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QUASIRAY DECOMPOSITION OF INFINITE GRAPHS

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Let Gr be an infinite connected graph with a set of vertices V. A subset $Q \subseteq V$ is called a quasiray if there exists a bijection $f\colon w \to V$ such that $d(f(i), f(i+1)) \le 3$ for every $i \in w$, where d is a path metric on V. A quasiray decomposition is applied to partition an infinite group into countably many large subsets.

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Пусть Gr — бесконечный связный граф с множеством вершин V. Подмножество $A \subseteq V$ называется квазилучом, если существует биекция $f\colon w \to V$ такая, что $d(f(i), f(i+1)) \le 3$ для всех $i \in w$, где $d(x,y)$ — длина кратчайшего пути между x и y. Квазилучевая декомпозиция применяется для разбиения бесконечной группы на счетное число больших подмножеств.

Let $\mathrm{Gr} = (V, e)$ be a connected graph with the set of vertices V and the set of edges E. For any $x, y \in V$ denote by $d(x,y)$ the length of the shortest path between x and y. For every subset $A \subseteq V$, denote by $\mathrm{Gr}[A]$ the graph $(A, E(A))$, where $E(A) = (A \times A) \cap E$. The diameter $\operatorname{diam} A$ is the supremum of $\{d(x,y) : (x,y) \in A\}$.

Consider a finite connected graph $\mathrm{Gr} = (V, E)$, $|V| = n$ and suppose that $x, y \in V$, $x \ne y$, $(x, y) \in E$. By the quasicycle lemma [1, Lemma 3], there exists a bijection $f\colon \{1, 2, \dots, n\} \to V$ such that $f(1) = x$, $f(n) = y$ and $d\big(f(i), f(i+1)\big) \le 3$ for every $i \in \{1, 2, \dots n-1\}$. Applying this lemma and arguments from [1, §4], it is not difficult to prove the following two theorems.

Theorem 1. *Let $\mathrm{Gr} = (V, E)$ be a countable connected graph. Then there exists a partition $\mathbf{A}$ of V into infinite subsets such that for every $A \in \mathbf{A}$ the following conditions hold:*

(i) there exists a bijection $f\colon w \to A$ such that $d\big(f(i), f(i+1)\big) \le 3$ for every $i \in \omega$;

(ii) $\mathrm{Gr}[A]$ is connected.

Theorem 2. *Let $\mathrm{Gr} = (V, E)$ be an infinite connected graph. Then there exists a partition $\mathbf{A}$ of V into infinite subsets such that for every $A \in \mathbf{A}$ the following conditions hold:*

(i) there exists a bijection $f\colon w \to A$ such that $d\big(f(i), f(i+1)\big) \le 3$ for every $i \in \omega$;

(ii) there exists $x \in V$ such that $\mathrm{Gr}[A \cup \{x\}]$ is connected.

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Theorem 3. *Let $G = (V, E)$ be an infinite connected graph. Then there exists a countable family $\{\mathbf{F}_n : n \in \omega\}$ of partitions of V with the following properties:*

(i) $|F| = n+1$ *and* $\operatorname{diam} F \leq 3n$ *for every* $F \in \mathbf{F}_n$*;*

(ii) $\mathbf{F}_m$ *is an enlargement of* $\mathbf{F}_n$ *provided that* $n+1$ *is a divisor of* $m+1$*, i.e. every cell of* $\mathbf{F}_m$ *is a union of cells of* $\mathbf{F}_n$*.*

Proof. Use a partition $\mathbf{A}$ from Theorem 2. Take any $x, y \in V$ and say that x, y are in the same cell of $\mathbf{F}_n$ if and only if $x, y \in A$ for some $A \in \mathbf{A}$ and there exists $k \in \omega$ such that $k(n+1) \leq f^{-1}(x) \leq k(n+1)+n$, $k(n+1) \leq f^{-1}(y) \leq k(n+1)+n$, where f is a bijection from (i) of Theorem 2. □

Theorem 4. *Let G be an infinite group with the identity e and let S be a finite subset of G generating an infinite subgroup, $S = S^{-1}$, $e \in S$. Then there exists a countable family $\{\mathbf{F}_n : n \in \omega\}$ of partitions of G with the following properties:*

(i) $|F| = n+1$ *and* $xy^{-1} \in S^{3n}$ *for all* $x, y \in F$ *and every* $F \in \mathbf{F}_n$*;*

(ii) $\mathbf{F}_m$ *is an enlargement of* $\mathbf{F}_n$ *provided that* $n+1$ *is a divisor of* $m+1$*.*

Proof. Consider the Cayley graph $\mathrm{Cay} = (G, E)$ of G determined by S, where $(x, y) \in E$ if and only if $x \neq y$ and $xy^{-1} \in S$. Then apply Theorem 3 to each connected component of Cay. □

Theorem 5. *Let G be an infinite group with the identity e and let S be a finite subset of G generating an infinite subgroup, $S = S^{-1}$, $e \in S$. Then, for every $n \in \omega$, there exists a partition $G = X_0 \cup X_1 \cup \cdots \cup X_n$ such that $G = S^{3n}X_i = X_iS^{3n}$ for every $i \in \{0, 1, \ldots, n\}$.*

Proof. Consider the partition $\mathbf{F}_n$ from Theorem 4. By transversality argument [2, Theorem 7.4.4], there exists a partition $G = X_0 \cup X_1 \cup \cdots \cup X_n$ such that $|X_i \cap F| = |X_i \cap F^{-1}| = 1$ for every $i \in \{0, 1, \ldots, n\}$ and every $F \in \mathbf{F}_n$. □

Theorem 6. *Let G be an infinite group with the identity e and let S be a finite subset of G generating an infinite subgroup, $S = S^{-1}$, $e \in S$. Then there exists a partition $G = \bigcup_{n\in\omega} X_n$ such that $G = S^{3\times(2^n)}X_n = X_nS^{3\times(2^n)}$ for every $n \in \omega$.*

Proof. Apply transversality argument to the family of partitions $\{F_2^n : n \in \omega\}$ from Theorem 4. □

Theorem 7. *Let G be a group and let H be a finite subgroup of G, $|H| = n+1$. Then there exists a partition $G = X_0 \cup X_1 \cup \cdots \cup X_n$ such that $G = HX_i = X_iH$ for every $i \in \{0, 1, \ldots, n\}$.*

Proof. Apply Theorem 7.4.4 from [2]. □

Theorem 8. *Let G be an infinite group and let $H_0 \subset H_1 \cdots \subset H_n \subset \cdots$ be an increasing chain of its subgroups, $|H_0| > 1$. Then there exists a partition $G = \bigcup_{n\in\omega} X_n$ such that $G = F_nX_n = X_nF_n$ for every $n \in \omega$.*

Proof. Apply Theorem 7.4.4 from [2]. □

A subset X of a group G is called *large* if there exists a finite subset F such that $G = FX = XF$. In [3] Bella and Malykhin posed the following question:

Does every infinite group contain two disjoint large subsets?

Theorem 9. *Every infinite group G can be partitioned into countably many large subsets.*

Proof. If every finite subset of G generates a finite subgroup, then there exists an increasing chain $H_0 \subset H_1 \subset \dots \subset H_n \subset \dots$ of finite subgroups. Apply Theorem 8. Otherwise, there exists a finite subset $S \subset G$ which generates an infinite subgroup. Apply Theorem 6. $\square$

REFERENCES

1. Protasov I. V. *Morphisms of ball's structures of groups anf graphs,* Ukr. Matem. Zhurn. **53** (2002), №6.
2. Ore O. Theory of graphs, Amer. Math. Soc. Colloquium Publications, Vol. XXXVIII, 1962.
3. Bella A., Malykhin V. I. *Small and others subsets of a group,* Q and A in General Topology **11** (1999), 183–187.

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