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N. A. Malaksiano*

ON THE SUPERPOSITIONS WITH FUNCTIONS FROM GEHRING'S, MUCKENHOUPT'S AND RELATED CLASSES

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A necessary and sufficient condition for a monotone external function such that the superposition with an arbitrary function from the fixed Gehring class belongs to the fixed Muckenhoupt class is found. The analogous questions for the limiting cases of Gehring and Muckenhoupt classes are solved as well.

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Для монотонности внешней функции найдено необходимое и достаточное условие такое, что суперпозици с произвольной функцией из фиксированого класса Геринга принадлежит фиксированому классу Макенхаупта. Решены также подобные вопросы для предельных случаев класса Геринга и Макенхаупта.

Introduction. We will deal only with non-negative measurable functions on a cube $Q_0 \subset \mathbb{R}^n$ and cubes, whose edges are parallel to the coordinate axes. Let $|E|$ denote the Lebesgue measure of a subset $E \subset Q_0$ and $f_E \equiv \fint_E f(x)dx \equiv \frac{1}{|E|}\int_E f(x)dx$.

A function f is said to belong to the Muckenhoupt class A_q $(q > 1)$ on the cube $Q_0 \subset \mathbb{R}^n$ if

$$A_q(f) \equiv A_q(f;Q_0) = \sup_{Q\subset Q_0} \left(\frac{1}{|Q|}\int_Q f(x)dx\right)\left(\frac{1}{|Q|}\int_Q f^{-\frac{1}{q-1}}(x)dx\right)^{q-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset Q_0$. Similarly, Gehring class G_p $(p > 1)$ on the cube $Q_0 \subset \mathbb{R}^n$ is defined by the following condition

$$G_p(f) \equiv G_p(f;Q_0) = \sup_{Q\subset Q_0} \left(\frac{1}{|Q|}\int_Q f^p(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q f(x)dx\right)^{-1} < \infty.$$

We will say that a function f belongs to the Muckenhoupt class A_1 on the cube $Q_0 \subset \mathbb{R}^n$ if

$$A_1(f) \equiv A_1(f;Q_0) = \sup_{Q\subset Q_0} \left(f_Q \cdot \left(\operatorname*{ess\,inf}_{x\in Q} f(x)\right)^{-1}\right) < \infty,$$

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Similarly, Gehring class G_∞ consists of functions f satisfying

$$G_\infty(f) \equiv G_\infty(f;Q_0) = \sup_{Q\subset Q_0}\left((f_Q)^{-1}\cdot \operatorname{ess\,sup}_{x\in Q} f(x)\right) < \infty.$$

We will say that a function f belongs to the Muckenhoupt class A_∞ if for some $\alpha > 0$

$$A_\infty(f) \equiv A_\infty(f;Q_0) = \sup_{Q\subset Q_0,\ E\subset Q} \frac{\int_E f(x)dx}{\int_Q f(x)dx}\cdot\left(\frac{|Q|}{|E|}\right)^\alpha < \infty,$$

where the supremum is taken over all cubes $Q \subset Q_0$ and all measurable sets $E \subset Q$.

It is obvious that if we take a function f from the Muckenhoupt (Gehring) class and multiply it by a constant or make a linear substitution of its argument, we will obtain a new function belonging to the same Muckenhoupt (Gehring) class. It is natural to ask the following questions:

1) For which function g does the condition $f \in A_q$ (or G_p) imply $f\cdot g \in A_{q_1}$ (or G_{p_1})?

2) For which non-linear substitution of the variable ψ does the condition $f \in A_q$ (or G_p) imply $f(\psi) \in A_{q_1}$ (or G_{p_1})?

3) For which external function φ does the condition $f \in A_q$ (or G_p) imply $\varphi(f) \in A_{q_1}$ (or G_{p_1})?

Questions 1) and 2) are answered by Johnson, Neugebauer and Buckley [1, 2, 3, 4, 5]. There is a number of papers, where the superpositions of the form $\varphi(f)$, with $f \in G_p$ (or A_q) and with a power function φ are studied (see, for example, [7]). But we don't know any work which answer the third question when φ is not a power function. The aim of this work is to find the necessary and sufficient conditions for a non-decreasing external function φ such that the superposition with an arbitrary function from the class X will belong to the class Y, where X is one of the classes A_∞, G_p, A_q, A_1 or G_∞ and Y is one of the classes A_∞, G_p or A_q. In particular the following theorems hold.

Theorem 1. *Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a non-decreasing function and $p_1 > 1$, $p_2 > 1$ be fixed. For any $f \in G_{p_1}$ the superposition $\varphi(f)$ belongs to the class G_{p_2} iff for every $\alpha > \frac{p_1}{p_2}$ there is K such that*

$$\varphi(ax) \le K\cdot x^\alpha\cdot\varphi(a) \quad \text{for all } a > 0 \text{ and } x \ge 1.$$

Theorem 2. *Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a non-decreasing function and $q_1 \ge q_2 > 1$ be fixed. For any $f \in A_{q_1}$ the superposition $\varphi(f)$ belongs to the class A_{q_2} iff for every $\alpha > \frac{q_2-1}{q_1-1}$ there is K such that*

$$\varphi(ax) \le K\cdot x^\alpha\cdot\varphi(a) \quad \text{for all } a > 0 \text{ and } x \ge 1.$$

Theorem 3. *Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a non-decreasing function and $q_2 \ge q_1 > 1$ be fixed. For any $f \in A_{q_1}$ the superposition $\varphi(f)$ belongs to the class A_{q_2} iff for every $\alpha > 1$ there is K such that*

$$\varphi(ax) \le K\cdot x^\alpha\cdot\varphi(a) \quad \text{for all } a > 0 \text{ and } x \ge 1.$$

Theorem 4. *Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a non-decreasing function and $p > 1$ be fixed. For any $f \in A_1$ the superposition $\varphi(f)$ belongs to the class G_p iff for every $a_0 > 0$ and every $\alpha > \frac{1}{p}$ there is K such that*

$$\varphi(ax) \le K\cdot x^\alpha\cdot\varphi(a) \quad \text{for all } a > a_0 \text{ and } x \ge 1.$$

These theorems can be proved by the same scheme so we will produce only the proof of Theorem 4.

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Preliminary results. We will use the non-increasing rearrangement of a function f on the set $E \subset \mathbb{R}^n$ which can be defined as follows

$$f^*(t) = \inf\{\alpha > 0 : |\{x \in E : f(x) > \alpha\}| \le t\}, \qquad 0 < t < |E|.$$

Clearly, the function f^* is non-increasing on $(0, |E|)$ and equimeasurable with f. We will need the following well-known property of rearrangement

$$\sup_{|E_1| \le t} \int_{E_1} f(x)dx = \int_0^t f^*(\tau)d\tau.$$

The following lemma easily follows from the definition of rearrangement.

Lemma 1. *If $\varphi : (0,\infty) \to (0,\infty)$ does not decrease, and function f is measurable on $E \subset \mathbb{R}^n$ then $\varphi(f^*(t)) = (\varphi(f))^*(t)$ a. e. on $(0, |E|)$.*

The following three lemmas can be found, for example, in [6].

Lemma 2. *For every $p > 1$ and $C > 1$ there is $M > 0$ such that $G_p(f; Q_0) \le C$ implies $(f \cdot \chi_Q)^*(|Q|/2) \ge M \cdot f_Q$ for all $Q \subset Q_0$.*

Lemma 3. *Let $p > 1$ be fixed. For every $C > 1$ there is $C_1 > 1$ such that $A_1(f; Q_0) \le C$ implies $G_p(f; Q_0) \le C_1$.*

Lemma 4. *If $f \in G_p$ then $f \in A_\infty$.*

The following theorem describes a fundamental property of G_p classes which plays an important role in many questions where Gehring classes are used.

Theorem A. (Gehring, [8]) *For every $p > 1$ and $C > 1$ there are $p_1 > p$ and $C_1 > 1$ such that $G_p(f) \le C$ implies $G_{p_1}(f) \le C_1$.*

Lemma 5. *If $f \subset G_p$ on the cube $Q_0 \subset \mathbb{R}^n$ then there are $\beta < \frac{1}{p}$ and C depending only on p, $G_p(f)$ and dimension n such that*

$$(f \cdot \chi_Q)^*(t) \le C \cdot \left(\frac{t}{|Q|}\right)^{-\beta} \cdot f_Q, \quad t \in (0, |Q|) \tag{1}$$

for every $Q \subset Q_0$. Conversely, if there are $\beta < \frac{1}{p}$ and C such that (1) holds for all cubes $Q \subset Q_0$, then $f \in G_p$.

In this lemma the first statement immediately follows from Lemma 4 and Theorem A, the second statement can be easily deduced from the definition of G_p classes.

Lemma 6. *Let φ be a function such that $f \in A_1$ implies $\varphi(f) \in G_p$ on $(0,1)$. Then for every fixed function $f_0 \in A_1$ and numbers $a_2 \ge a_1 > 0$ there is number $C = C(\varphi, f_0, p, a_1, a_2) > 1$, such that the condition $a_1 f_0(x) \le g(x) \le a_2 f_0(x)$, $x \in (0,1)$ implies $G_p(\varphi(g); (0,1)) \le C$.*

The important feature is that the number C does not depend on the choice of a function g in the lemma.

Proof. Let us assume the converse, i. e. there is a sequence of functions $\{g_k\}_{k=0}^{\infty}$ such that

$$a_1 f_0(x) \le g_k(x) \le a_2 f_0(x), \quad x \in (0,1), \ k \ge 0$$

and $G_p(\varphi(g_k);(0,1)) \ge k$. Define

$$\zeta(x) = \begin{cases} g_k\left(2^{k+1}x - 1\right), & x \in \left(2^{-k-1}, 2^{-k}\right), \quad k \in \{0,2,4,\dots\}; \\ g_k\left(2 - 2^{k+1}x\right), & x \in \left(2^{-k-1}, 2^{-k}\right), \quad k \in \{1,3,5,\dots\}. \end{cases}$$

For any even $k > 0$

$$G_p(\varphi(\zeta);(0,1)) \ge G_p\left(\varphi(\zeta); \left(2^{-k-1}, 2^{-k}\right)\right) =$$
$$= G_p\left(\varphi\left(g_k\left(2^{k+1}x - 1\right)\right); \left(2^{-k-1}, 2^{-k}\right)\right) = G_p(\varphi\left(g_k\right);(0,1)) \ge k,$$

consequently $\varphi(\zeta) \notin G_p$ on $(0,1)$.

By the definition of A_1 classes it is easy to verify that $\zeta \in A_1$ on $(0,1)$. Thus by the conditions of the lemma, this yields $\varphi(\zeta) \subset G_p$ on $(0,1)$. So we get the contradiction. □

Lemma 7. *Let φ be a function such that $f \in A_1$ implies $\varphi(f) \in G_p$ on $(0,1)$. Then for every fixed function $g \in A_1$ and number $a_0 > 0$ there is $C = C(\varphi, g, p, a_0) > 1$, such that $G_p(\varphi(a \cdot g);(0,1)) \le C$ for every $a \ge a_0$.*

Proof. Define

$$\zeta(x) = \begin{cases} a_0 2^{k/2} g\left(2^{k+1}x - 1\right), & x \in \left(2^{-k-1}, 2^{-k}\right), \quad k \in \{0,2,4,\dots\}; \\ a_0 2^{k/2} g\left(2 - 2^{k+1}x\right), & x \in \left(2^{-k-1}, 2^{-k}\right), \quad k \in \{1,3,5,\dots\}. \end{cases}$$

Using the definition, it is easy to prove that $\zeta \in A_1$ on $(0,1)$. Consequently, for every function $f_0 \in A_1$ there is C such that $G_p(\varphi(a_0 2^{k/2} \cdot g);(0,1)) \le C$ for every $k \in \{0,1,2,\dots\}$. To finish the proof it remains only to apply Lemma 6. □

Proof of Theorem 4.

Necessity. Let φ be such that $\varphi(f) \in G_p$ for every $f \in A_1$. Fix an *arbitrary* $\alpha > \frac{1}{p}$. Set $\gamma = \frac{1}{\alpha p}$. It is easy to verify that $f(x) \equiv x^{-\gamma} \in A_1$ on $(0,1)$. Fix an arbitrary $a_0 > 0$. By Lemma 7 there is C_1 such that $G_p(\varphi(a_1 \cdot f)) \le C_1$ for every $a_1 > 2^{-\gamma} a_0$. Fix an *arbitrary* $a_1 > 2^{-\gamma} a_0$. Lemma 5 implies the existence of $\beta < \frac{1}{p}$ and C_2, which depend only on p and C_1 such that for every $t \in (0,1)$

$$\left(\varphi(a_1 \cdot f)\right)^*(t) \le C_2 t^{-\beta} \int_0^1 \varphi(a_1 \cdot f(x))dx.$$

This inequality together with Lemmas 1 and 2 yield

$$\varphi(a_1 t^{-\gamma}) \overset{\text{a.e.}}{=} \left(\varphi(a_1 \cdot f)\right)^*(t) \le C_2 \cdot t^{-\beta} \int_0^1 \varphi(a_1 \cdot f(x))dx = C_2 \cdot \left(t^{-\gamma}\right)^{\frac{\beta}{\gamma}} \int_0^1 \varphi(a_1 \cdot f(x))dx \le$$

$$\leq C_2 \cdot \left(t^{-\gamma}\right)^{\frac{\beta}{\gamma}} C_3 \varphi\left(a_1 \cdot f\left(\frac{1}{2}\right)\right) \leq C_2 C_3 \cdot \left(t^{-\gamma}\right)^{\frac{1}{\gamma p}} \varphi\left(a_1 2^{\gamma}\right) = C_2 C_3 \cdot \left(t^{-\gamma}\right)^{\alpha} \varphi\left(a_1 2^{\gamma}\right).$$

Note that the number C_3 from Lemma 2 depends only on p and C_1, i.e. on α, p and φ. So we have proved that for every $t \in (0,1)$ and every $a_1 > 2^{-\gamma} a_0$

$$\varphi(a_1 t^{-\gamma}) \leq C_2 C_3 \cdot \left(t^{-\gamma}\right)^{\alpha} \varphi\left(a_1 \cdot 2^{\gamma}\right).$$

Redefine $x = t^{-\gamma} 2^{-\gamma}$, $a = a_1 2^{\gamma}$. We get

$$\varphi(ax) \leq C_2 C_3 \cdot 2^{\ \gamma\alpha} x^{\alpha} \varphi(a), \quad \text{for all } x \geq 1, \ a > a_0.$$

Define $K = C_2 C_3 \cdot 2^{\gamma\alpha}$. As we see from the proof, K depends only on p, α and φ.

The necessity is proved.

Sufficiency. Fix an arbitrary function $f \in A_1$. Fix an arbitrary cube $Q \subset Q_0$. By Lemmas 3 and 5 there are $\beta < 1$ and $C_1 > 0$ depending only on $A_1(f)$ such that

$$(f \cdot \chi_Q)^*(t) \leq C_1 \left(\frac{t}{|Q|}\right)^{-\beta} \cdot f_Q \ \text{ for every } t \in (0, |Q|). \tag{2}$$

Set $\alpha = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{p\beta}\right)$. It is obvious that $\frac{1}{p} < \alpha < \frac{1}{p\beta}$ and $\gamma \equiv \beta\alpha < \frac{1}{p}$. By the conditions of the theorem, for the chosen α and $a_0 \equiv \operatorname{ess\,inf}_{x \in Q_0} f(x)$ there is K such that $\varphi(ax) \leq K \cdot x^{\alpha} \varphi(a)$ for all $a > a_0$ and $x \geq 1$. Lemma 1 and inequality (2) for $t \leq \frac{|Q|}{2}$ imply

$$(\varphi(f) \cdot \chi_Q)^*(t) \overset{\text{a.e.}}{=} \varphi((f \cdot \chi_Q)^*(t)) = \varphi\left((f \cdot \chi_Q)^*\left(\frac{|Q|}{2}\right) \cdot \frac{(f \cdot \chi_Q)^*(t)}{(f \cdot \chi_Q)^*\left(\frac{|Q|}{2}\right)}\right) \leq$$

$$\leq K \left(\frac{(f \cdot \chi_Q)^*(t)}{(f \cdot \chi_Q)^*\left(\frac{|Q|}{2}\right)}\right)^{\alpha} \varphi\left((f \cdot \chi_Q)^*\left(\frac{|Q|}{2}\right)\right) \leq$$

$$\leq K \left(\frac{C_1 \left(\frac{t}{|Q|}\right)^{-\beta} f_Q}{C_2 f_Q}\right)^{\alpha} \varphi\left((f \cdot \chi_Q)^*\left(\frac{|Q|}{2}\right)\right) \leq$$

$$\leq K \cdot \left(\frac{C_1}{C_2}\right)^{\alpha} \cdot \left(\frac{t}{|Q|}\right)^{-\beta\alpha} \cdot 2 \fint_0^{|Q|} \left(\varphi(f) \cdot \chi_Q\right)^*(s) ds =$$

$$= 2K \cdot \left(\frac{C_1}{C_2}\right)^{\alpha} \cdot \left(\frac{t}{|Q|}\right)^{-\gamma} \fint_Q \varphi(f(x)) dx,$$

where the constant C_2, in view of Lemmas 3 and 2, depends only on $A_1(f)$. So

$$\left(\varphi(f) \cdot \chi_Q\right)^*(t) \leq 2K \cdot \left(\frac{C_1}{C_2}\right)^{\alpha} \left(\frac{t}{|Q|}\right)^{-\gamma} \fint_Q \varphi(f(x)) dx$$

for every $t \leq \frac{|Q|}{2}$. This implies

$$\left(\varphi(f) \cdot \chi_Q\right)^*(t) \leq C_3 \left(\frac{t}{|Q|}\right)^{-\gamma} \fint_Q \varphi(f(x)) dx$$

for every $t \in (0, |Q|)$, where $C_3 = 2K \cdot \left(\frac{C_1}{C_2}\right)^{\alpha} 2^{\gamma}$ depends only on p, β, φ and $A_1(f)$. Thus Lemma 5 implies $\varphi(f) \in G_p$.

The theorem is proved.

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Odessa National Mechnikov University

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