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THE INITIAL-BOUNDARY VALUE PROBLEM FOR NONLINEAR PSEUDOPARABOLIC SYSTEM

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The initial-boundary value problem for nonlinear pseudoparabolic system in an unbounded (with respect to space variables) domain is considered. The existence and the uniqueness of the solution of this problem are proved.

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Рассматривается смешанная задача для псевдопараболической системы в неограниченной (по пространственным переменным) области. Доказано существование и единственность решения этой задачи.

Recently several works both in mathematical and technical literature dealt with problems for pseudoparabolic equations. Many processes such as filtration of liquid through double-porous rocks [1], the heat emission in heterogeneous medium [2], the transfer of moisture in the soil [3], the diffusion in fissured medium with the absorption or part glut, the process of the glue setting [4] are described by pseudoparabolic equations and systems. Beginning from 1950 various problems for pseudoparabolic equations were studied in [5–10].

To our mind, nonlinear pseudoparabolic equations are not sufficiently investigated. Thereat we study the initial boundary value problem for a nonlinear pseudoparabolic system in unbounded (with respect to space variables) domain. Uniqueness and existence theorems are proved. Besides we establish that existence and uniqueness of a solution do not depend on the behavior of the solution as $|x| \to \infty$.

Let Ω be an unbounded domain in $\mathbb{R}^n$ and Γ be its boundary. From now on we make the assumption on the geometry of Ω: there exists a sequence of domains $\{\Omega^\tau\}_{\tau\in\Pi}$ (here Π is a countable subset of $\mathbb{R}_+$) satisfying the following conditions:

1) $\Omega = \bigcup_{\tau\in\Pi}\Omega^\tau$, $\Omega^\tau = \Omega\cap B_\tau$, B_τ is an n-dimensional ball with center at the origin and the radius τ;

2) $\partial\Omega^\tau = \Gamma_1^\tau \cup \Gamma_2^\tau$, where $\Gamma_1^\tau, \Gamma_2^\tau$ are piecewise smooth hypersurfaces; $\mathrm{mes}\{\Gamma_1^\tau\cap\Gamma_2^\tau\} = 0$, $\Gamma_1^\tau \neq \varnothing$, $\Gamma_1^\tau\cap\Gamma \neq \varnothing$, $\forall\tau\in\Pi$; $\Gamma = \bigcup_{\tau\in\Pi}\Gamma_1^\tau$.

Let $Q_T = \Omega\times(0,T)$, $T<\infty$; $S_T = \Gamma\times(0,T)$; $\Omega_\eta = Q_T\cap\{t=\eta\}$. We will denote by $L^p_{\mathrm{loc}}(\Omega)$ the space of functions that belong to $L^p(\Omega^\tau)$ for every $\tau\in\Pi$, $L^p_{\mathrm{loc}}(Q_T) =$

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$L^p\left((0,T);L^p_{\rm loc}(\Omega)\right)$, $p\geq 2$. For the Sobolev spaces we use the traditional notation H^1 and $H^1_{\rm loc}$.

Let us consider the system

$$G(u_t)-\sum_{i,j=1}^{n}\left(A_{ij}(x,t)u_{x_it}\right)_{x_j}-\sum_{i,j=1}^{n}\left(B_{ij}(x,t)u_{x_i}\right)_{x_j}+u=F_0(x,t)-\sum_{i=1}^{n}F_{i,x_i}(x,t) \quad (1)$$

with the conditions

$$u\big|_{\Gamma\times[0,T]}=0, \quad (2)$$

$$u\big|_{t=0}=u_0 \quad (3)$$

in the domain Q_T, where A_{ij}, B_{ij}, D are square matrices of the m-th order;

$$G(u_t)=(|u_{1t}|^{\gamma-2}u_{1t},\ldots,|u_{mt}|^{\gamma-2}u_{mt})^T,\ u=(u_1,\ldots,u_m)^T,$$

$$F_j=(F_j^1,\ldots,F_j^m)^T,\ j\in\{0,\ldots,n\};$$

$\gamma>2$; $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^m$; $|\cdot|$ is the norm in $\mathbb{R}^m$. With respect to a space variable x we also denote by $|x|$ the magnitude of x in $\mathbb{R}^n$.

Definition 1. Function u is a *solution* of (1)–(3) if:

1) $u\in L^2_{\rm loc}(Q_T)$; $u_t\in L^\gamma_{\rm loc}(Q_T)$; $u_{x_i}\in H^1\left((0,T);L^2_{\rm loc}(\Omega)\right)$, $i\in\{1,\ldots,n\}$;

2) u satisfies (3) almost everywhere in Ω;

3) the integral identity

$$\int_0^\eta\int_\Omega\left[(G(u_t),v)+\sum_{i,j=1}^{n}\left(A_{ij}(x,t)u_{x_it},v_{x_j}\right)+\sum_{i,j=1}^{n}\left(B_{ij}(x,t)u_{x_i},v_{x_j}\right)+(u,v)\right]dxdt=$$

$$=\int_0^\eta\int_\Omega\left[(F_0(x,t),v)+\sum_{i=1}^{n}(F_i(x,t),v_{x_i})\right]dxdt \quad (4)$$

holds for all $v\in C^\infty([0,T];C_0^\infty(\Omega))$ and for almost every $\eta\in[0,T]$.

We say that the coefficients of (1) satisfy conditions (A), (B), (D) if:

$$(A):\ a_0\sum_{i=1}^{n}|\xi^i|^2\leq\sum_{i,j=1}^{n}\left(A_{ij}(x,t)\xi^i,\xi^j\right),\quad a_0>0,\quad \forall(x,t)\in Q_T;$$

$$A_{ij}(x,t)=A_{ji}(x,t),\ A_{ij}(x,t)=A_{ij}^T(x,t),\ \forall(x,t)\in Q_T,\ \forall i,j\in\{1,...,n\};$$

$$A_{ij}\in L^\infty(Q_T),\ A_{ijx_i}\in L^\infty(Q_T),\ \forall i,j\in\{1,...,n\};$$

$$(B):\ b_0\sum_{i=1}^{n}|\xi^i|^2\leq\sum_{i,j=1}^{n}\left(B_{ij}(x,t)\xi^i,\xi^j\right),\quad b_0>0,\quad \forall(x,t)\in Q_T;$$

$$B_{ij}(x,t)=B_{ji}(x,t),\ B_{ij}(x,t)=B_{ij}^T(x,t),\quad \forall(x,t)\in Q_T,\ \forall i,j\{1,...,n\};$$

$$B_{ij},B_{ijt}\in L^\infty(Q_T),\quad \forall i,j\in\{1,...,n\};$$

$$(D):\ d_0|\xi|^2\leq(\xi,\xi),\quad d_0>0,\quad \forall(x,t)\in Q_T;$$

$$D(x,t)=D^T(x,t),\quad \forall(x,t)\in Q_T;\quad D,D_t\in L^\infty(Q_T)$$

for all vectors ξ, ξ^i, ξ^j from $\mathbb{R}^m$, $1\leq i,j\leq n$.

Theorem 1. *Let the coefficients of system (1) satisfy conditions* (A), (B), (D); $2 < \gamma < \frac{2n}{n-1}$ *if* $n > 1$ *and* $\gamma > 2$ *if* $n = 1$. *If there exists a solution of problem (1)–(3), the one is unique.*

Proof. Let u^1, u^2 be solutions of (1)–(3). For both u^1 and u^2 we consider integral identity (4). Put $u = u^1 - u^2$ and $v = u_t\psi^\alpha(x)$, where ψ is a function of $C_0^\infty(\Omega)$, $\alpha > 0$. Then from (4) we obtain

$$\int_0^\eta \int_\Omega \Big[\big(G(u_t^1) - G(u_t^2), u_t\psi^\alpha(x)\big) + \sum_{i,j=1}^n \Big(A_{ij}(x,t)u_{x_it} + B_{ij}(x,t)u_{x_i}, (u_t\psi^\alpha(x))_{x_j}\Big) + (u, u_t\psi^\alpha(x))\Big] dxdt = 0. \tag{5}$$

Now estimate every term of (5):

$$I_1 = \int_0^\eta \int_\Omega \big(G(u_t^1) - G(u_t^2), u_t\psi^\alpha\big)\, dxdt \ge 2^{2-\gamma} \int_0^\eta \int_\Omega |u_t|^\gamma \psi^\alpha dxdt,$$

$$I_2 = \int_0^\eta \int_\Omega \sum_{i,j=1}^n \Big(A_{ij}(x,t)u_{x_it}, (u_t\psi^\alpha)_{x_j}\Big)\, dxdt = \int_0^\eta \int_\Omega \sum_{i,j=1}^n \big(A_{ij}(x,t)u_{x_it}, u_{x_jt}\big)\,\psi^\alpha dxdt +$$
$$+ \int_0^\eta \int_\Omega \sum_{i,j=1}^n (A_{ij}(x,t)u_{x_it}, u_t)\,\alpha\psi^{\alpha-1}\psi_{x_j} dxdt \ge \int_0^\eta \int_\Omega a_0 \sum_{i=1}^n |u_{x_it}|^2 \psi^\alpha dxdt -$$
$$- \frac{\alpha A_0 \delta_1 n^2}{\gamma} \int_0^\eta \int_\Omega |u_t|^\gamma \psi^\alpha dxdt - \frac{\alpha A_0(\gamma-2)}{2\gamma\delta_1^{\frac{2}{\gamma-2}}}\eta\Psi_1,$$

$$I_3 = \int_0^\eta \int_\Omega \sum_{i,j=1}^n \Big(B_{ij}(x,t)u_{x_i}, (u_t\psi^\alpha)_{x_j}\Big)\, dxdt = \frac{1}{2}\int_{\Omega_\eta} \sum_{i,j=1}^n \big(B_{ij}(x,\eta)u_{x_i}, u_{x_j}\big)\,\psi^\alpha dx -$$
$$- \frac{1}{2}\int_0^\eta \int_\Omega \sum_{i,j=1}^n \big(B_{ijt}(x,t)u_{x_i}, u_{x_j}\big)\,\psi^\alpha dxdt + \int_0^\eta \int_\Omega \sum_{i,j=1}^n (B_{ij}(x,t)u_{x_i}, u_t)\,\alpha\psi^{\alpha-1}\psi_{x_j} dxdt \ge$$
$$\ge \frac{b_0}{2}\int_{\Omega_\eta} \sum_{i=1}^n |u_{x_i}|^2 \psi^\alpha dx - \frac{b_1}{2}\int_0^\eta \int_\Omega \sum_{i=1}^n |u_{x_i}|^2 \psi^\alpha dx -$$
$$- \frac{\alpha B_0 n}{2}\int_0^\eta \int_\Omega \sum_{i=1}^n |u_{x_i}|^2 \psi^\alpha dxdt - \frac{\alpha B_0\delta_2 n^2}{\gamma}\int_0^\eta \int_\Omega |u_t|^\gamma \psi^\alpha dxdt - \frac{\alpha B_0 n(\gamma-2)}{2\gamma\delta_2^{\frac{2}{\gamma-2}}}\eta\Psi_2,$$

$$I_4 = \int_0^\eta \int_\Omega (u, u_t)\,\psi^\alpha dxdt \ge \frac{d_0}{2}\int_{\Omega_\eta} |u|^2\psi^\alpha dx - \frac{d_1}{2}\int_0^\eta \int_\Omega |u|^2\psi^\alpha dxdt,$$

where $\delta_1>0$, $\delta_2>0$,

$$A_0=\max\left\{\sup_{Q_T}\sum_{i,j=1}^{n}\|A_{ij}(x,t)\|,\sup_{Q_T}\sum_{i,j=1}^{n}\|A_{ijx_i}(x,t)\|\right\};$$

$$\Psi_1=\sum_{i,j=1}^{n}\int\limits_{\Omega}\psi^{\alpha-\frac{2\gamma}{\gamma-2}}\left\{\left|\psi\psi_{x_j}\right|+\left|(\alpha-1)\psi_{x_i}\psi_{x_j}+\psi\psi_{x_ix_j}\right|\right\}^{\frac{\gamma}{\gamma-2}}dx;$$

$$B_0=\sup_{Q_T}\sum_{i,j=1}^{n}\|B_{ij}(x,t)\|;\qquad \Psi_2=\int\limits_{\Omega}\psi^{\alpha-\frac{2\gamma}{\gamma-2}}\sum_{i=1}^{n}\left|\psi_{x_i}\right|^{\frac{2\gamma}{\gamma-2}}dx;$$

$$\sum_{i,j=1}^{n}(B_{ijt}(x,t)\xi^j,\xi^i)\le b_1\sum_{i=1}^{n}|\xi^i|^2,\ \ (D_t(x,t)\xi,\xi)\le d_1|\xi|^2,\ \ \forall\xi,\xi^i,\xi^j\in\mathbb{R}^m(1\le i,j\le n).$$

Adding the estimates we get

$$\begin{aligned}&\frac{1}{2}\int\limits_{\Omega_\eta}\left(b_0\sum_{i=1}^{n}\left|u_{x_i}\right|^2+d_0\left|u\right|^2\right)\psi^\alpha dx+\int\limits_0^\eta\int\limits_{\Omega}\left[\left(g_0-\frac{\alpha A_0\delta_1n^2}{\gamma}-\frac{\alpha B_0\delta_2n^2}{\gamma}\right)\left|u_t\right|^\gamma+\right.\\&\left.+a_0\sum_{i=1}^{n}\left|u_{x_it}\right|^2-\left(\frac{\alpha B_0n}{2}+\frac{b_1}{2}\right)\sum_{i=1}^{n}\left|u_{x_i}\right|^2-\frac{d_1}{2}\left|u\right|^2\right]\psi^\alpha dxdt\le\\&\le\frac{\alpha A_0(\gamma-2)}{2\gamma\delta_1^{\frac{2}{\gamma-2}}}\eta\Psi_1+\frac{\alpha B_0(\gamma-2)n}{2\gamma\delta_2^{\frac{2}{\gamma-2}}}\eta\Psi_2,\quad g_0>0.\end{aligned}\tag{6}$$

Suppose that δ_1 and δ_2 satisfy $2\alpha(A_0\delta_1+B_0\delta_2)n^2\le\gamma g_0$. Then (6) implies

$$\begin{aligned}&\int\limits_{\Omega_\eta}\left(b_0\sum_{i=1}^{n}\left|u_{x_i}\right|^2+d_0\left|u\right|^2\right)\psi^\alpha dx+\int\limits_0^\eta\int\limits_{\Omega}\left(g_0\left|u_t\right|^\gamma+2a_0\sum_{i=1}^{n}\left|u_{x_it}\right|^2\right)\psi^\alpha dxdt\le\\&\le\int\limits_0^\eta\int\limits_{\Omega}\left((\alpha B_0n+b_1)\sum_{i=1}^{n}\left|u_{x_i}\right|^2+d_1\left|u\right|^2\right)\psi^\alpha dxdt+T\Psi,\end{aligned}$$

where $\Psi=\frac{\alpha A_0(\gamma-2)}{\gamma\delta_1^{\frac{2}{\gamma-2}}}\Psi_1+\frac{\alpha B_0(\gamma-2)n}{\gamma\delta_2^{\frac{2}{\gamma-2}}}\Psi_2$. Using the Gronwall lemma [4, p.191] from the last inequality we obtain

$$\int\limits_{\Omega_\eta}\left(\sum_{i=1}^{n}\left|u_{x_i}\right|^2+\left|u\right|^2\right)\psi^\alpha dx\le\frac{T}{\mu_0}\Psi e^{\frac{\mu_1}{\mu_0}T},\quad \mu_0=\min\{b_0;d_0\},\ \ \mu_1=\max\{\alpha B_0n+b_1;d_1\},$$

$$\int\limits_0^\eta\int\limits_{\Omega}\left(g_0\left|u_t\right|^\gamma+2a_0\sum_{i=1}^{n}\left|u_{x_it}\right|^2\right)\psi^\alpha dxdt\le\mu_2T\Psi,\quad \mu_2=1+\frac{\mu_1}{\mu_0}Te^{\frac{\mu_1}{\mu_0}}.$$

Let us introduce the function

$$\psi(x)=\begin{cases}\frac{R^2-|x|^2}{R}, & 0\le|x|\le R,\\ 0, & |x|>R.\end{cases}$$

Then

$$\psi_{x_i}(x) = \begin{cases} -\frac{2x_i}{R}, & 0 \le |x| \le R, \\ 0, & |x| > R, \end{cases} \qquad \psi_{x_ix_j}(x) = \begin{cases} -\frac{2\delta_{ij}}{R}, & 0 \le |x| \le R, \\ 0, & |x| > R, \end{cases}$$

where δ_{ij} is the Kronecker symbol. It is easy to see that $|\psi| \le 2R, \quad |\psi_{x_i}| \le 2, \quad |\psi_{x_ix_j}| \le \frac{2}{R}$. After computations

$$\Psi_1 = \sum_{i,j=1}^{n} \int_{\Omega} \psi^{\alpha-\frac{2\gamma}{\gamma-2}} \left\{ \left|\psi\psi_{x_i}\right| + \left|(\alpha-1)\psi_{x_i}\psi_{x_j} + \psi\psi_{x_ix_j}\right| \right\}^{\frac{\gamma}{\gamma-2}} dx \le$$

$$\le \sum_{i,j=1}^{n} \int_{B_R} (2R)^{\alpha-\frac{2\gamma}{\gamma-2}} (4R+4\alpha)^{\frac{\gamma}{\gamma-2}} dx = 2^{\alpha} n^2 R^{\alpha-\frac{2\gamma}{\gamma-2}+n} P_n (R+\alpha)^{\frac{\gamma}{\gamma-2}},$$

$$\Psi_2 = \int_{\Omega} \psi^{\alpha-\frac{2\gamma}{\gamma-2}} \sum_{i=1}^{n} \left|\psi_{x_i}\right|^{\frac{2\gamma}{\gamma-2}} dx \le \int_{B_R} (2R)^{\alpha-\frac{2\gamma}{\gamma-2}} n 2^{\frac{2\gamma}{\gamma-2}} dx = 2^{\alpha} n R^{\alpha-\frac{2\gamma}{\gamma-2}+n} P_n,$$

where $R \in \Pi$ and P_n is defined by the equality

$$\int_{B_R} dx = P_n R^n = \begin{cases} \frac{\pi^k}{k!} R^{2k}, & n = 2k, \\ \frac{2(2\pi)^k}{(2k+1)!!} R^{2k+1}, & n = 2k+1, \end{cases}$$

we get

$$\int_{\Omega_\eta} \sum_{i=1}^{n} \left(\left|u_{x_i}\right|^2 + \left|u\right|^2 \right) \left(\frac{R^2 - |x|^2}{R} \right)^{\alpha} dx \le$$

$$\le \frac{\alpha T(\gamma-2) 2^{\alpha} n^2}{\mu_0 \gamma} R^{\alpha-\frac{2\gamma}{\gamma-2}+n} P_n e^{\frac{\mu_1}{\mu_0}T} \left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}} (\alpha+R)^{\frac{\gamma}{\gamma-2}} + \frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}} \right),$$

$$\int_0^{\eta} \int_{\Omega} \left(g_0 \left|u_t\right|^{\gamma} + 2a_0 \sum_{i=1}^{n} \left|u_{x_it}\right|^2 \right) \left(\frac{R^2 - |x|^2}{R} \right)^{\alpha} dx dt \le$$

$$\le \frac{\alpha T(\gamma-2) 2^{\alpha} n^2}{\gamma} R^{\alpha-\frac{2\gamma}{\gamma-2}+n} P_n \left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}} (\alpha+R)^{\frac{\gamma}{\gamma-2}} + \frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}} \right) \mu_2$$

for almost all $\eta \in [0, T]$. Let us minorize the integrals in the left part of these inequalities by the integrals over domain $\Omega \cap B_{R_0} = \Omega^{R_0}$, $R_0 < R$, $R_0 \in \Pi$. Namely,

$$\int\limits_{\Omega_\eta^{R_0}} \left(\sum_{i=1}^n \left|u_{x_i}\right|^2 + \left|u\right|^2\right) (R-R_0)^\alpha \, dx \le \int\limits_{\Omega_\eta} \left(\sum_{i=1}^n \left|u_{x_i}\right|^2 + \left|u\right|^2\right) \left(\frac{R^2-|x|^2}{R}\right)^\alpha dx \le$$

$$\le \frac{\alpha T(\gamma-2)2^\alpha n^2}{\mu_0 \gamma} R^{\alpha - \frac{2\gamma}{\gamma-2}+n} P_n e^{\frac{\mu_1}{\mu_0}T} \left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}}(\alpha+R)^{\frac{\gamma}{\gamma-2}} + \frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}}\right),$$

$$\int\limits_0^\eta \int\limits_{\Omega^{R_0}} \left(g_0\left|u_t\right|^\gamma + 2a_0 \sum_{i=1}^n \left|u_{x_i t}\right|^2\right) (R-R_0)^\alpha \, dxdt \le$$

$$\le \int\limits_0^\eta \int\limits_{\Omega} \left(g_0\left|u_t\right|^\gamma + 2a_0 \sum_{i=1}^n \left|u_{x_i t}\right|^2\right) \left(\frac{R^2-|x|^2}{R}\right)^\alpha dxdt \le$$

$$\le \frac{\alpha T(\gamma-2)2^\alpha n^2}{\gamma} R^{\alpha - \frac{2\gamma}{\gamma-2}+n} P_n \left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}}(\alpha+R)^{\frac{\gamma}{\gamma-2}} + \frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}}\right) \mu_2.$$

Hence

$$\int\limits_{\Omega_\eta^{R_0}} \left(\sum_{i=1}^n \left|u_{x_i}\right|^2 + \left|u\right|^2\right) dx \le \frac{\alpha T(\gamma-2)2^\alpha n^2}{\mu_0 \gamma} R^{-\frac{2\gamma}{\gamma-2}+n} \left(\frac{R}{R-R_0}\right)^\alpha P_n e^{\frac{\mu_1}{\mu_0}T} \times$$

$$\times \left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}}(\alpha+R)^{\frac{\gamma}{\gamma-2}} + \frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}}\right),$$

$$\int\limits_0^\eta \int\limits_{\Omega^{R_0}} \left(g_0\left|u_t\right|^\gamma + 2a_0 \sum_{i=1}^n \left|u_{x_i t}\right|^2\right) dxdt \le \frac{\alpha T(\gamma-2)2^\alpha n^2}{\gamma} R^{-\frac{2\gamma}{\gamma-2}+n} \left(\frac{R}{R-R_0}\right)^\alpha P_n \times$$

$$\times \left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}}(\alpha+R)^{\frac{\gamma}{\gamma-2}} + \frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}}\right) \mu_2.$$

If $R > \alpha$, then for almost all $\eta \in [0,T]$ we have

$$\int\limits_{\Omega_\eta^{R_0}} \left(\sum_{i=1}^n \left|u_{x_i}\right|^2 + \left|u\right|^2\right) dx \le \frac{\alpha T(\gamma-2)2^\alpha n^2}{\mu_0 \gamma} R^{-\frac{2\gamma}{\gamma-2}+n} P_n e^{\frac{\mu_1}{\mu_0}T} \left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}}(2R)^{\frac{\gamma}{\gamma-2}} + \frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}}\right);$$

$$\int\limits_0^\eta \int\limits_{\Omega^{R_0}} \left(g_0\left|u_t\right|^\gamma + 2a_0 \sum_{i=1}^n \left|u_{x_i t}\right|^2\right) dxdt \le \frac{\alpha T(\gamma-2)2^\alpha n^2}{\gamma R^{\frac{2\gamma}{\gamma-2}-n}} P_n \left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}}(2R)^{\frac{\gamma}{\gamma-2}} + \frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}}\right) \mu_2.$$

Since $n < \frac{\gamma}{\gamma-1}$ for sufficiently large R, the left part of the estimates can be made sufficiently small. Thus for arbitrary fixed R_0 and $\eta \in [0,T]$ we obtain uniqueness of a solution in $\Omega^{R_0} \times [0,T]$. It is just the uniqueness of a solution in Q_T. □

Remark. If the coefficients A_{ij} do not depend on the space variable x then the imposed conditions on γ may be weakened. In this case the following restriction holds: $2 < \gamma < \frac{2n}{n-2}$ if $n > 2$ and $\gamma > 2$ if $n = 1, 2$.

Obtaining existence conditions is the next step in our consideration. For this we consider an auxiliary problem. We shall prove existence of a solution of the problem

$$\varepsilon u_{tt} + G(u_t) - \sum_{i,j=1}^{n} (A_{ij}(x,t)u_{tx_i})_{x_j} - \sum_{i,j=1}^{n} (B_{ij}(x,t)u_{x_i})_{x_j} + u =$$

$$= F_0^*(x,t) - \sum_{i=1}^{n} F_{i,x_i}^*(x,t), \qquad \varepsilon > 0, \tag{7_ε}$$

$$u\big|_{t=0} = u_0^*, \tag{8}$$

$$u\big|_{\Gamma^* \times (0,T)} = 0, \tag{9}$$

$$u_t\big|_{t=0} = 0, \tag{10}$$

in $Q_T^* = \Omega^* \times (0,T)$ (here Ω^* is a bounded subdomain of Ω, Γ^* is the boundary of Ω^*, $\Gamma^* \cap \Gamma \neq \varnothing$).

Definition 2. A function $u^{*,\varepsilon}$ is a solution of (7_ε)–(10) if:

1) $u^{*,\varepsilon} \in L^2(Q_T^*)$; $u_t^{*,\varepsilon} \in L^\gamma(Q_T^*)$; $u_{x_i}^{*,\varepsilon} \in H^1((0,T); L^2(\Omega^*))$, $(i \in \{1, ..., n\})$;

2) $u^{*,\varepsilon}$ satisfies (8), (10) almost everywhere in Ω^*;

3) the integral identity

$$\int_0^\eta \int_{\Omega^*} \Big[-\varepsilon\,(u_t^{*,\varepsilon}, v_t) + (G(u_t^{*,\varepsilon}), v) + \sum_{i,j=1}^{n} \big(A_{ij}(x,t)u_{x_i t}^{*,\varepsilon}, v_{x_j}\big) +$$

$$+ \sum_{i,j=1}^{n} \big(B_{ij}(x,t)u_{x_i}^{*,\varepsilon}, v_{x_i}\big) + (u^{*,\varepsilon}, v) \Big] dxdt = \tag{11}$$

$$= \int_0^\eta \int_{\Omega^*} \Big[(F_0^*(x,t), v) + \sum_{i=1}^{n} (F_i^*(x,t), v_{x_i}) \Big] dxdt - \varepsilon \int_{\Omega_\eta^*} (u_t^{*,\varepsilon}, v)\, dx$$

holds for all $v \in C^\infty([0,T]; C_0^\infty(\Omega^*))$ and for almost every $\eta \in [0,T]$.

Lemma. *Let the coefficients of (7_ε) satisfy conditions (A), (B), (D); $u_0^* \in H_0^1(\Omega^*)$; $F_0^* \in L^{\frac{\gamma}{\gamma-1}}(Q_T^*)$; $F_i^* \in L^2(Q_T^*)$, $i \in \{1, \ldots, n\}$. Then there exists a solution of (7_ε)–(10).*

Proof. Let $\{\varphi^{*,k}(x)\}$ be a basis in $H_0^1(\Omega^*) \cap L^\gamma(\Omega^*)$. We put $u^{*,\varepsilon,N} = \sum_{s=1}^{N} \varphi^{*,k}(x)c_k(t)$, where $c_k(t)$ can be found from the problem

$$\varepsilon\sum_{s=1}^{N} c''_s \int\limits_{\Omega^*} \left(\varphi^{*,s},\varphi^{*,k}\right) dx = -\int\limits_0^{\eta}\int\limits_{\Omega^*}\Bigg[\bigg(G\Big(\sum_{s=1}^{N} c'_s\varphi^{*,s}\Big),\varphi^{*,k}\bigg)+$$
$$+\sum_{i,j=1}^{n}\left(A_{ij}(x,t)\sum_{s=1}^{N} c'_s\varphi^{*,s}_{x_i},\varphi^{*,k}_{x_j}\right)+\sum_{i,j=1}^{n}\left(B_{ij}(x,t)\sum_{s=1}^{N} c_s\varphi^{*,s}_{x_i},\varphi^{*,k}_{x_j}\right)+ \tag{12}$$
$$+\left(c_s\varphi^{*,s},\varphi^{*,k}\right)-\left(F^*_0(x,t),\varphi^{*,k}\right)-\sum_{i=1}^{n}\left(F^*_i(x,t),\varphi^{*,k}_{x_i}\right)\Bigg]dxdt\equiv\hat{\phi}\left(c'_s,c_s,t\right),$$
$$c_k(0)=\left(u^*_0,\varphi^{*,k}\right)_{H^1_0(\Omega^*)\cap L^{\gamma}(\Omega^*)}, \tag{13}$$
$$c'_k(0)=0. \tag{14}$$

By the Carathéodory theorem [11, p. 54] and the lower estimates there exists a solution of (12)–(14). After calculations

$$I_5=\int\limits_0^{\eta}\int\limits_{\Omega^*}\varepsilon\left(u^{*,\varepsilon,N}_{tt},u^{*,\varepsilon,N}_t\right)dxdt=\frac{\varepsilon}{2}\int\limits_{\Omega^*_\eta}\left|u^{*,\varepsilon,N}_t\right|^2dx,$$

$$I_6=\int\limits_0^{\eta}\int\limits_{\Omega^*}\left(G(u^{*,\varepsilon,N}_t),u^{*,\varepsilon,N}_t\right)dxdt\geq g_1\int\limits_0^{\eta}\int\limits_{\Omega^*}\left|u^{*,\varepsilon,N}_t\right|^{\gamma}dxdt,\quad g_1>0,$$

$$I_7=\int\limits_0^{\eta}\int\limits_{\Omega^*}\sum_{i,j=1}^{n}\left(A_{ij}(x,t)u^{*,\varepsilon,N}_{x_it}+B_{ij}(x,t)u^{*,\varepsilon,N}_{x_i},u^{*,\varepsilon,N}_{x_jt}\right)dxdt\geq\frac{1}{2}\int\limits_{\Omega^*_\eta}\sum_{i=1}^{n}b_0\left|u^{*,\varepsilon,N}_{x_i}\right|^2dx-$$
$$-\frac{1}{2}\int\limits_{\Omega^*_0}\sum_{i,j=1}^{n}\left(B_{ij}(x,0)u^{*,\varepsilon,N}_{0x_i},u^{*,\varepsilon,N}_{0x_j}\right)dx+\int\limits_0^{\eta}\int\limits_{\Omega^*}\sum_{i=1}^{n}\left[a_0\left|u^{*,\varepsilon,N}_{x_it}\right|^2-\frac{b_1}{2}\left|u^{*,\varepsilon,N}_{x_i}\right|^2\right]dxdt,$$

$$I_8=\int\limits_0^{\eta}\int\limits_{\Omega^*}\left(u^{*,\varepsilon,N},u^{*,\varepsilon,N}_t\right)dxdt\geq\frac{1}{2}\int\limits_{\Omega^*_\eta}d_0\left|u^{*,\varepsilon,N}\right|^2dx-$$
$$-\frac{1}{2}\int\limits_{\Omega^*_0}\left(D(x,0)u^{*,\varepsilon,N}_0,u^{*,\varepsilon,N}_0\right)dx-\frac{d_1}{2}\int\limits_0^{\eta}\int\limits_{\Omega^*}\left|u^{*,\varepsilon,N}\right|^2dxdt,$$

$$I_9=\int\limits_0^{\eta}\int\limits_{\Omega^*}\left[\left(F^*_0(x,t),u^{*,\varepsilon,N}_t\right)+\sum_{i=1}^{n}\left(F^*_i(x,t),u^{*,\varepsilon,N}_{x_it}\right)\right]dxdt\leq$$
$$\leq\int\limits_0^{\eta}\int\limits_{\Omega^*}\left[\frac{\delta_3}{\gamma}\left|u^{*,\varepsilon,N}_t\right|^{\gamma}+\frac{\delta_4}{2}\sum_{i=1}^{n}\left|u^{*,\varepsilon,N}_{x_it}\right|^2\right]dxdt+$$
$$+\int\limits_0^{\eta}\int\limits_{\Omega^*}\left[\frac{\gamma-1}{\gamma\delta_3^{\frac{1}{1-\gamma}}}\left|F^*_0(x,t)\right|^{\frac{\gamma}{\gamma-1}}+\frac{1}{2\delta_4}\sum_{i=1}^{n}\left|F^*_i(x,t)\right|^2\right]dxdt,\quad\delta_3>0,\delta_4>0,$$

from (12) we obtain

$$\frac{1}{2}\int\limits_{\Omega_\eta^*}\left[\varepsilon\left|u_t^{*,\varepsilon,N}\right|^2+b_0\sum_{i=1}^{n}\left|u_{x_i}^{*,\varepsilon,N}\right|^2+d_0\left|u^{*,\varepsilon,N}\right|^2\right]dx+\int\limits_0^\eta\int\limits_{\Omega^*}\left[\left(g_1-\frac{\delta_3}{\gamma}\right)\left|u_t^{*,\varepsilon,N}\right|^\gamma+\right.$$
$$\left.+\left(a_0-\frac{\delta_4}{2}\right)\sum_{i=1}^{n}\left|u_{x_it}^{*,\varepsilon,N}\right|^2-\frac{d_1}{2}\left|u^{*,\varepsilon,N}\right|^2-\frac{b_1}{2}\sum_{i=1}^{n}\left|u_{x_i}^{*,\varepsilon,N}\right|^2\right]dxdt\le \tag{15}$$
$$\le\frac{1}{2}\int\limits_{\Omega_0^*}\left[\sum_{i,j=1}^{n}\left(B_{ij}(x,0)u_{0x_i}^{*,\varepsilon,N},u_{0x_j}^{*,\varepsilon,N}\right)+\left(D(x,0)u_0^{*,\varepsilon,N},u_0^{*,\varepsilon,N}\right)\right]dx+$$
$$+\int\limits_0^\eta\int\limits_{\Omega^*}\left[\frac{\gamma-1}{\gamma\delta_3^{\frac{1}{\gamma-1}}}\left|F_0^*(x,t)\right|^{\frac{\gamma}{\gamma-1}}+\frac{1}{2\delta_4}\sum_{i=1}^{n}\left|F_i^*(x,t)\right|^2\right]dxdt.$$

Let δ_3 and δ_4 be such that $g_1\gamma-\delta_3\ge 0$, $a_0-\delta_4\ge 0$. We again apply the Gronwall-Bellman lemma to (15) and deduce that

$\|u_t^{*,\varepsilon,N}\|$ is bounded in $L^\gamma(Q_T^*)$,
$\|u^{*,\varepsilon,N}\|$ is bounded in $L^\infty\left((0,T);L^2(\Omega^*)\right)$,
$\|u_{x_it}^{*,\varepsilon,N}\|$ is bounded in $L^2(Q_T^*)$, $i\in\{1,...,n\}$,
$\|u_{x_i}^{*,\varepsilon,N}\|$ is bounded in $L^\infty\left((0,T);L^2(\Omega^*)\right)$, $i\in\{1,...,n\}$,

Since Ω^* is a bounded domain, we have $L^\infty\left((0,T);L^2(\Omega^*)\right)\subset L^2(Q_T^*)$, $L^\gamma(Q_T^*)\subset L^2(Q_T^*)$. Therefore we can choose a subsequence $\{u^{*,\varepsilon,N_k}\}$ from the sequence $\{u^{*,\varepsilon,N}\}$ such that

$u^{*,\varepsilon,N_k}\to u^{*,\varepsilon}$ in $L^\infty\left((0,T);L^2(\Omega^*)\right)$ $*$-weakly,
$u_t^{*,\varepsilon,N_k}\to u_t^{*,\varepsilon}$ in $L^\gamma\left((0,T);L^\gamma(\Omega^*)\right)$ weakly, therefore
$u_t^{*,\varepsilon,N_k}\to u_t^{*,\varepsilon}$ in $L^2\left((0,T);L^2(\Omega^*)\right)$ weakly,
$u_{x_i}^{*,\varepsilon,N_k}\to u_{x_i}^{*,\varepsilon}$ in $L^\infty\left((0,T);L^2(\Omega^*)\right)$ $*$-weakly, $i\in\{1,...,n\}$,
$u_{x_it}^{*,\varepsilon,N_k}\to u_{x_it}^{*,\varepsilon}$ in $L^2\left((0,T);L^2(\Omega^*)\right)$ weakly, $i\in\{1,...,n\}$,
$|u_t^{*,\varepsilon,N_k}|^{\gamma-2}u_t^{*,\varepsilon,N_k}\to\chi$ in $L^{\gamma'}\left((0,T);L^{\gamma'}(\Omega^*)\right)$ weakly, $\frac{1}{\gamma'}+\frac{1}{\gamma}=1$, because of
$\left\|G(u_t^{*,\varepsilon,N_k})\right\|_{L^{\gamma'}\left((0,T);L^{\gamma'}(\Omega^*)\right)}\le g_2\left(\int_0^T\int_{\Omega^*}\left|u_t^{*,\varepsilon,N_k}\right|^\gamma dxdt\right)^{\frac{\gamma-1}{\gamma}}\le C=\text{const}.$

The premises and (7_ε) imply

$$\int\limits_0^\eta\int\limits_{\Omega^*}\left[-\varepsilon\left(u_t^{*,\varepsilon},v_t\right)+(\chi,v)+\sum_{i,j=1}^{n}\left(A_{ij}(x,t)u_{x_it}^{*,\varepsilon},v_{x_j}\right)+\sum_{i,j=1}^{n}\left(B_{ij}(x,t)u_{x_i}^{*,\varepsilon},v_{x_i}\right)+\right. \tag{16}$$
$$\left.+\left(u^{*,\varepsilon},v\right)\right]dxdt=\int\limits_0^\eta\int\limits_{\Omega^*}\left[\left(F_0^*(x,t),v\right)+\sum_{i=1}^{n}\left(F_i^*(x,t),v_{x_i}\right)\right]dxdt-\varepsilon\int\limits_{\Omega_\eta^*}\left(u_t^{*,\varepsilon},v\right)dx.$$

Let

$$(\forall v\in L^\gamma\left((0,T);L^\gamma(\Omega^*)\right)):\ X_{N_k}=\int\limits_0^\eta\int\limits_{\Omega^*}\left(G(u_t^{*,\varepsilon,N_k})-G(v),u_t^{*,\varepsilon,N_k}-v\right)e^{-\rho t}dxdt. \tag{17}$$

Then $X_{N_k} \geq 0$. According to (12)

$$\int_0^\eta \int_{\Omega^*} \Big(G(u_t^{*,\varepsilon,N_k}), u_t^{*,\varepsilon,N_k}\Big)\, e^{-\rho t} dxdt = -\frac{\varepsilon}{2} \int_{\Omega^*_\eta} \big|u_t^{*,\varepsilon,N_k}\big|^2 e^{-\rho t} dx + \int_0^\eta \int_{\Omega^*} \Bigg[\Big(F_0^*(x,t), u_t^{*,\varepsilon,N_k}\Big) +$$
$$+ \sum_{i=1}^n \Big(F_i^*(x,t), u_{x_i t}^{*,\varepsilon,N_k}\Big) - \frac{\rho\varepsilon}{2}\big|u_t^{*,\varepsilon,N_k}\big|^2 - \sum_{i,j=1}^n \Big(A_{ij}(x,t) u_{x_i t}^{*,\varepsilon,N_k}, u_{x_j t}^{*,\varepsilon,N_k}\Big) -$$
$$- \sum_{i,j=1}^n \Big(B_{ij}(x,t) u_{x_i}^{*,\varepsilon,N_k}, u_{x_j t}^{*,\varepsilon,N_k}\Big) - \Big(u^{*,\varepsilon,N_k}, u_t^{*,\varepsilon,N_k}\Big) \Bigg] e^{-\rho t} dxdt,$$

where ρ is a positive number such that

$$\sum_{i,j=1}^n \big((\rho B_{ij}(x,t) - B_{ijt}(x,t))\, \xi^i, \xi^j \big) \geq 0, \qquad \forall \xi^i, \xi^j \in \mathbb{R}^m, \ \ 1 \leq i,j \leq n.$$

Hence

$$X_{N_k} = -\frac{1}{2} \int_{\Omega^*_\eta} \Bigg[\varepsilon \big|u_t^{*,\varepsilon,N_k}\big|^2 + \sum_{i,j=1}^n \Big(B_{ij}(x,\eta) u_{x_i}^{*,\varepsilon,N_k}, u_{x_j}^{*,\varepsilon,N_k}\Big) \Bigg] e^{-\rho\eta} dx +$$
$$+ \int_0^\eta \int_{\Omega^*} \Bigg[\Big(F_0^*(x,t), u_t^{*,\varepsilon,N_k}\Big) + \sum_{i=1}^n \Big(F_i^*(x,t), u_{x_i t}^{*,\varepsilon,N_k}\Big) - \frac{\rho\varepsilon}{2}\big|u_t^{*,\varepsilon,N_k}\big|^2 -$$
$$- \sum_{i,j=1}^n \Big(A_{ij}(x,t) u_{x_i t}^{*,\varepsilon,N_k}, u_{x_j t}^{*,\varepsilon,N_k}\Big) - \Big(u^{*,\varepsilon,N_k}, u_t^{*,\varepsilon,N_k}\Big) -$$
$$- \frac{1}{2} \sum_{i,j=1}^n \Big((\rho B_{ij}(x,t) - B_{ijt}(x,t))\, u_{x_i}^{*,\varepsilon,N_k}, u_{x_j}^{*,\varepsilon,N_k}\Big) \Bigg] e^{-\rho t} dxdt -$$
$$- \int_0^\eta \int_{\Omega^*} \Big(G(u_t^{*,\varepsilon,N_k}), v\Big)\, e^{-\rho t} dxdt - \int_0^\eta \int_{\Omega^*} \Big(G(v), u_t^{*,\varepsilon,N_k} - v\Big)\, e^{-\rho t} dxdt,$$

and then

$$0 \leq \overline{\lim_{k\to\infty}} X_{N_k} \leq -\frac{1}{2} \int_{\Omega^*_\eta} \Bigg[\varepsilon \big|u_t^{*,\varepsilon}\big|^2 + \sum_{i,j=1}^n \Big(B_{ij}(x,\eta) u_{x_i}^{*,\varepsilon}, u_{x_j}^{*,\varepsilon}\Big) \Bigg] e^{-\rho\eta} dx +$$
$$+ \int_0^\eta \int_{\Omega^*} \Bigg[(F_0^*(x,t), u_t^{*,\varepsilon}) + \sum_{i=1}^n \big(F_i^*(x,t), u_{x_i t}^{*,\varepsilon}\big) - \frac{\rho\varepsilon}{2}\big|u_t^{*,\varepsilon}\big|^2 - \sum_{i,j=1}^n \Big(A_{ij}(x,t) u_{x_i t}^{*,\varepsilon}, u_{x_j t}^{*,\varepsilon}\Big) -$$
$$- \frac{1}{2} \sum_{i,j=1}^n \Big((\rho B_{ij}(x,t) - B_{ijt}(x,t))\, u_{x_i}^{*,\varepsilon}, u_{x_j}^{*,\varepsilon}\Big) - (u^{*,\varepsilon}, u_t^{*,\varepsilon}) \Bigg] e^{-\rho t} dxdt - \tag{18}$$
$$- \int_0^\eta \int_{\Omega^*} (\chi, v)\, e^{-\rho t} dxdt - \int_0^\eta \int_{\Omega^*} (G(v), u_t^{*,\varepsilon} - v)\, e^{-\rho t} dxdt.$$

From (16) we get

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{\Omega^*_\eta}\left[\varepsilon\left|u_t^{*,\varepsilon}\right|^2+\sum_{i,j=1}^{n}\left(B_{ij}(x,t)u^{*,\varepsilon}_{x_i},u^{*,\varepsilon}_{x_j}\right)\right]e^{-\rho\eta}dx+\\
&+\int\limits_0^\eta\int\limits_{\Omega^*}\Bigg[\left(F_0^*(x,t),u_t^{*,\varepsilon}\right)+\sum_{i=1}^{n}\left(F_i^*(x,t),u^{*,\varepsilon}_{x_it}\right)-\frac{\rho\varepsilon}{2}\left|u_t^{*,\varepsilon}\right|^2-\\
&-\sum_{i,j=1}^{n}\left(A_{ij}(x,t)u^{*,\varepsilon}_{x_it},u^{*,\varepsilon}_{x_jt}\right)-\frac{1}{2}\sum_{i,j=1}^{n}\left(\left(\rho B_{ij}(x,t)-B_{ijt}(x,t)\right)u^{*,\varepsilon}_{x_i},u^{*,\varepsilon}_{x_j}\right)-\\
&-\left(u^{*,\varepsilon},u_t^{*,\varepsilon}\right)\Bigg]e^{-\rho t}dxdt=\int\limits_0^\eta\int\limits_{\Omega^*}\left(\chi,u_t^{*,\varepsilon}\right)e^{-\rho t}dxdt.
\end{aligned}
\tag{19}
$$

By adding (18) to (19) we obtain $\int_0^\eta\int_{\Omega^*}\left(\chi-G(v),u_t^{*,\varepsilon}-v\right)e^{-\rho t}dxdt\ge 0$. Put $v=u_t^{*,\varepsilon}-\lambda w$, $\lambda>0$, where w is an arbitrary function from $L^\gamma\left((0,T);L^\gamma(\Omega^*)\right)$. Hereby

$$\lambda\int\limits_0^\eta\int\limits_{\Omega^*}\left(\chi-G(u_t^{*,\varepsilon}-\lambda w),w\right)e^{-\rho t}dxdt\ge 0,$$

and then $\int\limits_0^\eta\int\limits_{\Omega^*}\left(\chi-G(u_t^{*,\varepsilon}-\lambda w),w\right)e^{-\rho t}dxdt\ge 0$. If λ tends to zero, then

$$\left(\forall w\in L^\gamma\left((0,T);L^\gamma(\Omega^*)\right)\right):\qquad\int\limits_0^\eta\int\limits_{\Omega^*}\left(\chi-G(u_t^{*,\varepsilon}),w\right)e^{-\rho t}dxdt\ge 0,$$

(we can pass to the limit as $\lambda\to+0$ because of the Lebesgue theorem and semicontinuity of the operator G.) Thus $\chi=G(u_t^{*,\varepsilon})$. This implies that $u^{*,\varepsilon}$ is a solution of (7_ε)–(10), namely $u^{*,\varepsilon}$ satisfies (11). □

Estimate (15) holds for $u^{*,\varepsilon}$ and does not depend on ε. Thus we can choose a subsequence $\{u^{*,\varepsilon_m}\}$ such that $u^{*,\varepsilon_m}\to u^*$ in $L^2(Q_T^*)$ weakly. Whence, $u^{*,\varepsilon_m}_{x_i}\to u^*_{x_i}$ in $H^1\left((0,T);L^2(\Omega^*)\right)$ weakly and $u_t^{*,\varepsilon_m}\to u_t^*$ in $L^\gamma(Q_T^*)$ weakly.

As far as

$$
\begin{aligned}
\varepsilon_m\int\limits_{\Omega^*_\eta}\left(u_t^{*,\varepsilon_m},v\right)dx&\le\varepsilon_m\left(\int\limits_{\Omega^*_\eta}\left|u_t^{*,\varepsilon_m}\right|^2dx\right)^{\frac12}\left(\int\limits_{\Omega^*_\eta}|v|^2dx\right)^{\frac12}\le\\
&\le\frac{\varepsilon_m^{\frac43}}{2}\int\limits_{\Omega^*_\eta}\left|u_t^{*,\varepsilon_m}\right|^2dx+\frac{\varepsilon_m^{\frac23}}{2}\int\limits_{\Omega^*_\eta}|v|^2dx\to 0\quad\text{ if }\varepsilon_m\to 0,
\end{aligned}
$$

after passing to the limit in (11) as $\varepsilon_m\to 0$ we get that u^* is a solution of (7_0), (8), (10). Thus we proved existence of a solution (1)–(3) in a bounded cylinder.

Theorem 2. *Let the coefficients of (1) satisfy the conditions of Theorem 1 and*

$$u_0 \in H^1_{loc}(\Omega), \qquad F_0 \in L^{\frac{\gamma}{\gamma-1}}_{loc}(Q_T), \qquad F_i \in L^2_{loc}(Q_T), \ \ i \in \{1, ..., n\}.$$

Then a solution of problem (1)–(3) exists.

Proof. As far as domain Ω was worked in special way we can choose a sequence of embedded cylinders $Q^\tau_T = \Omega^\tau \times (0, T)$. Consider the problem

$$G(u_t) - \sum_{i,j=1}^{n} (A_{ij}(x,t)u_{x_i t})_{x_j} - \sum_{i,j=1}^{n} (B_{ij}(x,t)u_{x_i})_{x_i} + u = F^\tau_0(x,t) - \sum_{i=1}^{n} F^\tau_{i,x_i}(x,t), \quad (20_\tau)$$

$$u|_{t=0} = u^\tau_0, \quad u|_{\Gamma^\tau \times (0,T)} = 0, \quad (21_\tau)$$

for each cylinder Q^τ_T. Here

$$F^\tau_i(x,t) = \begin{cases} F_i(x,t), & (x,t) \in Q^\tau_T, \\ 0, & (x,t) \in Q_T \setminus Q^\tau_T, \end{cases} \qquad i \in \{0, ..., n\}.$$

As u^τ_0 we take $u_0(x)\zeta^\tau(|x|)$, where the function $\zeta^\tau \in C^\infty_0(\mathbb{R}^1)$ is such that $\zeta^\tau(r) = 1$ if $|r| \le \tau - \delta$, $\zeta^\tau(r) = 0$ if $|r| \ge \tau$, $0 \le \zeta^\tau(r) \le 1$ if $\tau - \delta < |r| < \tau$; a fixed number δ is small enough.

Problem (20_τ)–(21_τ) has a solution $u^\tau \in L^2(Q^\tau_T)$, $u^\tau_{x_i} \in H^1((0,T); L^2(\Omega^\tau))$, $u^\tau_t \in L^\gamma(Q^\tau_T)$. After extension of the solution by zero over all the domain Q_T the equality

$$\int_0^\eta \int_\Omega \Big[(G(u^\tau_t), v) + \sum_{i,j=1}^{n} \big(A_{ij}(x,t)u^\tau_{x_i t} + B_{ij}(x,t)u^\tau_{x_i}, v_{x_j}\big) + (u^\tau, v) \Big] dx dt =$$

$$= \int_0^\eta \int_\Omega \Big[(F^\tau_0(x,t), v) + \sum_{i=1}^{n} (F^\tau_i(x,t), v_{x_i}) \Big] dx dt \quad (22)$$

holds for every function $v \in C^\infty([0,T]; C^\infty_0(\Omega))$ and for almost every $\eta \in [0,T]$. We put $v = u^\tau_t \psi^\alpha(x)$, $\psi \in C^\infty_0(\Omega)$, $\alpha > 0$, in (22) and estimate every term. Therefore,

$$I_{10} = \int_0^\eta \int_\Omega (F^\tau_0(x,t), u^\tau_t \psi^\alpha)\, dx dt \le \int_0^\eta \int_\Omega \Big[\frac{\delta_5}{\gamma} |u^\tau_t|^\gamma + \frac{\gamma - 1}{\gamma \delta_5^{\frac{1}{\gamma-1}}} |F^\tau_0(x,t)|^{\frac{\gamma}{\gamma-1}} \Big] \psi^\alpha dx dt,$$

$$I_{11} = \int_0^\eta \int_\Omega \sum_{i=1}^{n} \big(F^\tau_i(x,t), (u^\tau_t \psi^\alpha)_{x_i}\big)\, dx dt \le \int_0^\eta \int_\Omega \sum_{i=1}^{n} \Big[\frac{\delta_6}{2} |u^\tau_{x_i t}|^2 + \frac{1}{2\delta_6} |F^\tau_i(x,t)|^2 \Big] \psi^\alpha dx dt +$$

$$+ \frac{\alpha}{2\delta_7^{\frac{2}{\gamma}}} \int_0^\eta \int_\Omega \sum_{i=1}^{n} |F^\tau_i(x,t)|^2 \psi^\alpha dx dt + \frac{\alpha \delta_7 n}{\gamma} \int_0^\eta \int_\Omega |u^\tau_t|^\gamma \psi^\alpha dx dt + \frac{\alpha(\gamma-2)}{2\gamma} T \Psi_2$$

where $\delta_5 > 0$, $\delta_6 > 0$, $\delta_7 > 0$. Hence, (22) implies

$$\frac{1}{2} \int_{\Omega_\eta} \Big[b_0 \sum_{i=1}^{n} |u^\tau_{x_i}|^2 + d_0 |u^\tau|^2 \Big] \psi^\alpha dx + \int_0^\eta \int_\Omega \Big[\Big(g_0 - \frac{\alpha n^2 (A_0 \delta_1 + B_0 \delta_2) + \delta_5 + \alpha n \delta_7}{\gamma} \Big) |u^\tau_t|^\gamma -$$

$$-\left(\frac{\alpha B_0 n}{2}+\frac{b_1}{2}\right)\sum_{i=1}^{n}\left|u^\tau_{x_i}\right|^2+\left(a_0-\frac{\delta_6}{2}\right)\sum_{i=1}^{n}\left|u^\tau_{x_it}\right|^2\psi^\alpha-\frac{d_1}{2}\left|u^\tau\right|^2\Bigg]\psi^\alpha dxdt\le\frac{T\hat{\Psi}+\Phi}{2},\quad(23)$$

where

$$\Phi=\int\limits_{Q_T}\left[2\frac{\gamma-1}{\gamma\delta_5^{\frac{1}{\gamma-1}}}|F_0^\tau(x,t)|^{\frac{\gamma}{\gamma-1}}+\left(\frac{\alpha}{\delta_7^{\frac{2}{\gamma}}}+\frac{1}{\delta_6}\right)\sum_{i=1}^{n}\left|F_i^\tau(x,t)\right|^2\right]\psi^\alpha dxdt+$$

$$+\int\limits_{\Omega_0}\left[\sum_{i,j=1}^{n}\left(B_{ij}(x,0)u^\tau_{0x_i},u^\tau_{0x_j}\right)+(D(x,0)u^\tau_0,u^\tau_0)\right]\psi^\alpha dx,$$

$$\hat{\Psi}=\frac{\alpha A_0(\gamma-2)}{\gamma\delta_1^{\frac{2}{\gamma-2}}}\Psi_1+\frac{\alpha B_0(\gamma-2)n}{\gamma\delta_2^{\frac{2}{\gamma-2}}}\Psi_2+\frac{\alpha(\gamma-2)}{\gamma}\Psi_2.$$

If δ_i $(i\in\{1,2,5,6,7\})$ such that

$$g_0-\frac{\alpha n^2(A_0\delta_1+B_0\delta_2)+\delta_5+\alpha n\delta_7}{\gamma}\ge\frac{g_0}{2},\qquad a_0-\delta_6\ge0,$$

then

$$\int\limits_{\Omega_\eta}\left(\sum_{i=1}^{n}\left|u^\tau_{x_i}\right|^2+\left|u^\tau\right|^2\right)\psi^\alpha dx\le\frac{T\hat{\Psi}+\Phi}{\mu_0}e^{\frac{\mu_1}{\mu_0}T},$$

$$\int\limits_0^\eta\int\limits_\Omega\left(g_0\left|u^\tau_t\right|^\gamma+a_0\sum_{i=1}^{n}\left|u^\tau_{x_it}\right|^2\right)\psi^\alpha dxdt\le\frac{T\hat{\Psi}+\Phi}{\mu_0}\mu_2.$$

If ψ is the function introduced in the proof of Theorem 1 for $R\in\Pi$, $R_0<R$, $R_0\in\Pi$, then

$$\int\limits_{\Omega_\eta^{R_0}}\left(\sum_{i=1}^{n}\left|u^\tau_{x_i}\right|^2+\left|u^\tau\right|^2\right)dx\le\frac{\alpha T(\gamma-2)2^\alpha n^2}{\mu_0\gamma}R^{-\frac{2\gamma}{\gamma-2}+n}P_ne^{\frac{\mu_1}{\mu_0}T}\times$$

$$\times\left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}}(2R)^{\frac{\gamma}{\gamma-2}}+\frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}}+\frac{1}{n}\right)+\frac{\Phi e^{\frac{\mu_1}{\mu_0}T}}{\mu_0(R-R_0)^\alpha};$$

$$\int\limits_0^\eta\int\limits_{\Omega^{R_0}}\left(g_0\left|u^\tau_t\right|^\gamma+a_0\sum_{i=1}^{n}\left|u^\tau_{x_it}\right|^2\right)dxdt\le\frac{\alpha T(\gamma-2)2^\alpha n^2}{\gamma}R^{-\frac{2\gamma}{\gamma-2}+n}P_n\times$$

$$\times\left(\frac{A_0}{\delta_1^{\frac{2}{\gamma-2}}}(2R)^{\frac{\gamma}{\gamma-2}}+\frac{B_0}{\delta_2^{\frac{2}{\gamma-2}}}+\frac{1}{n}\right)\mu_2+\frac{\Phi e^{\frac{\mu_1}{\mu_0}T}}{\mu_0(R-R_0)^\alpha}\mu_2.$$

Taking into account the assumptions of the theorem, the definition of u^τ_0, F^τ_i $(i\in\{0,...,n\})$ and the inequality $R>R_0$ we get

$$\int\limits_{\Omega_\eta^{R_0}}\left(\sum_{i=1}^{n}\left|u^\tau_{x_i}\right|^2+\left|u^\tau\right|^2\right)dx<\frac{\Phi_0e^{\frac{\mu_1}{\mu_0}T}}{\mu_0},\quad\int\limits_0^\eta\int\limits_{\Omega^{R_0}}\left(g_0\left|u^\tau_t\right|^\gamma+a_0\sum_{i=1}^{n}\left|u^\tau_{x_it}\right|^2\right)dxdt<\frac{\Phi_0e^{\frac{\mu_1}{\mu_0}T}}{\mu_0}\mu_2,$$

where

$$\Phi_0 = 2^\alpha \int\limits_{Q_T^R} \left[2\frac{\gamma-1}{\gamma\delta_5^{\frac{1}{\gamma-1}}} \left|F_0(x,t)\right|^{\frac{\gamma}{\gamma-1}} + \left(\frac{\alpha}{\delta_7^{\frac{2}{\gamma}}} + \frac{1}{\delta_6} \right) \sum_{i=1}^{n} \left|F_i(x,t)\right|^2 \right] dxdt +$$

$$+ 2^\alpha \int\limits_{\Omega_0^R} \left[\sum_{i,j=1}^{n} \left(B_{ij}(x,0)u_{0x_i}, u_{0x_j}\right) + \left(D(x,0)u_0, u_0\right) \right] dx.$$

It is easy to show that there exists a subsequence $\{u^{\tau_k}\}$ of the sequence $\{u^\tau\}$ such that
$u^{\tau_k} \to u$ in $L^2_{\rm loc}(Q_T)$ weakly,
$G(u_t^{\tau_k}) \to \chi$ in $L^{\gamma'}_{\rm loc}(Q_T)$ weakly,
$u_{x_i}^{\tau_k} \to u_{x_i}$ in $H^1\left((0,T); L^2_{\rm loc}(\Omega)\right)$ weakly, $(i \in \{1,...,n\})$.

Passage to the limit in (22) as $\tau_k \to \infty$ implies that $\chi = G(u_t)$, and finally that u is a solution of (1)–(3). □

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