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PARATOPOLOGICAL GROUPS, II

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We study relations between topologies on a group related the group operation. Some results on separation axioms and cardinal invariants of these topologies are obtained.

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Изучаются отношения между топологиями на группе связанные с групповой операцией. Получены результаты об аксиомах отделимости и кардинальных инвариантах этих топологий.

1. General properties

This paper is a continuation of the article [6]. Here we obtain some new results concerning paratopological and semitopological groups. We shall use the definitions from [6].

Let G be a group endowed with a topology τ. If the multiplication and the inversion in G are continuous then G is a *topological group*. If the multiplication in G is continuous then G is a *paratopological group*. If the multiplication in G is separately continuous then G is a *semitopological group*. If the multiplication is separately continuous and the inversion in G is continuous then G is a *quasitopological group*. In these cases the topology τ is called a *group topology*, a *paratopology*, a *semitopology*, and a *quasitopology* respectively. Remark that in the papers [4] and [7] a semitopological group is called m-topological, and a quasitopological group is called semitopological. Hence every topological group is paratopological and quasitopological and both these groups are semitopological. Clearly, all translations and interior automorphisms of a semitopological group are homeomorphisms.

Let $\mathcal{B}$ be a family of subsets of a group G containing the unit e. Put $\mathcal{B}' = \{xU : x \in G, U \in \mathcal{B}\}$. Consider the following Pontrjagin conditions for $\mathcal{B}$.

1. $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B}) : W \subset U \cap V$.
2. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^2 \subset U$.
3. $(\forall U \in \mathcal{B})(\forall x \in U)(\exists V \in \mathcal{B}) : xV \subset U$.
4. $(\forall U \in \mathcal{B})(\forall x \in G)(\exists V \in \mathcal{B}) : x^{-1}Vx \subset U$.
5. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^{-1} \subset U$.

The base $\mathcal{B}'$ defines a semitopology on a group G if and only if Conditions 1, 3, and 4 are satisfied. The base $\mathcal{B}'$ defines a paratopology on a group G if and only if Conditions 1,

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2, 3, and 4 are satisfied. The base $\mathcal{B}'$ defines a quasitopology on a group G if and only if Conditions 1, 3, 4, and 5 are satisfied. The base $\mathcal{B}'$ defines a group topology on G if and only if Conditions 1, 2, 3, 4, and 5 are satisfied.

The semitopological group G is Hausdorff if and only if $\bigcap\{UU^{-1} : U \in \mathcal{B}\} = \{e\}$.

Let $\mathcal{S}$ be a subbase of the topology τ on a group G. The group (G, τ) is semitopological if and only if for every points $x, y \in G$ and every neighborhood $U \in \mathcal{S}$ of the point xy there exist neighborhoods $V, W \in \mathcal{S}$ of the points x, y respectively such that $Vy \cup xW \subset U$. The group (G, τ) is paratopological if and only if for every points $x, y \in G$ and every neighborhood $U \in \mathcal{S}$ of the point xy there exist neighborhoods $V, W \in \mathcal{S}$ of the points x, y respectively such that $VW \subset U$. The group (G, τ) is topological if and only if for every points $x, y \in G$ and every neighborhood $U \in \mathcal{S}$ of the point xy^{-1} there exist neighborhoods $V, W \in \mathcal{S}$ of the points x, y respectively such that $VW^{-1} \subset U$. The group (G, τ) is quasitopological if and only if for every points $x, y \in G$ and every neighborhood $U \in \mathcal{S}$ of the point xy^{-1} there exist neighborhoods $V, W \in \mathcal{S}$ of the points x, y respectively such that $xW^{-1} \subset U$ and $Vy^{-1} \subset U$.

For a group G by $\mathcal{T}(G)$, $\mathcal{P}(G)$, $\mathcal{S}(G)$, $\mathcal{Q}(G)$ we denote the set of all group topologies, paratopologies, semitopologies and quasitopologies on the group G respectively.

Proposition 1.1. *Let $\tau \in \mathcal{S}(G)$ be a semitopology. Then there exist the following semitopologies on the group G:*

1. *The finest paratopology τ_p coarser than τ.*
2. *The finest quasitopology τ_q coarser than τ.*
3. *The finest group topology τ_g coarser than τ.*
4. *The coarsest quasitopology τ^q finer than τ.*

Proof. Let $\mathcal{S} = \{\tau' \in \mathcal{P}(G)(\mathcal{Q}(G), \mathcal{T}(G)$ respectively$) : \tau' \subset \tau\}$. For every semitopology $\tau' \in \mathcal{S}$ fix a base $\mathcal{B}'_\tau$ at the unit. Let $\mathcal{S}' \subset \mathcal{S}$ be a chain. Put $\mathcal{B}' = \bigcup\{\mathcal{B}'_\tau : \tau \in \mathcal{S}'\}$. Then $\mathcal{S}'$ is a base of a semitopology τ' on the group G. It is clear that $\tau' \in \mathcal{S}$. Thus the set $\mathcal{S}$ is inductive and hence contains a maximal element τ_1. Suppose that there exists another maximal element $\tau_2 \in \mathcal{S}$ different from τ_1. Let the semitopology τ_i have a base $\mathcal{B}_i$ at the unit. Then $\mathcal{B}_3 = \{U_1 \cap U_2 : U_i \in \mathcal{B}_i\}$ is a base at the unit of the semitopology τ_3. Thus τ_3 is an element of $\mathcal{S}$ greater than both τ_1 and τ_2. Therefore the set $\mathcal{S}$ contains the unique maximal element τ_1 which is clearly equal to τ_p (τ_q, τ_g respectively).

Let $\mathcal{B}$ be a base of the unit of the semitopology τ. Then $\mathcal{B}^q = \{U \cap U^{-1} : U \in \mathcal{B}\}$ is a base of the unit of the quasitopology τ^q. □

Proposition 1.2. *Let G_1, G_2 be groups and $\varphi\colon G_1 \to G_2$ be a homomorphism. Let $\tau_1 \in \mathcal{S}(G_1)$ and $\tau_2 \in \mathcal{S}(G_2)$. If the homomorphism $\varphi\colon (G_1, \tau_1) \to (G_2, \tau_2)$ is continuous then the homomorphisms $\varphi\colon (G_1, \tau_{1p}) \to (G_2, \tau_{2p})$, $\varphi\colon (G_1, \tau_{1q}) \to (G_2, \tau_{2q})$, $\varphi\colon (G_1, \tau_{1g}) \to (G_2, \tau_{2g})$ and $\varphi\colon (G_1, \tau_1^q) \to (G_2, \tau_2^q)$ are continuous.*

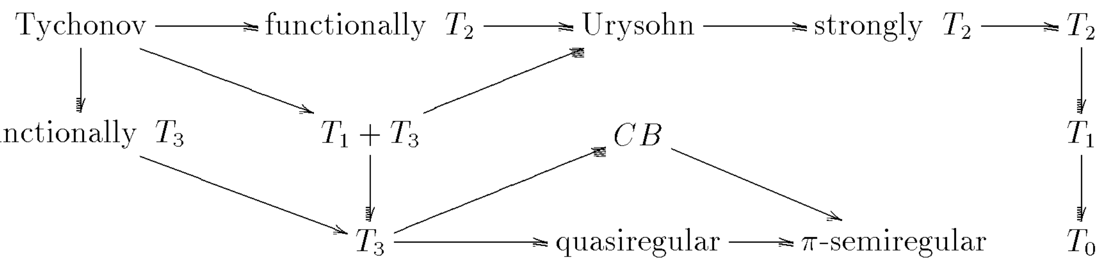
Proof. For the homomorphisms $\varphi\colon (G_1, \tau_{1q}) \to (G_2, \tau_{2q})$ and $\varphi\colon (G_1, \tau_1^q) \to (G_2, \tau_2^q)$ this is true because $\tau_{1q} = \tau_1 \wedge \tau_1^{-1}$ and $\tau_1^q = \tau_1 \vee \tau_1^{-1}$, similarly for the topology τ_2. For the homomorphism $\varphi\colon (G_1, \tau_{1p}) \to (G_2, \tau_{2p})$ this is true since the semitopology $\varphi^*(\tau_{2p})$ belongs to $\mathcal{P}(G_1)$ and hence $\varphi^*(\tau_{2p}) \subset \tau_{1p}$. For the homomorphism $\varphi\colon (G_1, \tau_{1g}) \to (G_2, \tau_{2g})$ this is true because the semitopology $\varphi^*(\tau_{2g})$ belongs to $\mathcal{G}(G_1)$ and hence $\varphi^*(\tau_{2g}) \subset \tau_{1g}$. □

A subsemigroup S of a group G is said to be *normal* if $x^{-1}Sx \subset S$ for every $x \in G$. For every normal submonoid S of the group G by τ_S we denote the paratopology with the base $\{xS : x \in G\}$. Then $\tau_S = \inf\{\tau \in \mathcal{P}(G) : S \in \tau\}$.

Example 1.3. Sorgenfrey arrow. $(\mathbb{R}, \tau_s) = (\mathbb{R}, \tau_{\mathbb{R}_+} \wedge \tau)$, where τ is the standard topology on $\mathbb{R}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. $(\mathbb{R}, \tau_s)$ is a zero-dimensional Hausdorff first countable paratopological group.

Example 1.4. Let $G = (\mathbb{R}^2, +)$. Put $\mathcal{B} = \{\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : (x-1/n)^2 + y^2 < 1/n^2\} : n \in \mathbb{N}\}$. Then $\mathcal{B}$ is a base at the zero of a paratopology on the group G.

Let X be a topological space. A space X is T_0 if for every distinct points $x, y \in X$ there exists an open set $U \subset X$ such that $U \cap \{x, y\}$ is a singleton. A space X is T_1 if for every distinct points $x, y \in X$ there exists an open set $x \in U \subset X \backslash \{y\}$. A space X is T_2 or *Hausdorff* if for every distinct points of X has disjoint open neighborhoods. A space X is *functionally Hausdorff* if for every distinct points $x, y \in X$ there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. A space X is *Urysohn* if every distinct points of X has disjoint closed neighborhoods. A space X is *strongly Hausdorff* if it is Hausdorff and from every infinite subset $A \subset X$ we can choose a sequence $\{x_n : n \in \omega\}$ such that the points x_n have pairwise disjoint neighborhoods in X. A space X is T_3 if every closed set $F \subset X$ and every point $x \in X \backslash F$ have disjoint neighborhoods. A space X is regular if it is T_1 and T_3. A set $U \in X$ is called *canonical* if $U = \operatorname{int} \overline{U}$. A space X is CB if X has a base consisting of canonical open sets. A space X is *semiregular* if it is CB and T_2. A space is *quasiregular* if every nonempty open set contains the closure of some nonempty open set. A space is *π-semiregular* if every nonempty open set contains the interior of the closure of some nonempty open set. A space X is *functionally* T_3 if for every if for every closed set $F \subset X$ every point $x \in X \backslash F$ there exists a continuous function $X \to [0,1]$ such that $f(x) = 0$ and $f|F = 1$. A space X is *completely regular* or *Tychonov* if it is T_1 and functionally T_3. The following inclusions are true. Every T_0+T_3 space is regular and every T_0+functionally T_3 space is Tychonov [1, Ex. 1.5.D].



It is well known that every topological group is T_3 and every T_0 topological group is completely regular. For paratopological groups the situation with the separation axioms is worse.

Proposition 1.5. *Every CB paratopological group is T_3.*

Proof. Let $\mathcal{B}$ be a base at the unit of a paratopological group G consisting of canonical open sets. For every neighborhood $U \in \mathcal{B}$ there exists a neighborhood $V \in \mathcal{B}$ such that $V^2 \subset U$. Then $V\overline{V} \subset \overline{U}$. Hence $\overline{V} \subset \operatorname{int} \overline{U} = U$. □

Proposition 1.6. *Every π-semiregular paratopological group is quasiregular.*

Proof. Let $\mathcal{B}$ be a base at the unit of a π-semiregular paratopological group G. For every neighborhood $U \in \mathcal{B}$ there exist a point $x \in U$ and a neighborhood $V \in \mathcal{B}$ such that $x \operatorname{int} \overline{V} \subset U$. Choose a neighborhood $W \in \mathcal{B}$ such that $W^2 \subset V$. Then $x\overline{W} \subset x\overline{W}W \subset x \operatorname{int} \overline{V}$ and hence G is quasiregular. □

Proposition 1.7. *Let G be a paratopological group such that* $\operatorname{int} U^{-1} \neq \varnothing$ *for every nonempty open set $U \subset G$. Then G is quasiregular.*

Proof. Let $\mathcal{B}$ be a base at the unit of such a group G. Let $U \in \mathcal{B}$ be an arbitrary neighborhood. Choose a neighborhood $V \in \mathcal{B}$ such that $V^2 \subset U$. There exist a neighborhood $W \in \mathcal{B}$ and a point $x \in V$ such that $W^{-1}x \subset V$. Then $\overline{V}x \subset VW^{-1}x \subset U$ and hence the group G is quasiregular. □

Example 1.8. Guran posed the following question: Let G be a Baire regular paratopological group and U be nonempty open subset of G. Is $\operatorname{int} U^{-1} \neq \varnothing$? We construct a zero-dimensional Hausdorff first countable Baire paratopological group G and a nonempty open set $U \subset G$ such that $\operatorname{int} U^{-1} = \varnothing$. Let G be a Tychonov product $\mathbb{R}^\omega$. Define a base at the zero $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$, where $U_n = \{(x_i) \in G : x_i = 0 \text{ for } i \leq n \text{ and } x_i \geq 0 \text{ for } i > n\}$. Since every set U_n is clopen, G is a zero-dimensional Hausdorff. It is clear that $\operatorname{int} U_1^{-1} = \varnothing$. We claim that G is Baire. Since every second category homogeneous space is Baire [3, Prop. 1.27] we show that G is of second category. Let $\{G_n\}$ be a decreasing sequence of dense open subsets of G. By induction we can construct sequences $\{x^n\} \subset G$ and $\{k_n\} \subset \mathbb{N}$ such that: 1) $x^{n+1} + U_{k_{n+1}} \subset G_n \cap (x^n + U_{k_n})$; 2) $k_{n+1} > k_n$.

Since $x^m \in x^n + U_{k_n}$ for every natural n and every $m \geq n$, the first k_n coordinates of the points x^n and x^m are the same. Hence there exists a unique point $x \in G$ such that for every natural n the first k_n coordinates of the points x^n and x are the same. By the construction $x_i \geq x_i^n$ for every natural i and n. Hence $x \in x^n + U_{k_n} \subset G_{n-1}$ for every natural $n > 1$ and therefore the space G is of second category.

Example 1.9. Let (G, τ) be a semitopological group. Define the topology τ_r on the group G as follows. Let $\mathcal{B}$ be a base of the unit of the topology τ. Put $\mathcal{B}_r = \{\operatorname{int} \overline{U} : U \in \mathcal{B}\}$. Verify Pontrjagin conditions for the topology $\mathcal{B}_r$.

1. Trivial.

3. Let $U \in \mathcal{B}$ and $x \in \operatorname{int} \overline{U}$. There exists a neighborhood $V \in \mathcal{B}$ such that $xV \subset \operatorname{int} \overline{U}$. Then $x\overline{V} \subset \overline{U}$. Hence $x \operatorname{int} \overline{V} \subset \operatorname{int} \overline{U}$.

4. Trivial.

Thus τ_r is a semitopology on the group G. Moreover, if τ is a paratopology then τ_r is a paratopology as well. Indeed, in this case we must verify Pontrjagin condition 2. Let $U \in \mathcal{B}$. There exists a neighborhood $V \in \mathcal{B}$ such that $V^2 \subset U$. Then $\overline{V}^2 \subset \overline{U}$ and since $\operatorname{int} \overline{V}$ is open, we see that $(\operatorname{int} \overline{V})^2 \subset \operatorname{int} \overline{U}$.

Clearly, τ_r does not depend on the choice of the basis $\mathcal{B}$ of a semitopology τ. Moreover, τ_r is a CB-semitopology. Indeed, it suffices to verify that for every neighborhood $U \in \mathcal{B}$ we have $\overline{\operatorname{int} \overline{U}}^{\tau_r} \subset \overline{U}$, because then $\operatorname{int}_{\tau_r} \overline{\operatorname{int} \overline{U}}^{\tau_r} \subset \operatorname{int} \overline{\operatorname{int} \overline{U}}^{\tau_r} \subset \operatorname{int} \overline{U}$. Let $y \in \overline{\operatorname{int} \overline{U}}^{\tau_r}$. Then for every neighborhood $V \in \mathcal{B}$ we have $y \operatorname{int} \overline{V} \cap \operatorname{int} \overline{U} \neq \varnothing$ and thus $yV \cap \operatorname{int} \overline{U} \neq \varnothing$. Hence $y \in \overline{\operatorname{int} \overline{U}} \subset \overline{U}$. Remark that if τ is Hausdorff then τ_r is T_1 and hence regular.

Engelking in [1, Ex. 1.7.8] considers a similar construction. A subset U of a topological space (G, τ) is called *canonically open* if $U = \operatorname{int} \overline{U}$. The interior of a closed set is canonically open. Stone and Katetov construct the topology τ_r on G generated by the base consisting of all canonically open sets of the space (G, τ). This definition coincides with our definition for the semitopological groups.

Let f be a function from a space X into a space Y. Then f will be called *δ-open* if for every nowhere dense subset N of Y, $f^{-1}(N)$ is nowhere dense. If a function $f \colon X \to Y$ is continuous and the image of each nonempty open subset of X is somewhere dense in Y then

f is δ-open [3, Prop. 4.4]. If f is a continuous δ-open function from a Baire space X onto a space Y then Y is a Baire space. Let U be an open subset of a semitopological group (G,τ). Then $\overline{U}^{\tau_r} \supset \overline{U} \supset \operatorname{int}\overline{U}$ hence U is somewhere dense in the topology τ_r and the identity map from (G,τ) to (G,τ_r) is δ-open. Hence if (G,τ) is Baire then (G,τ_r) is Baire as well.

Proposition 1.10. *Every Hausdorff semitopological group is Urysohn.*

Proof. Let (G,τ) be such a group. Example 1.9 implies that (G,τ_r) is regular. Since (G,τ_r) is Urysohn and $\tau_r \subset \tau$, we see that (G,τ) is Urysohn. □

Example 1.11. Put $G = \mathbb{Z}/2\mathbb{Z}$ with trivial topology. Then G is functionally T_3 not T_0 paratopological group.

Example 1.12. Put $G = (\mathbb{R},+)$ and $\tau = \{x + [0;\infty) : x \in \mathbb{R}\}$. Then G is T_0 not T_1 paratopological group.

Example 1.13. Put $G = (\mathbb{R},+)$ and $\tau = \{\{x\} \cup [y;\infty) : x,y \in \mathbb{R}\}$. Then G is T_1 not T_2 paratopological group.

Example 1.14. Put $G = \mathbb{T} \times \mathbb{Z}_2$ where $\mathbb{T}$ is the unit circle and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. For every $\varepsilon > 0$ put $U(\varepsilon) = \{(0,0)\} \cup (0;\varepsilon) \times \mathbb{Z}_2$. It easy to check that $\mathcal{B} = \{U(\varepsilon) : \varepsilon > 0\}$ is a base at the zero of a paratopology on the group G. Then G is T_1 quasiregular not T_2.

Example 1.15. Put $G = \mathbb{R}^2$. Define a base $\mathcal{B}$ at the unit of the group G putting $\mathcal{B} = \{\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : 0 < x,y < 1/n\} : n \in \mathbb{N}\}$. Then G is functionally T_2 quasiregular not T_3 paratopological group.

Example 1.16. The *p-arrow.* Let p be a natural number. Put $(\mathbb{Z},\tau) = (\mathbb{Z},\tau_{\mathbb{Z}_+} \wedge \tau_p)$, where τ_p has a base at the unit $\mathcal{B}_p = \{p^n\mathbb{Z} : n \in \mathbb{N}\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Then G is functionally T_2 not π-semiregular paratopological group.

A *subgroup* $(H,\tau|H)$ of a semitopological group (G,τ) is a subgroup of the group G endowed with the induced from (G,τ) topology $\tau|H$. Clearly, any subgroup of a semitopological (resp. paratopological, quasitopological, topological) group is again a semitopological (resp. paratopological, quasitopological, topological) group.

Remark 1.17. The closure of a subgroup of a paratopological group is not a paratopological group in general. Indeed, let G be the group from Example 1.12 and $H = \{0\}$. Then $\overline{H} = (-\infty;0]$ is not a subgroup.

Proposition 1.18. *Every element of a Hausdorff paratopological group is contained in a closed abelian subgroup.*

Proof. Let G be such a group and $x \in X$. We construct a transfinite sequence $\{G_\alpha\}$ of closed abelian subsemigroups of G by the following. Put $G_0 = \overline{\langle x\rangle}$. Let G_α be already defined for all ordinals $\alpha < \beta$. If β is a limit ordinal we put $G_\beta = \overline{\bigcup\{G_\alpha : \alpha < \beta\}}$. If $\beta = \alpha+1$ then put $G_\beta = \overline{\langle G_\alpha\rangle}$. There exists an ordinal α such that $G_\alpha = G_{\alpha+1}$. Then G_α is a closed abelian group containing the element x. □

The *product* of a family of semitopological groups $\{(G_\alpha,\tau_\alpha) : \alpha \in A\}$ is the product $\prod G_\alpha$ endowed with the Hausdorff product topology $\prod \tau_\alpha$. The *box product* of a family of semitopological groups $\{(G_\alpha,\tau_\alpha) : \alpha \in A\}$ is the product $\prod G_\alpha$ endowed with the box product topology τ_α. It is easy to see that both the product and the box product of a family of semitopological (resp. paratopological, quasitopological and topological) groups is again a semitopological (resp. paratopological, quasitopological and topological) group.

Example 1.19. Reznichenko posed the following question. *Can every Hausdorff paratopological group (G,τ) be embedded into the product $(G_1,\tau_1)\times(G_2,\tau_2)$ of paratopological groups where the topology τ_1^q is discrete and (G_2,τ_2) is a topological group?* This example shows that the answer is negative. Put $G=(\mathbb{R},+)$. Let ρ_1 denotes the standard topology of $\mathbb{R}$, ρ_2 denotes the topology induced on $\mathbb{R}$ by dense wrapping around the torus. Let $\mathcal{B}_i$ be a base at the zero of the topology ρ_i for $i=1,2$. Define a base $\mathcal{B}$ at the zero of a topology τ as follows: $\mathcal{B}=\{(U_1\cap(-\infty;0])\cup(U_2\cap[0;\infty)) : U_i\in\mathcal{B}_i\}$. Then τ is the supremum of the following two paratopologies: the topology which is on $\mathbb{R}$ wrapped dense around the torus and the topology with a base at the unit $\{-\varepsilon;\infty\}$, $\varepsilon>0$. Therefore τ is a paratopology on the group $\mathbb{R}$. It is easy to see that $\tau^q=\rho_1$. Suppose that there exists the embedding defined above. Let $\pi\colon G\to G_1$ be the standard projection. Proposition 1.2 implies that the map $\pi\colon(G,\tau^q)\to\ (G_1,\tau_1^q)$ is continuous. Since the topology τ_1^q is discrete and (G,τ^q) is connected then the map π is trivial. Therefore (G,τ) is a subgroup of a topological group (G_2,τ_2) and hence is a topological group, a contradiction. Therefore the embedding defined above for the group (G,τ) does not exist.

2. Cardinal invariants

A function φ defined in the class $\mathcal{C}$ of paratopological groups is called a *cardinal function* if it assigns to each member $G\in\mathcal{C}$ an infinite cardinal number $\varphi(G)$. Now we shall list the cardinal functions to be examined in what follows. Remark that in the definitions below $\min^*\{\cdot\}=\omega\cdot\min\{\cdot\}$ and $\sup^*\{\cdot\}=\omega\cdot\sup\{\cdot\}$.

Precompactness. Let λ be an ordinal. A semitopological group is left (right) λ-precompact if for every open set U there exists a set $A\subset G$ such that $|A|\le\lambda$ and $AU=G$ ($UA=G$). A semitopological group is *left (right) precompact* if for every open set U there exists a finite set $A\subset G$ such that $AU=G$ ($UA=G$). A subset K of a semitopological group is *left (right) precompact* if for every open set $U\subset G$ there exists a finite set $A\subset K$ such that $AU\supset K$ ($UA\supset K$). A subset K of a semitopological group is *left (right) hereditarily precompact* if every set $L\subset K$ is left (right) precompact. $\mathrm{pr}_l(G)=\min^*\{\lambda\in\mathrm{Card} : G$ is left λ-precompact$\}$. Similarly $\mathrm{pr}_r(G)$ is defined.

Cellurarity: $c(G)=\sup^*\{|\mathcal{U}| : \mathcal{U}$ is a disjoint family of open subsets of $G\}$.

Character: $\chi(G)=\min^*\{|\mathcal{B}| : \mathcal{B}$ is a neighborhood base at unit of $G\}$.

Density: $d(G)=\min^*\{|S| : S\subset G, \overline{S}=G\}$.

Network weight: $nw(G)=\min^*\{|\mathcal{U}| : \mathcal{U}$ is a network for $G\}$.

A family $\mathcal{B}$ of open sets is called a *π-base* at a point x if for every open set $U\ni x$ there exists a set $V\in\mathcal{B}$ such that $V\subset U$.

π-character: $\pi\chi(G)=\min^*\{|\mathcal{B}| : \mathcal{B}$ is a π-base at unit of $G\}$.

Pseudocharacter: $\psi(G)=\min^*\{|\mathcal{U}| : \mathcal{U}$ is a family of open sets and $\bigcap\mathcal{U}=\{e\}\}$. Let X be a subset of G. Then $\psi(X,G)=\min^*\{|\mathcal{U}| : \mathcal{U}$ is a family of open sets and $\bigcap\mathcal{U}=X\}$. A set $D_G=\{(x,x^{-1}) : x\in G\}$ is called a *diagonal* of the space G. Define the cardinal function $\Delta(G)=\psi(D_G,G^2)$.

Spread: $s(G)=\sup^*\{|S| : S\subset G,\ S$ is discrete as a subspace$\}$.

Weakly Lindelöf degree: $wl(G)=\min^*\{\lambda\in\mathrm{Card}$: in every open cover $\mathcal{V}$ there exists a subfamily $\mathcal{U}\subset\mathcal{V}$ such that $\overline{\mathcal{U}}=G$ and $|\mathcal{U}|\le\lambda\}$.

A family $\mathcal{B}$ of open sets is called a *π-base* of G if for every open set $U\subset G$ there exists a set $V\in\mathcal{B}$ such that $V\subset U$.

π-weight: $\pi w(G)=\min^*\{|\mathcal{B}| : \mathcal{B}$ is a π-base of $G\}$.

Weight: $w(G) = \min^*\{|\mathcal{U}| : \mathcal{U}$ is an open base of $G\}$.

Proposition 2.1. *Every left precompact paratopological group is right precompact. Every left ω-precompact Baire paratopological group is right ω-precompact.*

Proof. Let G be a left precompact paratopological group. Let $U \subset G$ be an arbitrary open set. There exists a finite set $A \subset G$ such that $AU = G$. Then $U^{-1}A^{-1} = G$ and hence $\varnothing \neq \operatorname{int} \overline{U} \subset \operatorname{int}(U^{-1})^2$. Hence there exists a finite set $B \subset G$ such that $B(U^{-1})^2 = G$. Then $U^2B^{-1} = G$ and hence the group G is left precompact. The proof of the second claim is similar. □

Question 2.2. (Guran) Is every left ω-precompact paratopological group a right ω-precompact?

Proposition 2.3. *Let (G, τ) be a paratopological SIN-group. Then*

$$wl(G, \tau) \leq \chi(G, \tau) l(G, \tau_g).$$

Proof. Let (G, τ) be such a group. If U, V are open in G then UV^{-1} is open in (G, τ_g) by Remark **??**. Let $\mathcal{U}$ be an arbitrary open cover of (G, τ) and $\mathcal{B}$ be a base at the unit of (G, τ) such that $|\mathcal{B}| = \chi(G, \tau)$. Then for every neighborhood $V \in \mathcal{B}$ the family $\{UV^{-1} : U \in \mathcal{U}\}$ is an open cover of (G, τ_g). Hence there exists a family $\mathcal{U}_V \subset U$ such that $|\mathcal{U}_V| \leq l(G, \tau_g)$ and $G = \bigcup\{U : U \in \mathcal{U}_V\}V^{-1}$. Put $\mathcal{U}' = \bigcup\{\mathcal{U}_V : V \in \mathcal{B}\}$. Then $|\mathcal{U}'| \leq \chi(G, \tau) l(G, \tau_g)$ and $\bigcup\{U : U \in \mathcal{U}'\}$ is dense in (G, τ). □

Proposition 2.4. (Guran [2]) *Let G be a paratopological group such that* $\operatorname{int} U^{-1} \neq \varnothing$ *for every nonempty open set $U \subset G$. Then* $\operatorname{pr}_l(G) \leq wl(G)$.

Example 2.5. Guran posed the following question: Let G be a paratopological group and $c(G) \leq \omega$ (respectively $wl(G) \leq \omega$). Is G ω-precompact? We construct a zero-dimensional Hausdorff first countable paratopological group G such that $c(G) \leq \omega$ and $wl(G) \leq \omega$ but G is not ω-precompact. We shall proceed similarly as in Example 1.8. Let G be a Tychonov product $\mathbb{R}^\omega$. Define a base at the zero $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$, where $U_n = \{(x_i) \in G : 0 \leq x_i < 1/n$ for $i \leq n$ and $x_i \geq 0$ for $i > n\}$. Since every set U_n is clopen then G is a zero-dimensional Hausdorff. Suppose that there exists a countable set $\{a_n\} \subset G$ such that $\{a_n\} + U_1 = G$. There exists a point $a \in G$ such that the n-th coordinate of the point a is smaller than that of the point a_n for every natural n. Clearly, $a \notin \{a_n\} + U_1$. Hence $\operatorname{pr}_l(G) > \omega$.

Suppose that there exists an uncountable family $\mathcal{U}$ be of disjoint open subsets of G. Without loss of generality we may suppose that each element has a form $x + U_n$ for some $x \in G$ and some natural n. Let $\mathcal{U}_n$ be all elements of $\mathcal{U}$ which has a form $x + U_n$ for some $x \in G$. The set of such x will be denoted as X_n. There exists n such that $\mathcal{U}_n$ and hence X_n is uncountable. Hence there exist elements $x, y \in X_n$ such that for every coordinate $1 \leq i \leq n$ we have $|x_i - y_i| < 1/n$. But then the point z defined as $z_i = \max(x_i, y_i)$ for every i is contained in $(x + U_n) \cap (y + U_n)$, a contradiction. Hence $c(G) = \omega$. Since (G, τ_g) is the Tychonoff product $\mathbb{R}^\omega$ with the standard topology, by Proposition 2.3, $wl(G) = \omega$.

Proposition 2.6. *Let G be a semitopological group. Then* $\operatorname{pr}_l(G) \leq d(G)$.

Proof. Let X be a dense subset of the group G. Then for every element $g \in G$ and every neighborhood $U \ni e$ of the unit there exists an element $x \in Ug^{-1} \cap X$. Then $g \in x^{-1}U$ and hence $G = X^{-1}U$. □

Proposition 2.7. *Let G be a paratopological group. Then $\pi w(G) = \pi\chi(G)\,\mathrm{pr}_l(G)$.*

Proof. Since $\pi w(G) \geq \pi\chi(G)\,\mathrm{pr}_l(G)$, it suffices to show that $\pi w(G) \leq \pi\chi(G)\,\mathrm{pr}_l(G)$. Let $\mathcal{B}$ be a π-base at the unit of G such that $|\mathcal{B}| \leq \pi\chi(G)$. For every set $V \in \mathcal{B}$ there exists a set A_V such that $|A_V| \leq \mathrm{pr}_l(G)$ and $V^{-1}A_V = G$. Put $\mathcal{A} = \{VA_V : V \in \mathcal{B}\}$. We show that the family $\mathcal{A}$ is a π-base of the group G. If U is an open subset of the group G then $e \in u^{-1}U$. Select a set $V \in \mathcal{B}$ such that $V^2 \subset Uu^{-1}$. Since $V^{-1}A_V = G$, there exists a point $a \in A_V$ such that $V^{-1}a \ni u$. Then $Va \in VVu \subset Uu^{-1}u = U$. □

A topological space is *pseudocompact* if every discrete family of open subsets of the space is finite. A Tychonov space is pseudocompact if and only if every real-valued continuous function on the space is bounded. Every countably compact space is pseudocompact.

Proposition 2.8. (Reznichenko, personal communication) *If G is a pseudocompact paratopological group then $c(G) \leq \omega$.*

Question 2.9. (Guran) Let G be a topological group and for every neighborhood $U \ni e$ there exists a countable set $F \subset G$ such that $UFU = G$. Is G ω-precompact?

Proposition 2.10. *Let G be a paratopological group, H be a subgroup of G. Then $\mathrm{pr}_l(G) \leq \mathrm{pr}_l(H)\,\mathrm{pr}_l(G/H)$.*

Proof. Consider an arbitrary neighborhood U of the unit. Then there exist sets $A, B \subset G$ such that $|A| \leq \mathrm{pr}_l(H)$, $|B| \leq \mathrm{pr}_l(G/H)$, $AU \supset H$ and $BHU = G$. Then $BAU^2 = G$. □

Proposition 2.11. *If G is a T_1 paratopological group then $w(G) = \chi(G)l(G^2)$.*

Proof. Let $\mathcal{B}$ be a base at the unit of the topology τ on the group G. Let $m\colon G \times G \to G$ be the multiplication on the group G. Then $D = m^{-1}(e)$ is a closed subset of the space $G \times G$ and hence $l(D) \leq l(G^2)$. Let τ^q be a group topology on the group G with a base $\mathcal{B}^q = \{U \cap U^{-1} : U \in \mathcal{B}\}$. Let $\pi\colon G \times G \to G$, $(x, y) \mapsto x$ be the projection onto the first coordinate. Then $\pi|D : D \to (G, \tau^q)$ is a homeomorphism. Therefore $nw(G, \tau) \leq w(G, \tau^q) \leq \chi(G, \tau^q)\,\mathrm{pr}_l(G, \tau^q) \leq \chi(G, \tau)l(D) \leq \chi(G)l(G^2)$. By Proposition 2.3 from [6] $w(G) \leq nw(G)\chi(G) \leq \chi(G)l(G^2) \leq w(G)$. □

Proposition 2.12. (Y. Q. Chen) *If G is a Hausdorff semitopological group then $\Delta(G) \leq \pi\chi(G)$.*

Proof. Let $\mathcal{B}$ be a π-base at the unit. Since G is Hausdorff, $\bigcap\{UU^{-1} : U \in \mathcal{B}\} = \{e\}$. For every set $U \in \mathcal{B}$ put $D_U = \bigcup\{Ug \times Ug : g \in G\}$. Then $xy^{-1} \in UU^{-1}$ for every point $(x, y) \in D_U$. Indeed, there exists an element $g \in G$ such that $x, y \in Ug$. Thus $x = u_x g$ and $y = u_y g$ for some points $u_x, u_y \in U$. Then $xy^{-1} = u_x g(u_y g)^{-1} = u_x u_y^{-1} \in UU^{-1}$. Hence $\bigcap\{D_U : U \in \mathcal{B}\} = \{(x, x) : x \in G\}$. □

Example 2.13. Spread of a paratopological group has a bad behavior with respect to multiplication. Let $G = (\mathbb{R}, \tau_s)$ be the Sorgenfrey arrow. Then $s(G) = \omega$ but $s(G^2) = 2^\omega$.

Example 2.14. Let G be a free abelian group over an infinite set X. For every element $x \in X$ let $U(x)$ is the semigroup generated by the set $X \backslash \{x\}$. Put $\mathcal{B} = \{U(x) : x \in X\}$. Then $\mathcal{B}$ is a subbase at the zero of a paratopology τ on the group G.

It is easy to show that every neighborhood of the unit of every topological group contains a G_δ-subgroup. In our case $U(x)$ contains only trivial subgroups. Hence if X is uncountable then in every neighborhood $U(x)$ there are no G_δ-subgroups. The group G is Hausdorff zero-dimensional and all the sets $U(x)$ are clopen. The sets $U(x) - x$ are pairwise disjoint and hence $c(G) = |X|$. Also $\mathrm{pr}_l(G) = |X|$. Indeed, suppose that $\mathrm{pr}_l(G) = \lambda < |X|$. Choose an arbitrary element $x \in X$. Let $A + U(x) = G$ and $|A| = \lambda$. Let B be the set of all elements of X contained in the words of the set A. Then $|B| \le \omega\lambda < |X|$ and $-y \notin A + U(x)$ for every element $y \in X \backslash B$.

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