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ON INVARIANT INCLUSION HYPERSPACES FOR ITERATED FUNCTION SYSTEMS

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A notion of invariant set of an iterated function systems is considered for the inclusion hyperspaces. It is shown that the open set condition is sufficient for the preservation of supports by the operation of passing to the invariant element of an iterated function system.

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Понятие инвариантного множества итерированной системы функций рассмотрено для гиперпространств включения. Показано что условие открытого множества является достаточным для сохранения носителя операцией перехода к инвариантному элементу итерированной системы функций.

Let (X, d) be a locally compact metric space. A *contraction* on X is a map $f\colon X \longrightarrow X$ such that there exists $\alpha < 1$ with the property: $d(f(x), f(y)) < \alpha d(x, y)$ for every $x, y \in X$. An *iterated function system* (IFS) on X is a finite sequence of contractions on X.

If $\Psi = \{f_1, \dots, f_n\}$ is an IFS on X, then let $\hat{\Psi}$ denote the transformation of $\exp X$ defined by

$$\hat{\Psi}(E) = \bigcup_{i=1}^{n} f_i(E) \text{ for } E \subset X.$$

It is well known (see [1]), that there exists a unique compact set $A_\Psi \subset X$ such that $\hat{\Psi}(A_\Psi) = A_\Psi$. This set will be called *invariant* for the IFS $(f_1, \dots, f_n)$.

An IFS $(f_1, \dots, f_n)$ is said to satisfy the *open set condition* (OSC) if there exists a nonempty open set $U \subset X$ such that $\bigcup_{i=1}^{n} f_i(U) \subset U$ and the family $\{f_1(U), \dots, f_n(U)\}$ is disjoint (see [1, p. 121]).

A nonempty family $\mathcal{A}$ of nonempty closed subsets of X is called an *inclusion hyperspace* [2] if the following holds: for every $A \in \mathcal{A}$ and every closed subset $B \subset X$ with $A \subset B$, we have $B \in \mathcal{A}$.

For every inclusion hyperspace $\mathcal{A}$ we denote by $\mathrm{supp}(\mathcal{A})$ the *support* of $\mathcal{A}$, i.e. the minimal with respect to inclusion set with the following property: there exists a family $\mathcal{B}$ of closed subsets of $\mathrm{supp}(\mathcal{A})$ such that

$$\mathcal{A} = \{A \subset X \text{ is closed} \mid A \supset B \text{ for some } B \in \mathcal{B}\}.$$

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Following [3] we denote by GX the set of all inclusion hyperspaces on X with compact supports. If $\mathcal{A}$,$\mathcal{B} \in GX$, let

$$\hat{d}(\mathcal{A},\mathcal{B}) = \inf\{\varepsilon > 0 \mid \overline{O_\varepsilon(A)} \in \mathcal{B} \text{ for every } A \in \mathcal{A}, \text{and } \overline{O_\varepsilon(B)} \in \mathcal{A} \text{ for every } B \in \mathcal{B}\}.$$

For every continuous map $f\colon X \longrightarrow X$ define the map $Gf\colon GX \longrightarrow GX$ by the formula

$$Gf(\mathcal{A}) = \left\{B \subset X \text{ is closed} \mid B \supset \overline{f(A)} \text{ for some } A \in \mathcal{A}\right\}.$$

Proposition 1. *If f is a contraction, then so is Gf.*

Proof. Straightforward. □

If A is a (closed) subset of X then we identify GA with the subspace $Gi(GA)$ of GX, where $i\colon A \to X$ is an embedding.

Now let $(f_1, \dots, f_n)$ be an IFS on a locally compact metric space (X, d). Fix $\mathcal{A} \in G\{1, 2, \dots, n\}$.

Define the map $\Phi = \Phi_{\mathcal{A};f_1,\dots,f_n}\colon GX \longrightarrow GX$ by the formula

$$\Phi(\mathcal{B}) = \{A \subset X \text{ is closed} \mid \text{there exists } C \in \mathcal{A} \text{ such that } A \in Gf_i(\mathcal{B}) \text{ for every } i \in C\}.$$

Proposition 2. *The map $\Phi\colon GX \longrightarrow GX$ is a contraction.*

Proof. Follows from the fact that the natural map $\mu\colon G^2Y \to GY$, $\mu(\mathfrak{A}) = \bigcup\{\bigcap \alpha \mid \alpha \in \mathfrak{A}\}$ (defined for compact metric spaces Y) is a contraction (see, e. g., [4]).

□

Recall that for every IFS there exists an invariant set (see [1, p. 119]).

Theorem 3. *There is a unique fixed point of the map Φ. This fixed point is the limit of the sequence whose elements are the images of the iterations of the map Φ, $(\Phi^i(\mathcal{B}))_{i=1}^\infty$, for any $\mathcal{B} \in GX$.*

Proof. Note first that we cannot apply directly the Banach fixed point theorem, because the space GX is not necessarily locally compact. Denote by B the invariant set for the IFS $(f_1, \dots, f_n)$. Then it is easy to see that, for every $\mathcal{B} \in GX$ with $\operatorname{supp}(\mathcal{B}) \subset B$ and for every m we have $\operatorname{supp}\Phi^m\mathcal{B}) \subset B$. Then we apply the Banach fixed point theorem to the restriction of the map Φ onto GB. □

The abovementioned fixed point will be called the *invariant* inclusion hyperspace of the IFS $(f_1, f_2, \dots, f_n)$ with respect to $\mathcal{A} \in G\{1, \dots, n\}$.

For the IFS $(f_1, f_2, \dots, f_n)$ and $\mathcal{A} \in G\{1, \dots, n\}$ we denote by $\mathcal{B}$ the corresponding invariant inclusion hyperspace and by B the invariant set (for the IFS $(f_1, f_2, \dots, f_n)$). Consider the following question: when $\operatorname{supp}(\mathcal{B}) = B$? The following example demonstrates that this is not necessarily the case.

Example 4. Let $X = \mathbb{R}^1$, $n = 3$, $\mathcal{A} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Consider the IFS on X consisting of the functions $f_1(x) = f_2(x) = \frac{x}{3}$, $f_3(x) = 1 + \frac{x-1}{3}$.

Then it is easy to see that the invariant inclusion hyperspace for the IFS (f_1, f_2, f_3) with respect to $\mathcal{A}$ is $\mathcal{B} = \{A \subset \mathbb{R} \text{ is closed} \mid 0 \in A\}$ with $\operatorname{supp}(\mathcal{B}) = \{0\}$ while the invariant set for (f_1, f_2, f_3) is the standard middle-third Cantor set.

The following theorem gives a sufficient condition for the equality.

Theorem 5. *Suppose an IFS $(f_1, f_2, \dots, f_n)$ on a locally compact metric space satisfies the OSC and $\mathcal{A} \in G\{1, \dots, n\}$, $\mathrm{supp}(\mathcal{A}) = \{1, \dots, n\}$. Then the support of the invariant inclusion hyperspace for the IFS $(f_1, f_2, \dots, f_n)$ with respect to $\mathcal{A}$ coincides with the invariant set for the IFS $(f_1, f_2, \dots, f_n)$.*

Proof. For the IFS $(f_1, f_2, \dots, f_n)$ and $\mathcal{A} \in G\{1, \dots, n\}$ we denote by $\mathcal{B}$ the corresponding invariant inclusion hyperspace and by P the invariant set for the IFS $(f_1, f_2, \dots, f_n)$. One can easily verify that $\mathrm{supp}(\mathcal{B}) \subseteq P$.

Let $\mathcal{B}_P$ denote the inclusion hyperspace such that $\mathcal{B}_P = \{B \cap P \mid B \in \mathcal{B}\}$. Clearly, $\mathcal{B}_P \in GP$ and $\mathrm{supp}(\mathcal{B}_P) = \mathrm{supp}(\mathcal{B})$.

For any sequence $\{j_1, ..., j_k\}$ with $1 \le j_i \le n$ and any subset F of X let $F_{j_1 \dots j_k} = f_{j_1}(f_{j_2}(...(f_{j_k}(F))...))$.

Fix any set $A_{i_0} \in \mathcal{A}$. For any $i_1 \in A_{i_0}$ fix any set $A_{i_0 i_1} \in \mathcal{A}$. For any $i_2 \in A_{i_0 i_1}$ fix any set $A_{i_0 i_1 i_2} \in \mathcal{A}$ etc. Let $A_k = \{A_{i_0 i_1 \dots i_k} \in \mathcal{A} \mid i_j \in A_{i_0 i_1 \dots i_{j-1}}, 1 \le j \le k\}$ for $k \in \mathbb{N}$.

Let V be an open subset of X given by the OSC.

The sequence $(A_k)_{k=0}^{\infty}$ generates the sequence of sets

$$
\begin{aligned}
W_1 &= \bigcup_{i_1 \in A_{i_0}} \overline{V}_{i_1}, \\
W_2 &= \bigcup_{i_1 \in A_{i_0}} \bigcup_{i_2 \in A_{i_0 i_1}} \overline{V}_{i_1 i_2}, \\
&\dots, \\
W_n &= \bigcup_{i_1 \in A_{i_0}} \bigcup_{i_2 \in A_{i_0 i_1}} \bigcup_{i_3 \in A_{i_0 i_1 i_2}} \dots \bigcup_{i_n \in A_{i_0 i_1 i_2 \dots i_{n-1}}} \overline{V}_{i_1 i_2 \dots i_n}, \\
&\dots .
\end{aligned}
$$

Denote by $\mathcal{W}$ the set of all sequences $(W_i)_{i=1}^{\infty}$ that can be obtained as the result of the described procedure (for all the possible choices of the sequences $(A_k)_{k=0}^{\infty}$).

Claim 6. Every set $B \in \mathcal{B}_P$ can be written as $B = \bigcap_{i=1}^{\infty} W_i$ for some $(W_i)_{i=1}^{\infty} \in \mathcal{W}$.

Indeed, since $\mathcal{B}_P$ is the fixed point of the map Φ, by the construction of the map Φ we see that there is $A_{i_0} \in \mathcal{A}$ such that $B \subset \bigcup_{i_1 \in A_{i_0}} \overline{V}_{i_1}$ and $B \cap \overline{V}_{i_1} \neq \varnothing$ for all $i_1 \in A_{i_0}$. By the same argument, for any $i_1 \in A_{i_0}$ there is $A_{i_0 i_1} \in \mathcal{A}$ such that $B \cap \overline{V}_{i_1} \subset \bigcup_{i_2 \in A_{i_0 i_1}} \overline{V}_{i_1 i_2}$ and $B \cap \overline{V}_{i_1} \cap \overline{V}_{i_1 i_2} \neq \varnothing$ for all $i_2 \in A_{i_0 i_1}$, etc.

Then $W_1 \supset W_2 \supset \dots \supset W_n \supset \dots$, where W_i are as above, and, by the construction, we have $\bigcap_{i=1}^{\infty} W_i = B$.

Fix any $x \in P$. Construct the sets W_i by the method described above but such that for all i, $x \in W_i$ and the sets $A_{i_0 i_1 \dots i_j} \in \mathcal{A}$ are minimal sets (with respect to the inclusion) for $\mathcal{A}$. Then $x \in \bigcap_{i=1}^{\infty} W_i = B \in \mathcal{B}_P$.

Claim 7. B is a minimal set for $\mathcal{B}_P$.

Suppose that there is $M \in \mathcal{B}_P$ such that $M \subset B$, $M \neq B$. In this case, for any $k \in \mathbb{N}$ there is $i'_k \in A_{i_0 i_1 \dots i_{k-1}}$ such that $\overline{V}_{i_1 i_2 \dots i'_k} \cap M = \varnothing$. Thus, there is $A' \subset A_{i_0 i_1 \dots i_{k-1}}$ such that

$M = \bigcap_{i=1}^{\infty} W_i'$, where $(W_j')_{j=1}^{\infty}$ is constructed as described above, for a sequence $(A_j')_{j=0}^{\infty}$ such that $A' \in A_{k-1}'$. However, this is impossible, because $A_{i_0 i_1 \dots i_{k-1}}$ is a minimal set for $\mathcal{A}$.

Since B is a minimal set for $\mathcal{B}_P$, we see that $B \subset \mathrm{supp}(\mathcal{B}_P)$ and therefore $x \in \mathrm{supp}(\mathcal{B}_P)$. The fact that $\mathrm{supp}(\mathcal{B}_P) = \mathrm{supp}(\mathcal{B})$ completes the proof of Theorem 5. □

The analogous results are also valid for the superextensions and the spaces of complete linked systems (see, e. g., [5] for the definitions).

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