converse is not true: the Sorgenfrey line, being hereditarily quarter-stratifiable, is not semi-stratifiable.

First, we give characterizations of the quarter-stratifiability in terms of open covers as well as of lower semi-continuous multivalued functions. We recall that a multivalued function $\mathcal{F}\colon X \to Y$ is *lower semi-continuous* if for every open set $U \subset Y$ its "preimage" $\mathcal{F}^{-1}(U) = \{x \in X : \mathcal{F}(x) \cap U \neq \varnothing\}$ is open in X. For a topological space X by X_d we denote X endowed with the discrete topology.

1.4. Theorem. *For a space X the following statements are equivalent:*

(1) X is quarter-stratifiable;

(2) there exists a sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of open covers of X and a sequence $\{s_n\colon \mathcal{U}_n \to X\}_{n \in \mathbb{N}}$ of functions such that $s_n(U_n) \to x$ if $x \in U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$;

(3) there exists a sequence $\{\mathcal{F}_n\colon X \to X_d\}_{n \in \mathbb{N}}$ of lower semi-continuous multivalued functions tending to the identity map of X in the sense that for every point $x \in X$ and every neighborhood $O(x) \subset X$ of x there is $n_0 \in \mathbb{N}$ such that $\mathcal{F}_n(x) \subset O(x)$ for every $n \geq n_0$.

Proof. (1) $\Rightarrow$ (2) Assume that (X, τ) is quarter-stratifiable and fix a function $g\colon \mathbb{N} \times X \to \tau$ from Definition 1.3. Let $\mathcal{U}_n = \{g(n, x) : x \in X\}$ and for every $U \in \mathcal{U}_n$ find $x \in X$ with $U = g(n, x)$ and let $s_n(U) = x$. Clearly, the so-defined sequences $\{\mathcal{U}_n\}$ and $\{s_n\}$ satisfy our requirements.

(2) $\Rightarrow$ (3) For every $n \in \mathbb{N}$ define $\mathcal{F}_n\colon X \to X_d$ letting $\mathcal{F}_n(x) = \{s_n(U) : x \in U \in \mathcal{U}_n\}$ for $x \in X$. Clearly, $\mathcal{F}_n$ is lower-semicontinuous and $\mathcal{F}_n(x) \to x$ for each $x \in X$.

(3) $\Rightarrow$ (1) For each $(n, x) \in \mathbb{N} \times X$ let $g(n, x) = \mathcal{F}_n^{-1}(x)$ and check that the so-defined function $g\colon \mathbb{N} \times X \to \tau$ fits. □

Given a topological space X, by $l(X)$ and $d(X)$ we denote respectively the Lindelöf number and the density of X.

Like other classes of generalized metric spaces, the class of quarter-stratifiable spaces has many nice inheritance properties.

1.5. Theorem. *Let X be a quarter-stratifiable space.*

(1) Every open subspace and every retract of X is quarter-stratifiable.

(2) If X is Hausdorff, then it has a G_δ-diagonal and countable pseudo-character.

(3) If X is Hausdorff, then every paracompact Čech-complete subspace and every countably compact subspace of X is metrizable.

(4) $d(X) \le l(X)$.

(5) If $f\colon Y \to X$ *is a condensation with sequentially continuous inverse* f^{-1}*, then* Y *is quarter-stratifiable.*

(6) If $f\colon X \to Y$ *is a finite-to-one open surjective map, then the space* Y *is quarter-stratifiable.*

Proof. Using Theorem 1.4, fix a map $g\colon \mathbb{N}\times X \to \tau$, a sequence $\{\mathcal{F}_n\colon X \to X_d\}_{n\in\mathbb{N}}$ of lower-semicontinuous multivalued functions tending to the identity map of X, and sequences $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers and functions $\{s_n\}_{n\in\mathbb{N}}$ satisfying the conditions of Definition 1.3 and Theorem 1.4.

1) Given an open subspace Y of X, let $\mathcal{V}_n = \{U\cap Y : U \in \mathcal{U}_n\}$, $n\in\mathbb{N}$. For each $V\in\mathcal{V}_n$ find a set $U(V)\in\mathcal{U}_n$ with $V = U(V)\cap Y$ and let $\tilde{s}_n(V) = s_n(U(V))$ if $s_n(U)\in Y$ and $\tilde{s}_n(V)$ be any point of Y, otherwise. Clearly, $\mathcal{V}_n$ is an open cover of Y. Now fix any point y of Y and take a sequence $V_n\in\mathcal{V}_n$ with $y\in V_n$ for $n\in\mathbb{N}$. Since $s_n(U(V_n))\to y$, we get $s_n(U(V_n))\in Y$ for all n beginning from some n_0. Then $\tilde{s}_n(V_n) = s_n(U(V_n))$ for $n\ge n_0$ and thus $\tilde{s}_n(V_n)\to y$, which proves that Y is a quarter-stratifiable space.

Now suppose that a subspace Y of X is a retract of X and let $r\colon X\to Y$ be the corresponding retraction. One can readily prove that $\{r\circ\mathcal{F}_n|Y : Y\to Y_d\}$ is a sequence of lower-semicontinuous multivalued functions tending to the identity map of Y.

2) Assume that the quarter-stratifiable space X is Hausdorff. Let $G_n = \bigcup_{U\in\mathcal{U}_n} U\times U \subset X\times X$. We claim that $\bigcap_{n\in\mathbb{N}} G_n$ coincides with the diagonal of $X\times X$. To show this, take any distinct points $x,y\in X$ and take disjoint neighborhood $O(x), O(y)$ of x and y, respectively. It follows from the choice of the sequences $\{\mathcal{U}_n\}$ and $\{s_n\}$ that there is $n\in\mathbb{N}$ such that $s_m(U_m)\in O(x)$ and $s_m(V_m)\in O(y)$ for all $m\ge n$ and $x\in U_m\in\mathcal{U}_m$, $y\in V_m\in\mathcal{U}_m$. We claim that $(x,y)\notin G_n$. Assuming the converse, we would find a set $U_n\in\mathcal{U}_n$ with $(x,y)\in U_n\times U_n$. Then $x,y\in U_n$ and $s_n(U_n)\in O(x)\cap O(y)$, a contradiction. Thus X has a G_δ-diagonal, which implies the countability of the pseudo-character of X.

3) If X is Hausdorff and $Y\subset X$ is a paracompact Čech-complete or countably compact subspace of X, then Y has a G_δ-diagonal and by [3, 8.2] or [8, 2.14] is metrizable.

4) To see that $d(X)\le l(X)$ we may take the subset $D = \{s_n(U) : U\in\mathcal{V}_n, n\in\mathbb{N}\}$, where $\mathcal{V}_n$ is a subcover of $\mathcal{U}_n$ of size $|\mathcal{V}_n|\le l(X)$ for $n\in\mathbb{N}$. Clearly, D is a dense subset in X of size $|D|\le l(X)$.

5) If $f\colon Y\to X$ is a condensation with sequentially continuous inverse f^{-1}, we may consider the function g_Y assigning to each pair $(n,y)\in\mathbb{N}\times Y$ the open set $f^{-1}(g(n,f(y)))$ of Y. One may easily show that the so-defined function g_Y turns Y into a quarter-stratifiable space.

6) Suppose $f\colon X\to Y$ is an open finite-to-one surjective map. It can be shown that for every $n\in\mathbb{N}$ the multivalued map $f\circ\mathcal{F}_n\circ f^{-1}\colon Y\to Y_d$ defined by $f\circ\mathcal{F}_n\circ f^{-1} = \bigcup_{x\in f^{-1}(y)} f(\mathcal{F}_n(x))$ for $y\in Y$ is lower-semicontinuous. Moreover, the sequence $\{f\circ\mathcal{F}_n\circ f^{-1}\}_{n\in\mathbb{N}}$ tends to the identity map of Y. Then according to Theorem 1.4, the space Y is quarter-stratifiable. □

Next, we show that the quarter-stratifiability is stable with respect to certain set-theoretic operations.

1.6. Theorem.

(1) The countable product of quarter-stratifiable spaces is quarter-stratifiable.

(2) A space X is quarter-stratifiable, provided $X = A \cup B$ is a union of two quarter-stratifiable subspaces of X, one of which is a closed G_δ-set in X.

(3) A space X is quarter-stratifiable, provided X can be covered by a σ-locally finite collection of quarter-stratifiable closed G_δ-subspaces of X.

Proof. 1) Let X_i, $i \in \mathbb{N}$, be a sequence of quarter-stratifiable spaces. For every $i \in \mathbb{N}$, fix a point $*_i \in X_i$ and a sequence $\{\mathcal{F}_{i,n} : X_i \to (X_i)_d\}_{n\in\mathbb{N}}$ of lower-semicontinuous multivalued functions tending to the identity map of X_i as $n \to \infty$. Let $X = \prod_{i\in\mathbb{N}} X_i$ and for every $n \in \mathbb{N}$ identify the product $\prod_{i=1}^n X_i$ with the subspace $\{(x_i)_{i\in\mathbb{N}} : x_i = *_i \text{ for } i > n\}$ of X. For every $n \in \mathbb{N}$ define a multivalued function $\mathcal{F}_n \colon X \to X_d$ as follows: $\mathcal{F}_n((x_i)_{i\in\mathbb{N}}) = \prod_{i=1}^n \mathcal{F}_{i,n}(x_i)$. It is easy to see that each function $\mathcal{F}_n$ is lower semicontinuous and the sequence $\{\mathcal{F}_n\}$ tends to id_X.

2) Suppose $X = A \cup B$, where A, B are quarter-stratifiable subspaces and B is a closed G_δ-set in X. Fix a decreasing sequence $\{O_n(B)\}_{n\in\mathbb{N}}$ of open subsets of X with $B = \bigcap_{n\in\mathbb{N}} O_n(B)$. By Theorem 1.5(1), the open subspace $A \setminus B = X \setminus B$ of A is quarter-stratifiable and thus admits a function $g_{A\setminus B} \colon \mathbb{N} \times (A \setminus B) \to \tau$ into the topology τ of X such that $\bigcup_{x\in A\setminus B} g_{A\setminus B}(n, x) \supset A \setminus B$ for each $n \in \mathbb{N}$, and $x \in g_{A\setminus B}(n, x_n) \;\Rightarrow\; x_n \to x$ for each $x, x_n \in A \setminus B$, $n \in \mathbb{N}$.

Using the quarter-stratifiability of B find a function $g_B \colon \mathbb{N} \times B \to \tau$ such that $B \subset \bigcup_{x\in B} g_B(n, x) \subset O_n(B)$ for every $n \in \mathbb{N}$, and $x \in g_B(n, x_n) \;\Rightarrow\; x_n \to x$ for each $x, x_n \in B$, $n \in \mathbb{N}$. It is easy to verify that the function $g \colon \mathbb{N} \times X \to \tau$ defined by

$$g(n,x) = \begin{cases} g_{A\setminus B}(n,x) & \text{if } x \in A \setminus B, \\ g_B(n,x) & \text{if } x \in B \end{cases}$$

turns X into a quarter-stratifiable space.

3) Let $\{X_i\}_{i\in\mathcal{I}}$ be a σ-locally finite collection of quarter-stratifiable closed G_δ-subspaces of a space X. Write $\mathcal{I} = \bigcup_{k\in\mathbb{N}} \mathcal{I}_k$ so that the collection $\{X_i\}_{i\in\mathcal{I}_k}$ is locally finite for every $k \in \mathbb{N}$. Without loss of generality, $\mathcal{I}_k \subset \mathcal{I}_{k+1}$ for every k. For every k fix an open cover $\mathcal{W}_k$ of X whose any element $W \in \mathcal{W}_k$ meets only finitely many of the sets X_i's with $i \in \mathcal{I}_k$. We may additionally require that the cover $\mathcal{W}_k$ is inscribed into the cover $\mathcal{W}_{k-1}$ for $k > 1$.

Since X_i's are G_δ-sets in X, for every $i \in \mathcal{I}$ we may find a decreasing sequence $\{O_n(X_i)\}_{n\in\mathbb{N}}$ of open subsets

$$O_n(X_i) \subset \mathcal{S}t(X_i, \mathcal{W}_n) = \{W \in \mathcal{W}_n : W \cap X_i \neq \varnothing\}$$

with $\bigcap_{n\in\mathbb{N}} O_n(X_i) = X_i$.

Let $\leq$ be any well-ordering of the index set $\mathcal{I}$ such that $i < j$ for every $i, j \in I$ with $i \in I_k \not\ni j$ for some k. It follows from the local finiteness of the collections $\{X_i\}_{i\in\mathcal{I}_k}$ that for every $i \in \mathcal{I}$ the set $\bigcup_{j<i} X_j$ is closed in X. Using the quarter-stratifiability of open subsets of X_i's, for every $i \in \mathbb{N}$ fix a sequence $\{\mathcal{U}_{i,n}\}_{n\in\mathbb{N}}$ of open covers of the set $Y_i = X_i \setminus \bigcup_{j<i} X_j$ and a sequence $\{s_{i,n} \colon \mathcal{U}_{i,n} \to Y_i\}$ of functions such that $s_{i,n}(U_n) \to y$ for every $Y_i \ni y \in U_n \in \mathcal{U}_{i,n}$, $n \in \mathbb{N}$. Without loss of generality, we may assume that each $U \in \mathcal{U}_{i,n}$ is an open subset of X lying in $O_n(X_i) \setminus \bigcup_{j<i} X_j$. Now for every $n \in \mathbb{N}$ consider the open cover $\mathcal{U}_n = \bigcup_{i\in\mathcal{I}} \mathcal{U}_{i,n}$ and the function $s_n = \bigcup_{i\in\mathcal{I}} s_{i,n} \colon \mathcal{U}_n \to X$.

To show that the space X is quarter-stratifiable, it suffices to verify that for every $x \in X$ and a sequence $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, with $x \in U_n$ we have $s_n(U_n) \to x$. Let $i = \min\{j \in \mathcal{I} : x \in X_i\}$. Then $x \in Y_i$. For every $n \in \mathbb{N}$ find $i_n \in \mathcal{I}$ with $U_n \in \mathcal{U}_{i_n,n}$. Since $x \in U_n \subset$

$O_{i_n}(X_i)\setminus\bigcup_{j<i_n} X_j$, we get $i_n \le i$ for all $n \in \mathbb{N}$. It follows from the choice of the cover $\mathcal{W}_n$ and the sets $O_n(X_j)$ that $\bigcap_{n=1}(\bigcup_{j<i} O_n(X_j)) = \bigcup_{j<i} X_j$. Consequently, $x \notin \bigcap_{n=1}^{\infty}(\bigcup_{j<i} O_n(X_j))$ and there is $n_0 \in \mathbb{N}$ such that $x \notin \bigcup_{j<i} O_n(X_j)$ for every $n \ge n_0$. Since $x \in U_n \subset O_n(X_{i_n})$ for every $n \in \mathbb{N}$, we conclude that $i_n = i$ for $n \ge n_0$. Then $x \in U_n \in \mathcal{U}_{i,n}$ for all $n \ge n_0$ and $s_n(U_n) = s_{i,n}(U_n) \to x$, which completes the proof of the quarter-stratifiability of X. □

1.7. Remark. The G_δ-condition in Theorem 1.6(2) is essential: the one-point compactification $\alpha\Gamma$ of an uncountable discrete space Γ may be written as the union $\alpha\Gamma = \{\infty\} \cup \Gamma$ of two metrizable subspaces, one of which is closed. But $\alpha\Gamma$, being a non-metrizable compactum, is not quarter-stratifiable, see Theorem 1.5(3). Yet, we do not know the answer to the following question.

1.8. Question. *Is a (regular) space X quarter-stratifiable if it is a union of two closed quarter-stratifiable subspaces?*

According to Theorem 1.5(1), every open subspace of a quarter-stratifiable space is quarter-stratifiable. It is not true for closed subspaces of quarter-stratifiable spaces: every G_δ-diagonal space is homeomorphic to a closed subset of a quarter-stratifiable T_1-space.

1.9. Theorem. *Every space X with G_δ-diagonal is homeomorphic to a closed subspace of a quarter-stratifiable T_1-space Y with $|Y \setminus X| \le \max\{l(X), d(X)\}$.*

Proof. Suppose X is a space with G_δ-diagonal. By Theorem 2.2 of [8], X admits a sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ of covers such that $\{x\} = \bigcap_{n\in\mathbb{N}} \mathcal{S}t(x, \mathcal{V}_n)$ for every $x \in X$. Replacing each $\mathcal{V}_n$ by a suitable subcover, we may assume that $|\mathcal{V}_n| \le l(X)$ for every n, and each cover $\mathcal{V}_n$ consists of non-empty subsets of X. Let $D \subset X$ be a dense subset of X with $|D| \le d(X)$. For every $n \in \mathbb{N}$ and every $V \in \mathcal{V}_n$ pick a point $c(V) \in V \cap D$. Let $S_n = \{0\} \cup \{1/i : i \ge n\}$ denote a tail of the convergent sequence S_1.

Consider the subset $Y = X \times \{0\} \cup \{(c(V), 1/n) : V \in \mathcal{V}_n,\ n \in \mathbb{N}\} \subset X \times S_1$. It is clear that $|Y \setminus X| \le \min\{|\bigcup_{n\in\mathbb{N}} \mathcal{V}_n|, |D|\} \le \min\{l(X), d(X)\}$. Identify X with the subset $X \times \{0\}$ of Y. Define a topology τ on Y letting $U \subset Y$ be open if and only if $U \cap X$ is open in X and for every $x \in U \cap X$ there is $n_0 \in \mathbb{N}$ such that $\{(c(V), 1/n) : x \in V \in \mathcal{V}_n,\ n \ge n_0\} \subset U$. Thus X is homeomorphic to a closed subspace of Y, while all points of $Y \setminus X$ are isolated. Since each one-point subset of Y is closed, Y is a T_1-space. To show that Y is a quarter-stratifiable space, consider open covers $\mathcal{U}_n = \{\{y\}, (V \times S_n) \cap Y : y \in Y \setminus X,\ V \in \mathcal{V}_n\}$, $n \in \mathbb{N}$, of Y. For every $U \in \mathcal{U}_n$ let

$$s_n(U) = \begin{cases} y & \text{if } U = \{y\} \text{ for some } y \in Y \setminus X; \\ (c(V), 1/n) & \text{if } U = (V \times S_n) \cap Y \text{ for } V \in \mathcal{V}_n. \end{cases}$$

It follows from the choice of the topology τ on Y that $x \in U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, implies $s_n(U_n) \to x$ i.e., Y is a quarter-stratifiable T_1-space. □

1.10. Problem. *Describe the class of subspaces of regular (Tychonoff) quarter-stratifiable T_1-spaces.*

2. Metrically quarter-stratifiable spaces

In this section we introduce and study metrically quarter-stratifiable spaces forming a class, intermediate between the class of paracompact quarter-stratifiable Hausdorff spaces and the class of submetrizable quarter-stratifiable spaces. We start with defining a quarter-stratifying topology.

2.1. Definition. A topology τ' on a topological space (X,τ) is called *quarter-stratifying* if there exists a quarter-stratifying function $g\colon \mathbb{N}\times X\to\tau$ for X such that $g(\mathbb{N}\times X)\subset\tau'$.

A topological space X is defined to be *metrically quarter-stratifiable* if it admits a weaker metrizable quarter-stratifying topology.

The following theorem characterizes metrically quarter-stratifiable spaces.

2.2. Theorem. *For a space X the following statements are equivalent:*

(1) X admits a weaker metrizable quarter-stratifying topology;

(2) X admits a weaker paracompact Hausdorff quarter-stratifying topology;

(3) there exists a weaker metrizable topology τ_m on X, a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of covers of X by τ_m-open subsets and a sequence $\{s_n\colon \mathcal{U}_n\to X\}_{n\in\mathbb{N}}$ of functions such that $s_n(U_n)\to x$ in X if $x\in U_n\in\mathcal{U}_n$, $n\in\mathbb{N}$;

(4) there exists a weaker paracompact Hausdorff topology τ_p on X, a sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ of covers of X by τ_p-open subsets and a sequence $\{\tilde{s}_n\colon \mathcal{V}_n\to X\}_{n\in\mathbb{N}}$ of functions such that $\tilde{s}_n(V_n)\to x$ in X if $x\in V_n\in\mathcal{V}_n$, $n\in\mathbb{N}$;

Proof. The equivalence (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) can be proved by analogy with the corresponding equivalences in Theorem 1.4; the implication (3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (3) Assume that τ_p is a weaker paracompact Hausdorff topology on X, $\{U_n\}_{n\in\mathbb{N}}$ is a sequence of τ_p-open covers of X, and $\{s_n\colon \mathcal{U}_n\to X\}_{n\in\mathbb{N}}$ is a sequence of functions such that $s_n(U_n)\to x$ in X if $x\in U_n\in\mathcal{U}_n$, $n\in\mathbb{N}$. Let $l_1(\mathcal{U}_n)$ denote the Banach space of all absolutely summable functions $f\colon \mathcal{U}_n\to\mathbb{R}$ equipped with the norm $\|f\|=\sum_{U\in\mathcal{U}_n}|f(U)|$.

Using the paracompactness of the topology $\tau_p\subset\tau$, for every $n\in\mathbb{N}$ find a partition of unity $\{\lambda_U\colon X\to[0,1]\}_{U\in\mathcal{U}_n}$ such that $\lambda_U^{-1}(0,1]\subset U$ for $U\in\mathcal{U}_n$. Observe that this partition of unity can be regarded as a continuous map $\Lambda_n\colon X\to l_1(\mathcal{U}_n)$ acting as $\Lambda_n(x)=(\lambda_U(x))_{U\in\mathcal{U}_n}$ (the continuity of Λ_n follows from the local finiteness of the cover $\{\lambda_U^{-1}(0,1] : U\in\mathcal{U}_n\}$).

Let $\Lambda\colon X\to\prod_{n\in\mathbb{N}} l_1(\mathcal{U}_n)$ denote the diagonal product of the maps Λ_n. By analogy with the proof of Theorem 1.5(2), show that the map Λ is injective. Then the weakest topology τ_m on X for which the map Λ is continuous is metrizable. Observe that for each $n\in\mathbb{N}$ and $U\in\mathcal{U}_n$ the set $V(U)=\lambda_U^{-1}(0,1]$ is τ_m-open. Let $\mathcal{V}_n=\{V(U) : U\in\mathcal{U}_n\}$. For every $V\in\mathcal{V}_n$ find $U\in\mathcal{U}_n$ with $V=V(U)$ and let $\tilde{s}_n(V)=s_n(U)$. One can easily check that the metrizable topology τ_m on X and the sequences $\{V_n\}_{n\in\mathbb{N}}$, $\{\tilde{s}_n\}_{n\in\mathbb{N}}$ satisfy condition (3). □

In the following theorem we collect some elementary properties of metrically quarter-stratifiable spaces.

2.3. Theorem.

(1) Each paracompact Hausdorff quarter-stratifiable space is metrically quarter-stratifiable.

(2) Each metrically quarter-stratifiable space is submetrizable and quarter-stratifiable.

(3) Every open subspace of a metrically quarter-stratifiable space is metrically quarter-stratifiable.

(4) If X is a metrically quarter-stratifiable space and $f\colon Y\to X$ is a condensation with sequentially continuous inverse f^{-1}, then the space Y is metrically quarter-stratifiable.

(5) *The product* $X = \prod_{n \in \mathbb{N}} X_n$ *of a countable collection of metrically quarter-stratifiable spaces* X_n, $n \in \mathbb{N}$, *is a metrically quarter-stratifiable space.*

(6) *A space* X *is metrically quarter-stratifiable if there exist a weaker paracompact Hausdorff topology* τ_p *and two subspaces* $A, B \subset X$ *such that* $A \cup B = X$, A *is a closed* G_δ*-set in* (X, τ_p) *and the topologies induced by the topology* τ_p *on* A *and* B *are quarter-stratifying.*

(7) *A space* X *is metrically quarter-stratifiable if there exist a weaker paracompact Hausdorff topology* τ_p *on* X *and a* σ*-locally finite cover* $\mathcal{C}$ *of* (X, τ_p) *by closed* G_δ*-subspaces such that the topology induced by* τ_p *on each* $C \in \mathcal{C}$ *is quarter-stratifying (with respect to the original topology of* C*);*

(8) *A space* X *is metrically quarter-stratifiable if there is a weaker metrizable topology* τ_m *on* X *and a cover* $\mathcal{C}$ *of* X*, well-ordered by the inclusion relation, such that the topology* τ_m *induces the original topology on each* $C \in \mathcal{C}$*;*

(9) *Every submetrizable space is homeomorphic to a metrically quarter- stratifiable space.*

Proof. All statements (except (8)) easily follow from the definitions or can be proved by analogy with the corresponding properties of the quarter-stratifiable spaces.

To prove statement (8), fix a continuous metric d on X and a well-ordered (by the inclusion relation) cover $\mathcal{C}$ of X such that d induces the original topology on each $C \in \mathcal{C}$. Let $\mathcal{U}_n$ denotes the collection of all open $1/n$-balls with respect to the metric d. For every $U \in \mathcal{U}_n$ let $C(U)$ be the smallest set $C \in \mathcal{C}$ meeting U and let $s_n(U) \in U \cap C(U)$ be any point.

Fix any $x \in X$ and a sequence $x \in U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$. We have to show that $s_n(U_n) \to x$. Let $C(x)$ be the smallest set $C \in \mathcal{C}$ containing the point x. Since $x \in C(x) \cap U_n$, we conclude that $s_n(U_n) \in C(x)$. By the choice of the sets U_n, $\mathrm{diam}(U_n) \le 2/n$. Then $d(x, s_n(U_n)) \le \mathrm{diam}(U_n) \le 2/n$, $n \in \mathbb{N}$, and thus $s_n(U_n) \to x$ (because d induces the topology of $C(x)$). □

According to Theorem 2.3(2) each metrically quarter-stratifiable space is submetrizable and quarter-stratifiable. We do not know if the converse is also true.

2.4. Question. *Is every submetrizable quarter-stratifiable space metrically quarter-stratifiable?*

3. Some Examples

In this section we collect some examples exposing the difference between the class of (metrizable) quarter-stratifiable spaces and other classes of generalized metric spaces. It is known that every Moore space is semi-stratifiable and each collectively normal Moore space is metrizable, see [8]. Yet, there exist Moore spaces which are not submetrizable, see [18]. Thus we have

3.1. Example. *There exists a quarter-stratifiable (Moore) space which is not metrically quarter-stratifiable.*

Theorems 1.9 and 2.3(9) show that the class of quarter-stratifiable spaces is much wider than that of semi-stratifiable spaces. Using these theorems many wild (metrically) quarter-stratifiable spaces may be constructed. But in general, so-constructed spaces are not hereditarily quarter-stratifiable.

3.2. Example. *The Sorgenfrey line Z is not semi-stratifiable but every subspace of Z is Lindelöf, separable, and metrically quarter-stratifiable.*

Proof. Recall that the Sorgenfrey line Z is the semi-interval $[0,1)$ endowed with the topology generated by the base consisting of all semi-intervals $[a,b)$ where $0 \le a < b \le 1$, see [5, 1.2.2]. Since Z embeds into a linearly ordered space ("two arrows" of Aleksandrov [5, 3.10.C]), Z is monotonically normal, see [8, 5.21]. If Z would be semi-stratifiable, then Z, being monotonically normal, would be stratifiable according to Theorem 5.16 of [8]. Since each stratifiable space is a σ-space [8, 5.9] and Lindelöf σ-spaces have countable network [8, 4.4], this would imply that Z has countable network, which is a contradiction. Therefore, the Sorgenfrey line Z is not semi-stratifiable.

Next, we show that every subspace X of Z is quarter-stratifiable. For every $n \in \mathbb{N}$ consider the finite open cover $\mathcal{U}_n = \{X \cap [\frac{k-1}{n}, \frac{k}{n}) : 0 < k \le n\}$ of X. For every element $U \in \mathcal{U}_n$ choose a point $s_n(U) \in X$ as follows. If the set $\uparrow U = \{x \in X : x \ge \sup U\}$ is not empty, then let $s_n(U)$ be any point of X with $s_n(U) < \inf \uparrow U + \frac{1}{n}$; otherwise, let $s_n(U)$ be any point of X. It can be shown that $s_n(U_n) \to x$ for every $x \in X$ and every sequence $x \in U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$. Thus the space X is quarter-stratifiable. It is well known that the Sorgenfrey line is hereditarily Lindelöf and hereditarily separable which implies that the space X is Lindelöf and separable. Then X is also paracompact ([5, 5.1.2]) and hence, being quarter-stratifiable, is metrically quarter-stratifiable. □

Since the metrical quarter-stratifiability is productive, we get

3.3. Example. *The square of the Sorgenfrey line is metrically quarter-stratifiable but not normal.*

Another example of a metrically quarter-stratifiable non-normal space is the Nemytski plane. Unlike to (semi-)stratifiable spaces, metrically quarter-stratifiable spaces need not be perfect (or paracompact).

3.4. Example. *A separable zero-dimensional metrically quarter-stratifiable Tychonov space which is neither perfect nor Lindelöf.*

Proof. Consider the Cantor cube $2^\omega = \{0,1\}^\omega$, i.e., the set of all functions $\omega \to \{0,1\}$. There is a natural partial order on 2^ω: $x \le y$ iff $x(i) \le y(i)$ for each $i \in \omega$. For $k \in \{0,1\}$ let $2^\omega_k = \{(x_i)_{i\in\omega} : \exists n \in \omega \text{ with } x(i) = k \text{ for all } i \ge n\}$. For a point $x \in 2^\omega$ and $n \in \omega$ let $U(x,n) = \{y \in 2^\omega : y \ge x \text{ and } y(i) = x(i) \text{ for } i \le n\}$. On the set 2^ω consider the topology τ generated by the base $\{U(x,n) : x \in 2^\omega,\ n \in \omega\}$. Clearly, the space $X = (2^\omega, \tau)$ is Tychonoff, zero-dimensional, and separable (2^ω_1 is a countable dense set in X).

Let us show that the space X is metrically quarter-stratifiable. For every $n \in \omega$ identify 2^n with the subset $\{(x_i) \in 2^\omega : x_i = 0 \text{ for } i \ge n\} \subset 2^\omega$ and consider the finite cover $\mathcal{U}_n = \{U(x,n) : x \in 2^n\}$ of 2^ω. It is clear that each set $U(x,n) \in \mathcal{U}_n$ is open is the product topology of 2^ω which is metrizable. To each $U \in \mathcal{U}_n$ assign the point $s_n(U) = \max U \in 2^\omega_1$. One may easily verify that $s_n(U_n) \to x$ for every $x \in X$ and a sequence $x \in U_n \in \mathcal{U}_n$, $n \in \omega$. Thus the space X is metrically quarter-stratifiable.

Now we show that the space X is not perfect. Note that 2^ω_0 is a closed subset of X. We claim that it is not a G_δ-subset of X. Assume conversely that $2^\omega_0 = \bigcap_{n\in\mathbb{N}} O_n$, where O_n are open subsets of X. Since the topology of X coincides with the usual product topology at every point of the set 2^ω_0, we may assume that each set O_n is open in the product topology of

2^ω. Then by the Baire theorem, the intersection $\bigcap_{n\in\mathbb{N}} O_n$ is a dense G_δ-subset in 2^ω, which contains a point $x \notin 2^\omega_0$, a contradiction.

By Theorem 2.3(3) the open subspace $Y = X \setminus 2^\omega_0$ of X is metrically quarter-stratifiable. Since X is regular and 2^ω_0 is not a G_δ-set in X, we conclude that the space Y is not Lindelöf.

Then the product $X \times Y$ is a separable Tychonov zero-dimensional metrically quarter-stratifiable space which is neither perfect nor Lindelöf. □

3.5. Question. *Is there a hereditarily quarter-stratifiable space which is not perfect?*

We recall that a topological space X is *finally compact* if $l(X) \le \aleph_0$, i.e., every open cover of X admits a countable subcover. Finally compact regular spaces are called *Lindelöf.*

3.6. Example. *A hereditarily finally compact submetrizable uncountable space whose uncountable subspaces are neither separable nor quarter-stratifiable.*

Proof. Let X be the interval $(0,1)$ with the topology generated by the sets of the form $(a,b)\setminus C$, where $0 \le a < b \le 1$ and C is a countable set. Clearly, the space X is submetrizable and hereditarily finally compact. Yet, every countable subset of X is closed which implies that every uncountable subspace Y of X is not separable. Since the finally compact quarter-stratifiable spaces are separable, see Theorem 1.6(4), we conclude that the space Y is not quarter-stratifiable. □

The space constructed in Example 3.6 is not regular. There is also a regular submetrizable non-quarter-stratifiable space.

Given a subset $B \subset \mathbb{R}$ denote by $\mathbb{R}_B$ the real line $\mathbb{R}$ endowed with the topology consisting of the sets $U \cup A$, where U open in $\mathbb{R}$ and $A \subset B$, see [5, 5.1.22, 5.5.2]. It is known that for every $B \subset \mathbb{R}$ the space $\mathbb{R}_B$ is hereditarily paracompact; $\mathbb{R}_B$ is perfect if and only if B is a G_δ-set in $\mathbb{R}$. If $B \subset \mathbb{R}$ is a Bernstein set, then the space $\mathbb{R}_B$ is Lindelöf, see [5, 5.5.4]. Recall that a subset $B \subset \mathbb{R}$ is called a *Bernstein set* if $C \cap B \ne \varnothing \ne C \setminus B$ for every uncountable compactum $C \subset \mathbb{R}$. Bernstein sets can be easily constructed by transfinite induction, see [5, 5.4.4].

3.7. Example. *If $B \subset \mathbb{R}$ is not σ-compact, then the space $\mathbb{R}_B$ is not quarter-stratifiable. Consequently, if $B \subset \mathbb{R}$ is the set of irrationals, then $\mathbb{R}_B$ is a hereditarily paracompact submetrizable perfect zero-dimensional space which is neither separable nor quarter-stratifiable; if $B \subset \mathbb{R}$ is a Bernstein set, then $\mathbb{R}_B$ is a submetrizable hereditarily paracompact Lindelöf zero-dimensional space which is neither perfect nor separable nor quarter-stratifiable.*

Proof. Assume that $B \subset \mathbb{R}$ is not σ-compact. Suppose the space $\mathbb{R}_B$ is quarter-stratifiable and fix a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of $\mathbb{R}_B$ and a sequence $\{s_n\colon \mathcal{U}_n \to \mathbb{R}_B\}_{n\in\mathbb{N}}$ of functions such that $s_n(U_n) \to x$ for every $x \in U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$. Since the real line is hereditarily Lindelöf and the topology of $\mathbb{R}_B$ coincides with the usual topology at the points of the set $\mathbb{R}\setminus B$, we may find countable subcovers $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, of the set $\mathbb{R}\setminus B$. Observe that $C = \{s_n(V) : V \in \mathcal{V}_n,\ n \in \mathbb{N}\}$ is countable while the intersection $G = \bigcap_{n\in\mathbb{N}} \mathcal{S}t(\mathbb{R}\setminus B, \mathcal{V}_n)$ is a G_δ-set in the usual topology of $\mathbb{R}$. Since B is not σ-compact, there is a point $x \in B\cap G\setminus C$. Choose for every $n \in \mathbb{N}$ an element $V_n \in \mathcal{V}_n$ with $V_n \ni x$. Then $x \ne s_n(V_n)$ for all n and by the definition of the topology of $\mathbb{R}_B$, $s_n(V_n) \not\to x$. This contradiction shows that the space $\mathbb{R}_B$ is not quarter-stratifiable. □

4. A generalization of Kuratowski-Montgomery Theorem

In this section we apply metrically quarter-stratifiable spaces to generalize a theorem of Kuratowski [14] and Montgomery [17] which asserts that for metrizable spaces X, Y, Z a function $f\colon X \times Y \to Z$ is Borel measurable of class $\alpha+1$, α a countable ordinal, if f is continuous with respect to the first variable and Borel measurable of class α with respect to the second variable.

At first, we recall the definitions of the multiplicative and additive Borel classes. Given a topological space X let $\mathcal{A}_0(X)$ and $\mathcal{M}_0(X)$ denote the classes of all open and all closed subsets of X, respectively. Assuming that for a countable ordinal α the classes $\mathcal{A}_\beta(X)$, $\mathcal{M}_\beta(X)$, $\beta<\alpha$, are already defined, let $\mathcal{A}_\alpha(X) = \{\bigcup_{n=1}^\infty M_n : M_n \in \bigcup_{\beta<\alpha}\mathcal{M}_\beta(X)$ for all $n\in\mathbb{N}\}$ and $\mathcal{M}_\alpha(X) = \{M \subset X : X\setminus M \in \mathcal{A}_\alpha(X)\}$.

We say that a function $f\colon X\to Y$ between topological spaces is Borel measurable of class α if the preimage $f^{-1}(U)$ of any open set $U\subset Y$ belongs to the additive class $\mathcal{A}_\alpha(X)$ (of course, this is equivalent to saying that the preimage $f^{-1}(F)$ of any closed subset $F\subset Y$ belongs to the multiplicative class $\mathcal{M}_\alpha(X)$).

The set of all Borel measurable functions $f\colon X\to Y$ of class α is denoted by $H_\alpha(X,Y)$. For topological spaces X,Y,Z and a countable ordinal α let $CH_\alpha(X\times Y,Z)$ denote the set of all functions $f\colon X\times Y\to Z$ which are continuous with respect to the first variable and are Borel measurable of class α with respect to the second variable. The Kuratowski-Montgomery Theorem states that $CH_\alpha(X\times Y,Z)\subset H_{\alpha+1}(X\times Y,Z)$ for metrizable spaces X,Y,Z and a countable ordinal α.

A subset A of a topological space X is called a *$\overline{G}_\delta$-set* if $A=\bigcap_{n\in\mathbb{N}}U_n=\bigcap_{n\in\mathbb{N}}\bar{U}_n$ for some open sets $U_n\subset X$, $n\in\mathbb{N}$. Observe that every $\overline{G}_\delta$-set is a closed G_δ-set and every closed G_δ-set in a normal space is a $\overline{G}_\delta$-set. Consequently, every closed subset of a perfectly normal space is a $\overline{G}_\delta$-set.

4.1. Theorem. *The inclusion $CH_\alpha(X\times Y,Z)\subset H_{\alpha+1}(X\times Y,Z)$ holds for any countable ordinal α, any metrically quarter-stratifiable space X, any topological space Y, and any topological space Z whose every closed subset is a $\overline{G}_\delta$-set.*

Proof. Using Theorem 2.2, fix a weaker metrizable topology τ_m on the metrically quarter-stratifiable space X, a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of τ_m-open covers of X and a sequence $\{s_n\colon\mathcal{U}_n\to X\}_{n\in\mathbb{N}}$ of functions such that $s_n(U_n)\to x$ if $x\in U_n\in\mathcal{U}_n$, $n\in\mathbb{N}$. According to the classical Stone Theorem [21], we may assume that each cover $\mathcal{U}_n$ is locally finite and σ-discrete.

Fix any function $f\in CH_\alpha(X\times Y,Z)$. To show that $f\in H_{\alpha+1}(X\times Y,Z)$ we have to verify that the preimage $f^{-1}(F)$ of any closed subset $F\subset Z$ belongs to the class $\mathcal{M}_{\alpha+1}(X\times Y)$.

The set F, being $\overline{G}_\delta$-set, can be written as $F=\bigcap_{n=1}^\infty\overline{W}_n$, where $W_n\subset W_{n-1}$ are open neighborhoods of F. For every $n\in\mathbb{N}$ and $U\in\mathcal{U}_n$ let $F_U\colon Y\to Z$ be the map defined by $f_U(y)=f(s_n(U),y)$ for $y\in Y$. It is clear that $f_U\in H_\alpha(Y,Z)$ which yields $f_U^{-1}(W_m)\in\mathcal{A}_\alpha(Y)$ for every m. Since the set U is functionally open in X, we get $U\times f_U^{-1}(W_m)\in\mathcal{A}_\alpha(X\times Y)$, $m\in\mathbb{N}$. Using this observation and the σ-discreteness of the cover $\mathcal{U}_n$, show that $A_{n,m}=\bigcup\limits_{U\in\mathcal{U}_n}U\times f_U^{-1}(W_n)\in\mathcal{A}_\alpha(X\times Y)$.

Now to get $f^{-1}(F)\in\mathcal{M}_{\alpha+1}(X\times Y)$ it suffices to verify that $f^{-1}(F)=\bigcap_{m=1}^\infty\bigcup_{n\ge m}A_{n,m}$.

To verify the inclusion $f^{-1}(F)\subset\bigcap_{m\in\mathbb{N}}\bigcup_{n\ge m}A_{n,m}$, fix any $m\in\mathbb{N}$ and $(x,y)\in X\times Y$ with $f(x,y)\in F$. Let $U_n\in\mathcal{U}_n$, $n\in\mathbb{N}$, be a sequence of open sets containing the point x.

Since $s_n(U_n) \to x$ and f is continuous with respect to the first variable, we may find $k \geq m$ with $f(s_k(U_k), y) = f_{U_k}(y) \in W_m$. Thus

$$(x, y) \in U_k \times f_{U_k}^{-1}(W_m) \subset A_{k,m} \subset \bigcup_{n \geq m} A_{n,m}$$

for every m.

Next, assume that $(x, y) \notin f^{-1}(F)$. Then $f(x, y) \notin F$ and there is $k \in \mathbb{N}$ with $\overline{W}_k \not\ni f(x, y)$. By the choice of the covers $\mathcal{U}_n$ and the continuity of f with respect to the first variable, there is $m \geq k$ such that $f(s_n(U_n), y) \notin \overline{W}_k$ for any $n \geq m$ and $x \in U_n \in \mathcal{U}_n$. Since $W_m \subset W_k$, we get $(x, y) \notin \bigcup_{n \geq m} A_{n,m}$. □

4.2. Remark. The inclusion $CC(X \times Y, Z) \subset H_1(X \times Y, Z)$ is not true without restrictions on spaces X, Y, Z. If $X = Y = \{\infty\} \cup \Gamma$ is the one-point compactification of un uncountable discrete space Γ, then the function $f \colon X \times Y \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 1, & \text{if } x = y \in \Gamma; \\ 0, & \text{otherwise} \end{cases}$$

is separately continuous but does not belong to the class $H_1(X \times Y, \mathbb{R})$.

5. A generalization of the Rudin theorem

In [19] W.Rudin proved that every separately continuous function $f \colon X \times Y \to Z$ defined on the product of a metrizable space X and a topological space Y and acting into a locally convex topological vector space Z belongs to the first Baire class $B_1(X \times Y, Z)$. In [15] it was proved that the metrizability of X Rudin Theorem can be replaced by the paracompactness and the σ-metrizability; moreover, if the space X is finite-dimensional then the local convexity of Z is superfluous, see [11].

In this section we shall prove that the Rudin Theorem is still valid if X is replaced by any metrically quarter-stratifiable space and Z by a locally convex equiconnected space Z.

We remind that an *equiconnected space* is a pair (Z, λ) consisting of a topological space Z and a continuous map $\lambda \colon Z \times Z \times [0, 1] \to Z$ such that $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$, and $\lambda(x, x, t) = x$ for every $x, y \in X$, $t \in [0, 1]$. For a subset $A \subset Z$ of an equiconnected space (Z, λ) let $\lambda^0(A) = A$ and $\lambda^n(A) = \lambda(\lambda^{n-1}(A) \times A \times [0, 1])$ for $n \geq 1$. Let also $\lambda^\infty(A) = \bigcup_{n \in \mathbb{N}} \lambda^n(A)$. An equiconnected space (Z, λ) is called *locally convex* if for every point $z \in Z$ and a neighborhood $O(z) \subset Z$ of z there is a neighborhood $U \subset Z$ of z with $\lambda^\infty(U) \subset O(z)$.

Equiconnected spaces are tightly connected with absolute extensors. We remind that a space Z is an *absolute extensors for a class* $\mathcal{C}$ of topological spaces if every continuous map $f \colon B \to Z$ from a closed subspace B of a space $C \in \mathcal{C}$ admits a continuous extension $\bar{f} \colon C \to Z$ over all C. It is known that each equiconnected space is an absolute extensor for strongly countable-dimensional stratifiable spaces, while each locally convex equiconnected space is an absolute extensor for the class of stratifiable spaces. Moreover, a stratifiable space Z is an absolute extensor for stratifiable spaces if and only if Z is admits an equiconnecting function λ turning Z into a locally convex equiconnected spaces, see [3], [4], [10].

Obvious examples of (locally convex) equiconnected spaces are convex subsets of (locally convex) linear topological spaces and their retracts. Also every contractible topological group G is an equiconnected space: an equiconnecting function λ on G can be defined by $\lambda(x,y,t) = h_t(xy^{-1})\cdot h_t(e)^{-1}\cdot y$, where e is the neutral element of G and $\{h_t\colon G\to G\}_{t\in[0,1]}$ is a contraction of G with $h_0 = \mathrm{id}$ and $h_1(G) = \{e\}$. Let us recall that a topological space X is *contractible* if it admits a homotopy $h\colon X\times[0,1]\to X$ such that $h(x,0) = x$ and $h(x,1) = *$ for all $x\in X$ and some fixed point $*\in X$.

Given topological spaces X and Y denote by $C_p(X,Y)$ the subspace of the Tychonov product Y^X, consisting of all continuous functions from X to Y. Let $B_0(X,Y) = C_p(X,Y)$ and by transfinite induction for every ordinal $\alpha > 0$ define the Baire class $B_\alpha(X,Y)$ to be the sequential closure of the set $\bigcup_{\xi<\alpha} B_\xi(X,Y)$ in Y^X.

Now let us look at the Rudin theorem from the following point of view. Actually, this theorem states that every continuous function $f\colon X\to C_p(Y,Z)$ from a metrizable space X is a pointwise limit of "jointly continuous" functions. This observation leads to the following question: *Under which conditions every continuous map $f\colon X\to Z$ is a pointwise limit of some "nice" functions, and what should be understood under a "nice" function?*

In case of an equiconnected space (Z,λ) under "nice" functions we shall understand so-called piecewise-linear functions which are defined as follows. Let $PL(X,Z)$ be the smallest non-empty subset of $C_p(X,Z)$ satisfying the conditions:

(1) for every point $z_0\in Z$ and functions $f\in PL(X,Z)$, $\alpha\in C(X,[0,1])$ the function $g(x) = \lambda(f(x), z_0, \alpha(x))$ belongs to $PL(X,Z)$;

(2) a function $f\colon X\to Z$ belongs to $PL(X,Z)$ if there is an open cover $\mathcal{U}$ of X such that for every $U\in\mathcal{U}$ there is a function $g\in PL(X,Z)$ with $f|U = g|U$.

Equivalently, the set $PL(X,Z)$ can be defined constructively as the set of functions $f\colon X\to Z$ for which there is an open cover $\mathcal{U}$ of X such that for every $U\in\mathcal{U}$ there are points $z_0,\dots,z_n\in Z$ and functions $\alpha_1,\dots,a_n\in C(X,[0,1])$ such that $f|U = f_n|U$, where $f_0\equiv z_0$ and $f_{i+1}(x) = \lambda(f_i(x), z_i, \alpha_i(x))$ for $0\le i<n$.

By $PL_{(1)}(X,Z)$ we denote the sequential closure of $PL(X,Z)$ in Z^X.

We recall that a topological space X is defined to be *strongly countable-dimensional* if X can be represented as the countable union $\bigcup_{n=1}^\infty X_n$ of closed subspaces with $\dim X_n<\infty$ for all n. According to [6, 5.1.10] a paracompact space X is strongly countable-dimensional if and only if X can be represented as the countable union $X = \bigcup_{n=1}^\infty X_n$ of closed subspaces such that every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ with the property $\operatorname{ord}\mathcal{V}|X_n\le n$ for every $n\in\mathbb{N}$ (that is, every point $x\in X_n$ belongs to at most n elements of the cover $\mathcal{V}$).

5.1. Theorem. *Let X be a metrically quarter-stratifiable T_1-space and (Z,λ) be an equiconnected space. If X is paracompact and strongly countable-dimensional or Z is locally convex, then $C_p(X,Z)\subset PL_{(1)}(X,Z)$.*

Proof. Fix any function $f\in C_p(X,Z)$. Using Theorem 2.2, find a weaker paracompact topology τ_p on X, a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of τ_p-open covers of X and a sequence $\{s_n\colon\mathcal{U}_n\to X\}_{n\in\mathbb{N}}$ of functions such that $(x\in U_n\in\mathcal{U}_n,\ n\in\mathbb{N})\Rightarrow(s_n(U_n)\to x)$. Since the space (X,τ_p) is paracompact, we may assume that each cover $\mathcal{U}_n$ is locally finite and contains no empty set.

If the space X is paracompact and strongly countable-dimensional, then according to the above-mentioned Theorem 5.1.10 of [6], we may additionally assume that X is represented as the union $X = \bigcup_{m=1}^\infty X_m$ of closed subsets such that $\operatorname{ord}\mathcal{V}_n|X_m\le m$ for every $n,m\in\mathbb{N}$.

For every $n\in\mathbb{N}$ take any partition of unity $\{\alpha_{n,U}\colon X\to[0,1]\}_{U\in\mathcal{U}_n}$ subordinated to $\mathcal{U}_n$, that is $\alpha_{n,U}^{-1}(0,1]\subset U$ for each $U\in\mathcal{U}_n$ and $\sum_{U\in\mathcal{U}_n}\alpha_{n,U}\equiv 1$.

Let $\xi_n : X \to N(\mathcal{U}_n)$, $\xi_n : x \mapsto \sum_{U\in\mathcal{U}_n} \alpha_{n,U}(x)\cdot U$, be the canonical map into the nerve of the cover $\mathcal{U}_n$, see [7, VI. §3]. Next, we construct a map $\eta_n : N(\mathcal{U}_n) \to Z$ as follows. Denote by $N^{(k)}(\mathcal{U}_n)$ the k-skeleton of $N(\mathcal{U}_n)$, where $k \geq 0$. The map η_n will be defined by induction. Let $\eta_n(U) = f(s_n(U))$ for every $U \in N^{(0)}(\mathcal{U}_n) = \mathcal{U}_n$. Suppose that η_n is already defined on the k-skeleton $N^{(k)}(\mathcal{U}_n)$ of $N(\mathcal{U}_n)$. We shall extend η_n onto $N^{(k+1)}(\mathcal{U}_n)$. Take any point $x \in N^{(k+1)}(\mathcal{U}_n)\setminus N^k(\mathcal{U}_n)$ and find a k-dimensional simplex $\sigma \ni x$. Fix any vertex v of σ. The point x can be uniquely written as $x = tv + (1-t)y$, where $t \in [0,1]$ and $y \in \sigma \cap N^{(k)}(\mathcal{U}_n)$. Let $\eta_n(x) = \lambda(\eta_n(y), \eta_n(v), t)$ and observe that the so-extended map $\eta_n : N^{(k+1)}(\mathcal{U}_n) \to Z$ is continuous.

The inductive construction yields a continuous map $\eta_n : N(\mathcal{U}_n) \to Z$ which has the following property: $\eta_n(\sigma) \subset \lambda^\infty(\eta_n(\sigma^{(0)}) = \lambda^\infty\big(\bigcup_{U\in\sigma^{(0)}} f(s_n(U))\big)$ for every simplex σ of $N(\mathcal{U}_n)$. Observe also that $\eta_n \circ \xi_n \in PL(X, Z)$.

We claim that $\eta_n \circ \xi_n \to f$, provided X is strongly countable-dimensional or Z is locally convex. To show this, fix any point $x \in X$ and a neighborhood $O(f(x)) \subset Z$ of $f(x)$.

First, we consider the case when the space X is paracompact and strongly countable-dimensional. In this case $x \in X_m$ for some $m \in \mathbb{N}$ and $\operatorname{ord}\mathcal{U}_n|X_m \leq m$ for each n, which implies $\xi_n(x) \in N^{(m)}(\mathcal{U}_n)$ for every n. Using the continuity of the map λ, find a neighborhood $O_1 \subset Z$ of $f(x)$ such that $\lambda^m(O_1) \subset O(f(x))$. By the choice of the sequences $\{\mathcal{U}_n\}$ and $\{s_n\}$, there is $n_0 \in \mathbb{N}$ such that $s_n(U_n) \in f^{-1}(O_1)$ for every $n \geq n_0$ and $U_n \in \mathcal{U}_n$ with $U_n \ni x$. It follows from the construction of the map η_n that $\eta_n \circ \xi_n(x) \in \lambda^m(O_1) \subset O(f(x))$ for every $n \geq n_0$. Thus $\eta_n \circ \xi_n \to f$.

Now suppose that the equiconnected space (Z, λ) is locally convex. Then we may find a neighborhood $O_1 \subset Z$ of $f(x)$ with $\lambda^\infty(O_1) \subset O(f(x))$ and a number $n_0 \in \mathbb{N}$ such that $s_n(U_n) \in f^{-1}(O_1)$ for every $n \geq n_0$ and $x \in U_n \in \mathcal{U}_n$. It follows from the construction of the map η_n that $\eta_n \circ \xi_n(x) \subset \lambda^\infty(O_1) \subset O(f(x))$ for every $n \geq n_0$. Thus in both cases the function sequence $\{\eta_n \circ \xi_n\}_{n=1}^\infty \subset PL(X, Z)$ tends to f, which proves that $f \in PL_{(1)}(X, Z)$. □

Now we use Theorem 5.1 to generalize the Rudin Theorem as well as the results of [15], [16], [12], [11]. By $CC(X \times Y, Z)$ the set of separately continuous functions $X \times Y \to Z$ is denoted.

5.2. Corollary. *Let X be a metrically quarter-stratifiable space, Y be a topological space, and (Z, λ) be an equiconnected space. If X is paracompact and strongly countable-dimensional or Z is locally convex, then $CC(X \times Y, Z) \subset B_1(X \times Y, Z)$.*

Proof. In obvious way the equiconnecting function λ induces an equiconnecting function on $C_p(Y, Z)$. Moreover, the equiconnected space $C_p(Y, Z)$ is locally convex if so is Z. Hence, we may apply Theorem 5.1 to conclude that $C_p(X, C_p(Y, Z)) \subset PL_{(1)}(X, C_p(Y, Z))$. Observe that the space $C_p(X, C_p(Y, Z))$ may be identified with $CC(X \times Y, Z)$, while every element of $PL(X, C_p(Y, Z))$ is jointly continuous as a function $X \times Y \to Z$. This yields that $CC(X \times Y, Z) \subset B_1(X \times Y, Z)$. □

For an ordinal α and topological spaces X, Y, Z denote by $CB_\alpha(X \times Y, Z)$ the set of functions $f : X \times Y \to Z$ such that for every $x_0 \in X$ and $y_0 \in Y$ we have $f(\cdot, y_0) \in C(X, Z)$ and $f(x_0, \cdot) \in B_\alpha(Y, Z)$. Thus $CB_0(X \times Y, Z) = CC(X \times Y, Z)$. Generalizing the Rudin Theorem, V. K. Maslyuchenko et al. [15] proved that $CB_\alpha(X \times Y, Z) \subset B_{\alpha+1}(X \times Y, Z)$ for any countable ordinal α, any metrizable space X, any topological space Y and any locally convex space Z, see [15]. Below we have a further generalization.

5.3. Theorem. *Let X be a metrically quarter-stratifiable space, Y be a topological space and Z be a contractible space. Then $CB_\alpha(X\times Y,Z)\subset B_{\alpha+1}(X\times Y,Z)$ for every countable ordinal $\alpha>0$.*

Proof. Let $\alpha\ge 1$ be a countable ordinal and $f\in CB_\alpha(X\times Y,Z)$. Fix a non-decreasing sequence (α_n) of ordinals with $\alpha=\sup_n\alpha_n+1$ and let $h\colon Z\times[0,1]\to Z$ be a contraction of Z with $h(z,0)=z$ and $h(z,1)=*$ for all $z\in Z$ and some fixed point $*\in Z$.

Using Theorem 2.2, find a weaker metrizable topology τ_m on X, a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of τ_m-open cover of X and a sequence $\{s_n\colon\mathcal{U}_n\to X\}_{n\in\mathbb{N}}$ of maps such that $s_n(U_n)\to x$ for every $x\in U_n\in\mathcal{U}_n$, $n\in\mathbb{N}$. Using the paracompactness of (X,τ_m), for every $n\in\mathbb{N}$ find a partition of unity $\{\alpha_{n,U}\colon(X,\tau_m)\to[0,1]\}_{U\in\mathcal{U}_n}$ subordinated to $\mathcal{U}_n$, that is, $\alpha_{n,U}^{-1}(0,1]\subset U$ for $U\in\mathcal{U}_n$. Then $\{\alpha_{n,U}^{-1}(0,1]\}_{U\in\mathcal{U}_n}$ is a locally finite cover of X by functionally open sets.

Fix any well-ordering $\le$ on the set $\mathcal{U}_n$. For every $U\in\mathcal{U}_n$ and $m\in\mathbb{N}$ consider the closed set $F_{m,U}=\alpha_{n,U}^{-1}[1/m,1]\setminus\bigcup_{V<U}\alpha_{n,V}^{-1}(0,1]$. Clearly, $\{F_{m,U}\}_{U\in\mathcal{U}_n}$ is a discrete collection of closed subsets of (X,τ_m) for every $m\in\mathbb{N}$. Using the collective normality of (X,τ_m), for every $(m,U)\in\mathbb{N}\times\mathcal{U}_n$ we may find a continuous function $\beta_{m,U}\colon(X,\tau_m)\to[0,1]$ such that $\beta_{m,U}^{-1}(1)\supset F_{m,U}$ and $\{\beta_{m,U}^{-1}(0,1]\}_{U\in\mathcal{U}_n}$ is a discrete collection in (X,τ_m) for every $m\in\mathbb{N}$.

For every $U\in\mathcal{U}_n$ find a function sequence $\{f_{m,U}\}_{m\in\mathbb{N}}\subset\bigcup_{\xi<\alpha}B_\xi(Y,Z)$ with $\lim_{m\to\infty}f_{m,U}(y)=f(s_n(U),y)$ for every $y\in Y$. Without loss of generality, we may assume that $f_m\in B_{\alpha_m}(Y,Z)$ for every m. Then the function $g_{m,n}\colon X\times Y\to Z$ defined by the formula

$$g_{n,m}(x,y)=\begin{cases}h(f_{m,U}(y),1-\beta_{m,U}(x)), & \text{if } \beta_{m,U}(x)>0 \text{ for some } U\in\mathcal{U}_n;\\ *, & \text{otherwise}\end{cases}$$

belongs to the class $B_{\alpha_m}(X\times Y,Z)$ for every $m,n\in\mathbb{N}$. Let us show that the limit $\lim_{m\to\infty}g_{m,n}(x,y)$ exists for every $(x,y)\in X\times Y$ and $n\in\mathbb{N}$. Indeed, given $(x,y)\in(X,Y)$ and $n\in\mathbb{N}$ let $U_n=\min\{U\in\mathcal{U}_n:\alpha_{n,U}(x)>0\}$. Then $x\in F_{m,U_n}$ for all sufficiently large m and $g_{m,n}(x)=h(f_{m,U_n}(y),1-\beta_{m,U_n}(x))=f_{m,U_n}(y)\to f(s_n(U_n),y)$ as $m\to\infty$. Thus $g_n(x,y)=\lim_{m\to\infty}g_{m,n}(x,y)=f(s_n(U_n),y)$ exists and $g_n\in B_\alpha(X\times Y,Z)$. Next, since $s_n(U_n)\to x$, we get $f(s_n(U_n),y)\to f(x,y)$ which yields that $f=\lim_{n\to\infty}g_n$ and $f\in B_{\alpha+1}(X\times Y,Z)$. □

The following corollary generalizes a result from [15].

5.4. Corollary. *Let $X_1,\dots,X_n$ be metrically quarter-stratifiable spaces, Y be a topological space, and (Z,λ) be an equiconnected space. If X_1 is paracompact and strongly countable-dimensional or Z is locally convex, then every separately continuous function $f\colon X_1\times\dots\times X_n\times Y\to Z$ belongs to the n-th Baire class.*

We do not know if the conditions on X_1 or Z are essential.

5.5. Question. *Does every separately continuous function $f\colon X\times Y\to Z$ defined on the product of metrizable compacta and acting into a linear metric space belong to the first Baire class?*

5.6. Remark. Actually, according to [15], the Rudin Theorem holds in a more strong form: $\bar{C}C(X\times Y,Z)\subset B_1(X\times Y,Z)$ for any metrizable space X, topological space Y, and a locally convex space Z, where $\bar{C}C(X\times Y,Z)$ stands for the set of functions for which there is a dense subset $D\subset X$ such that for every $x_0\in D$ and every $y_0\in Y$ the functions $f(x_0,\cdot)$ and $f(\cdot,y_0)$ are continuous. It is interesting to remark that the metrizability of X is essential in this stronger form of the Rudin Theorem and can not be replaced by stratifiability; namely $\bar{C}C(A \times [0,1], \mathbb{R}) \not\subset B_1(A \times [0,1], \mathbb{R})$, where A is the Arens fan.

We recall the definition of the *Arens fan* A, see [5, 1.6.19]. Consider the following points of the real line: $a_0 = 0$, $a_n = \frac{1}{n}$ and $a_{nm} = \frac{1}{n} + \frac{1}{m}$, where $1 \le n^2 \le m$. Let $A_k = \{a_n : n \ge 0\} \cup \{a_{nm} : k \le n \le n^2 \le m\}$ for $k \in \mathbb{N}$. On the union $A = \bigcup_{k\in\mathbb{N}} A_k$ consider the strongest topology inducing the original topology on each compactum A_k. Clearly, the so-defined space A is countable and stratifiable. The space A is known as an example of a sequential space which is not a Fréchet-Urysohn space.

5.7. Example. *$\bar{C}C(A \times [0,1], \mathbb{R}) \not\subset B_1(A \times [0,1], \mathbb{R})$ for the Arens fan A.*

Proof. We shall construct a function $f \in \bar{C}C(A \times [0,1], \mathbb{R})$ which is not of the first Baire class. Take any function $f_0 \in B_2([0,1], \mathbb{R}) \setminus B_1([0,1], \mathbb{R})$. Write $f_0 = \lim_{n\to\infty} f_n$ where $\{f_n\}_{n\in\mathbb{N}} \subset B_1([0,1], \mathbb{R})$. In its turn, represent each f_n as a pointwise limit $f_n = \lim_{m\to\infty} f_{nm}$ of continuous functions. Now consider the map $f\colon A \times [0,1] \to \mathbb{R}$:

$$f(a, y) = \begin{cases} f_0(y), & \text{if } a = a_0; \\ f_n(y), & \text{if } a = a_n; \\ f_{nm}(y), & \text{if } a = a_{nm}. \end{cases}$$

It is easy to see that $f \in \bar{C}C(A \times [0,1], \mathbb{R})$ but $f \notin B_1(A \times [0,1], \mathbb{R})$. □

6. Rudin spaces

In light of the mentioned generalizations of the Rudin Theorem it is natural to introduce the following

6.1. Definition. A topological space X is defined to be *Rudin* if for arbitrary topological space Y every separately continuous function $f\colon X \times Y \to \mathbb{R}$ belongs to the first Baire class.

In the following theorem we collect all fact concerning Rudin spaces we know at the moment. Let us recall that a subspace Y of a topological space X is *t-embedded*, if there is a continuous extender $E\colon C_p(Y) \to C_p(X)$, that is a map E such that $E(f)|Y = f$ for every $f \in C_p(Y)$, see [1, III.§2.1]. Here $C_p(X) = C_p(X, \mathbb{R})$.

6.2. Theorem.

(1) *A space X is Rudin if and only if the calculation map $c_X : X \times C_p(X) \to \mathbb{R}$, $c_X\colon (x, f) \mapsto f(x)$, is of the first Baire class.*

(2) *A space X is Rudin if the space $C_p(X)$ is Rudin.*

(3) *Every metrically quarter-stratifiable space is Rudin.*

(4) *If a space X is Rudin, then each t-embedded subspace of X is Rudin.*

(5) *A space X is Rudin if $X = A \cup B$, where A, B are Rudin closed subspaces of X and A is a functionally closed retract of X.*

(6) *If a Tychonov space X is Rudin, then $d(X) \le l(X)$. Consequently, each compact Rudin space is separable.*

Proof. 1) The "only if" part of the first statement is trivial. To prove the "if" part, assume that the evaluation map $c\colon X \times C_p(X) \to \mathbb{R}$ is of the first Baire class. Observe that every separately continuous map $f\colon X \times Y \to \mathbb{R}$ can be seen as a continuous map $F\colon Y \to C_p(X)$ such that $f(x,y) = c(x, F(y))$ for $(x,y) \in X \times Y$. Now it is clear that f is of the first Baire class.

2) The second statement follows immediately from the first one.

3) According to Corollary 4.2, each metrically quarter-stratifiable space is Rudin.

4) Suppose X is a Rudin space and Y a t-embedded subspace of X. Let $E\colon C_p(Y) \to C_p(X)$ be the corresponding extender. Let $c_X\colon X \times C_p(X) \to \mathbb{R}$, $c_Y\colon Y \times C_p(Y) \to \mathbb{R}$ be the evaluation functions for the spaces X and Y, respectively. Taking into account that the function c_X is of the first Baire class and $c_Y(y,f) = c_X(y, E(f))$ for every $(y,f) \in Y \times C_p(Y)$ we conclude that c_Y is a function of the first Baire class either.

5) Suppose $X = A \cup B$, where A, B are Rudin closed subspaces of X and A is a functionally closed retract in X. Let $r\colon X \to A$ be a retraction and $\psi\colon X \to [0,1]$ a map with $\psi^{-1}(0) = A$. Since the spaces A, B are Rudin, there are sequences of maps $\alpha_n\colon A \times C_p(A) \to \mathbb{R}$ and $\beta_n\colon B \times C_p(B) \to \mathbb{R}$, $n \in \mathbb{N}$, tending to the evaluation functions c_A and c_B of the spaces A, B, respectively. For every $n \in \mathbb{N}$ consider the function $\psi_n = \min\{n\psi, 1\}$ and define the map $c_n\colon X \times C_p(X) \to \mathbb{R}$ by the formula

$$c_n(x,f) = \begin{cases} \alpha_n(x, f|A), & \text{if } x \in A; \\ (1-\psi_n(x))\alpha_n(r(x), f|A) + \psi_n(x)\beta_n(x, f|B), & \text{if } x \in B. \end{cases}$$

It is easy to see that the maps c_n are well-defined and continuous, and the sequence $\{c_n\}$ tends to the evaluation function of X.

6) Suppose X is a Rudin space. Let $\{c_n\}_{n\in\mathbb{N}} \subset C(X \times C_p(X), \mathbb{R})$ be a sequence of continuous functions tending to the evaluation function $c\colon X \times C_p(X) \to \mathbb{R}$. Then $c^{-1}(0) = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} c_m^{-1}(-1/n, 1/m)$ is a G_δ-set in $X \times C_p(X)$. Let $\{O_n\}_{n\in\mathbb{N}}$ be a sequence of open sets in $X \times C_p(X)$ with $\bigcap_{n=1} O_n = c^{-1}(0)$. By the continuity of the functions c_n, for every $x \in X$ and $n \in \mathbb{N}$ we may find a neighborhood $U_n(x)$ and a finite subset $F_n(x) \subset X$ such that $U_n(x) \times F_n(x)^\perp \subset O_n$, where $A^\perp = \{f \in C_p(X)\colon f|A \equiv 0\}$ for a subset $A \subset X$. For every $n \in \mathbb{N}$ choose a subset $X_n \subset X$ of size $|X_n| \leq l(X)$ with $\bigcup_{x\in X_n} U_n(x) = X$. We claim that $D = \cup\{F_n(x) : x \in X_n,\ n \in \mathbb{N}\}$ is a dense set in X. Otherwise, we would find a function $f \in C_p(X)$ with $f|D \equiv 0$ and $f(x) \neq 0$ for some $x \in X$. It is easy to see that $(x,f) \in \bigcap_{n\in\mathbb{N}} O_n$ but $(x,f) \notin c^{-1}(0)$, a contradiction, which shows that D is dense in X and $d(X) \leq |D| \leq l(X)$. □

There are many open question about the structure of Rudin spaces.

6.3. Question. *Is there a Rudin space which is not quarter-stratifiable? Is a (compact) space X Rudin, if $\{(x,f) : f(x) = 0\}$ is a G_δ-set in $X \times C_p(X)$? Is every (fragmentable, scattered or Rosenthal) compact Rudin space metrizable?*

Let us remark that according to [2, II.6.1] each Corson Rudin compactum, being separable, is metrizable. Concerning the last question let us remark that there are non-metrizable compact spaces X such that $CC(X \times Y, \mathbb{R}) \subset B_1(X \times Y, \mathbb{R})$ for every compact space Y. The following result belongs to G. Vera [22].

6.4. Theorem. *A compact space X has countable Souslin number if and only if for every compact space Y every separately continuous function $f\colon X \times Y \to \mathbb{R}$ is of the first Baire class.*

Proof. Suppose X has a countable Souslin number and $f\colon X \times Y \to \mathbb{R}$ is a separately continuous function which can be seen as a continuous map $F\colon Y \to C_p(X)$. Then the image $F(Y) \subset C_p(X)$ is a compact subset of the space $C_p(X)$ over a compactum with countable Souslin number. According to Theorem 14 [1, II.§2] all such compact subsets are metrizable. By the Rudin Theorem, the restriction of the separately continuous evaluation function $c\colon X \times C_p(X) \to \mathbb{R}$ onto $X \times F(Y)$ is of the first Baire class, which implies that the map $f(x,y) = c(x, F(y))$ is of the first Baire class either.

Now assume that the Souslin number of a compactum X is uncountable. Then there exists an uncountable family $\mathcal{U}$ of pairwise disjoint nonempty open sets in X. For every $U \in \mathcal{U}$ fix a continuous function $f_U\colon X \to [0,1]$ such that $f_U|X \setminus U \equiv 0$ and $\max f_U = 1$. Denote by 0 the origin of the vector space $C_p(X)$. It is easy to see that the set $Y = \{0\} \cup \{f_U : U \in \mathcal{U}\}$ in $C_p(X)$ is compact and the evaluation function $c\colon X \times Y \to \mathbb{R}$, $c\colon (x,f) \mapsto f(x)$, is separately continuous. Yet, since $c^{-1}(1/2,1]$ is not an F_σ-set in $X \times Y$, the function c is not of the first Baire class. In fact, one may show that $c \notin \bigcup_{\alpha<\omega_1} B_\alpha(X \times Y, Z)$, i.e., c is not Baire measurable. □

6.5. Remark. It follows from Theorems 6.2 and 6.4 that any non-separable compact space X with countable Souslin number is not Rudin but for every compact space Y every separately continuous function $f\colon X \times Y \to \mathbb{R}$ belongs to the first Baire class.

Finally let us prove one more result related to Baire classification of separately continuous functions.

6.6. Theorem. *Let X, Y, Z be compact Hausdorff spaces. If X is separable and Y has countable Souslin number, then every separately continuous function $f\colon X \times Y \times Z \to \mathbb{R}$ belongs to the second Baire class.*

Proof. Fix any separately continuous function $f\colon X \times Y \times Z \to \mathbb{R}$, which can be seen as a continuous map $F\colon X \to CC(Y \times Z, \mathbb{R})$ into the space of separately continuous functions endowed with the product topology. According to a recent result of S. Gulko and G. Sokolov [9] if the space Y has countable Souslin number, then every compactum in $CC(Y \times Z, \mathbb{R})$ is Corson. Now if X is separable and Y has countable Souslin number, then $F(X) \subset CC(Y \times Z, \mathbb{R})$ is a separable Corson compactum. Since the separable Corson compacta are metrizable, see [2, II.6.1], we conclude that the compactum $F(X)$ is metrizable. By [15] (see also Corollary 5.4), the evaluation map $c\colon F(X) \times Y \times X \to \mathbb{R}$, $c\colon (g,y,z) \mapsto g(y,z)$, being separately continuous, belongs to the second Baire class. Then the function f belongs to the second Baire class because $f(x,y,z) = c(F(x), y, z)$. □

6.6. Remark. The class of Rudin spaces includes the class of Lebesgue spaces introduced by O. Sobchuk in [20].

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Faculty of Mechanics and Mathematics, Lviv National University,
Universytetska 1, Lviv, 79000, Ukraine
tbanakh@franko.lviv.ua

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