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PARATOPOLOGICAL GROUPS, II


We study relations between topologies on a group related the group operation. Some results on separation axioms and cardinal invariants of these topologies are obtained.

1. General properties

This paper is a continuation of the article [6]. Here we obtain some new results concerning paratopological and semitopological groups. We shall use the definitions from [6].

Let $G$ be a group endowed with a topology $\tau$. If the multiplication and the inversion in $G$ are continuous then $G$ is a topological group. If the multiplication in $G$ is continuous then $G$ is a paratopological group. If the multiplication in $G$ is separately continuous then $G$ is a semitopological group. If the multiplication is separately continuous and the inversion in $G$ is continuous then $G$ is a quasitopological group. In these cases the topology $\tau$ is called a group topology, a paratopology, a semitopology, and a quasitopology respectively. Remark that in the papers [4] and [7] a semitopological group is called $m$-topological, and a quasitopological group is called semitopological. Hence every topological group is paratopological and quasitopological and both these groups are semitopological. Clearly, all translations and interior automorphisms of a semitopological group are homeomorphisms.

Let $\mathcal{B}$ be a family of subsets of a group $G$ containing the unit $e$. Put $\mathcal{B}^\prime = \{xU : x \in G, U \in \mathcal{B}\}$. Consider the following Pontrjagin conditions for $\mathcal{B}$.

1. $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B}) : W \subseteq U \cap V$.
2. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^2 \subseteq U$.
3. $(\forall U \in \mathcal{B})(\forall x \in U)(\exists V \in \mathcal{B}) : xV \subseteq U$.
4. $(\forall U \in \mathcal{B})(\forall x \in G)(\exists V \in \mathcal{B}) : x^{-1}Vx \subseteq U$.
5. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^{-1} \subseteq U$.

The base $\mathcal{B}^\prime$ defines a semitopology on a group $G$ if and only if Conditions 1, 3, and 4 are satisfied. The base $\mathcal{B}^\prime$ defines a paratopology on a group $G$ if and only if Conditions 1,
2, 3, and 4 are satisfied. The base $B'$ defines a quasitopology on a group $G$ if and only if Conditions 1, 3, 4, and 5 are satisfied. The base $B'$ defines a group topology on $G$ if and only if Conditions 1, 2, 3, 4, and 5 are satisfied.

The semitopological group $G$ is Hausdorff if and only if $\bigcap \{UU^{-1} : U \in B\} = \{e\}$.

Let $S$ be a subbase of the topology $\tau$ on a group $G$. The group $(G, \tau)$ is semitopological if and only if for every points $x, y \in G$ and every neighborhood $U \in S$ of the point $xy$ there exist neighborhoods $V, W \in S$ of the points $x, y$ respectively such that $V y \cup x W \subset U$. The group $(G, \tau)$ is paratopological if and only if for every points $x, y \in G$ and every neighborhood $U \in S$ of the point $xy$ there exist neighborhoods $V, W \in S$ of the points $x, y$ respectively such that $V W \subset U$. The group $(G, \tau)$ is topological if and only if for every points $x, y \in G$ and every neighborhood $U \in S$ of the point $xy^{-1}$ there exist neighborhoods $V, W \in S$ of the points $x, y$ respectively such that $V W^{-1} \subset U$. The group $(G, \tau)$ is quasitopological if and only if for every points $x, y \in G$ and every neighborhood $U \in S$ of the point $xy^{-1}$ there exist neighborhoods $V, W \in S$ of the points $x, y$ respectively such that $x W^{-1} \subset U$ and $V y^{-1} \subset U$.

For a group $G$ by $T(G), P(G), S(G), Q(G)$ we denote the set of all group topologies, paratopologies, semitopologies and quasitopologies on the group $G$ respectively.

**Proposition 1.1.** Let $\tau \in S(G)$ be a semitopology. Then there exist the following semitopologies on the group $G$:

1. The finest paratopology $\tau_p$ coarser than $\tau$.
2. The finest quasitopology $\tau_q$ coarser than $\tau$.
3. The finest group topology $\tau_g$ coarser than $\tau$.
4. The coarsest quasitopology $\tau^q$ finer than $\tau$.

**Proof.** Let $S = \{\tau' \in P(G)(Q(G), T(G)) \text{ respectively} : \tau' \subset \tau\}$. For every semitopology $\tau' \in S$ fix a base $B'_\tau$ at the unit. Let $S' \subset S$ be a chain. Put $B' = \bigcup\{B'_\tau : \tau \in S'\}$. Then $S'$ is a base of a semitopology $\tau'$ on the group $G$. It is clear that $\tau' \in S$. Thus the set $S$ is inductive and hence contains a maximal element $\tau_1$. Suppose that there exists another maximal element $\tau_2 \in S$ different from $\tau_1$. Let the semitopology $\tau_i$ have a base $B_i$ at the unit. Then $B_3 = \{U_1 \cap U_2 : U_i \in B_i\}$ is a base at the unit of the semitopology $\tau_3$. Thus $\tau_3$ is an element of $S$ greater than both $\tau_1$ and $\tau_2$. Therefore the set $S$ contains the unique maximal element $\tau_1$ which is clearly equal to $\tau_p$ ($\tau_q, \tau_g$ respectively).

Let $B$ be a base of the unit of the semitopology $\tau$. Then $B^q = \{U \cap U^{-1} : U \in B\}$ is a base of the unit of the quasitopology $\tau^q$.

**Proposition 1.2.** Let $G_1, G_2$ be groups and $\varphi : G_1 \to G_2$ be a homomorphism. Let $\tau_1 \in S(G_1)$ and $\tau_2 \in S(G_2)$. If the homomorphism $\varphi : (G_1, \tau_1) \to (G_2, \tau_2)$ is continuous then the homomorphisms $\varphi : (G_1, \tau_1) \to (G_2, \tau_2)$, $\varphi : (G_1, \tau_2) \to (G_2, \tau_2)$, $\varphi : (G_1, \tau_2) \to (G_2, \tau_2)$ and $\varphi : (G_1, \tau_2) \to (G_2, \tau_2)$ are continuous.

**Proof.** For the homomorphisms $\varphi : (G_1, \tau_1) \to (G_2, \tau_2)$ and $\varphi : (G_1, \tau_2) \to (G_2, \tau_2)$ this is true because $\tau_1 = \tau_1 \wedge \tau_1^{-1}$ and $\tau_2 = \tau_1 \vee \tau_1^{-1}$, similarly for the topology $\tau_2$. For the homomorphism $\varphi : (G_1, \tau_1) \to (G_2, \tau_2)$ this is true since the semitopology $\varphi^*(\tau_2)$ belongs to $P(G_1)$ and hence $\varphi^*(\tau_2) \subset \tau_1$. For the homomorphism $\varphi : (G_1, \tau_2) \to (G_2, \tau_2)$ this is true because the semitopology $\varphi^*(\tau_2)$ belongs to $S(G_1)$ and hence $\varphi^*(\tau_2) \subset \tau_2$.

A subsemigroup $S$ of a group $G$ is said to be normal if $x^{-1} S x \subset S$ for every $x \in G$. For every normal submonoid $S$ of the group $G$ by $\tau_S$ we denote the paratopology with the base $\{xS : x \in G\}$. Then $\tau_S = \inf\{\tau \in P(G) : S \in \tau\}$.

\[\]
Example 1.3. Sorgenfrey arrow. $(\mathbb{R}, \tau_s) = (\mathbb{R}, \tau_{\mathbb{R}_+} \land \tau)$, where $\tau$ is the standard topology on $\mathbb{R}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. $(\mathbb{R}, \tau_s)$ is a zero-dimensional Hausdorff first countable paratopological group.

Example 1.4. Let $G = (\mathbb{R}^2, +)$. Put $\mathcal{B} = \{(0,0)\} \cup \{(x, y) \in \mathbb{R}^2 : (x - 1/n)^2 + y^2 < 1/n^2\} : n \in \mathbb{N}\}$. Then $\mathcal{B}$ is a base at the zero of a paratopology on the group $G$.

Let $X$ be a topological space. A space $X$ is $T_0$ if for every distinct points $x, y \in X$ there exists an open set $U \subset X$ such that $U \cap \{x, y\}$ is a singleton. A space $X$ is $T_1$ if for every distinct points $x, y \in X$ there exists an open set $x \in U \subset X \\setminus \{y\}$. A space $X$ is $T_2$ or Hausdorff if for every distinct points of $X$ has disjoint open neighborhoods. A space $X$ is functionally Hausdorff if for every distinct points $x, y \in X$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$. A space $X$ is Urysohn if every distinct points of $X$ has disjoint closed neighborhoods. A space $X$ is strongly Hausdorff if it is Hausdorff and from every infinite subset $A \subset X$ we can choose a sequence $\{x_n : n \in \omega\}$ such that the points $x_n$ have pairwise disjoint neighborhoods in $X$. A space $X$ is $T_3$ if every closed set $F \subset X$ and every point $x \in X \setminus F$ have disjoint neighborhoods. A space $X$ is regular if it is $T_1$ and $T_3$. A set $U \subset X$ is called canonical if $U = \text{int} \overline{U}$. A space $X$ is $CB$ if $X$ has a base consisting of canonical open sets. A space $X$ is semiregular if it is $CB$ and $T_2$. A space is quasiregular if every nonempty open set contains the closure of some nonempty open set. A space is $\pi$-semiregular if every nonempty open set contains the interior of the closure of some nonempty open set. A space $X$ is functionally $T_3$ if for every if for every closed set $F \subset X$ every point $x \in X \setminus F$ there exists a continuous function $X \to [0, 1]$ such that $f(x) = 0$ and $f|F = 1$. A space $X$ is completely regular or Tychonov if it is $T_1$ and functionally $T_3$. The following inclusions are true. Every $T_0+T_3$ space is regular and every $T_0+functionally T_3$ space is Tychonov [1, Ex. 1.5.D].

\[ \begin{array}{c}
\text{Tychonov} \quad \text{functionally} \quad T_2 \quad \text{strongly} \quad T_2 \quad T_2 \\
\text{functionally} \quad T_3 \quad T_1 + T_3 \quad CB \quad T_1 \\
T_3 \quad \text{quasiregular} \quad \pi\text{-semiregular} \quad T_0
\end{array} \]

It is well known that every topological group is $T_3$ and every $T_0$ topological group is completely regular. For paratopological groups the situation with the separation axioms is worse.

**Proposition 1.5.** Every $CB$ paratopological group is $T_3$.

**Proof.** Let $\mathcal{B}$ be a base at the unit of a paratopological group $G$ consisting of canonical open sets. For every neighborhood $U \in \mathcal{B}$ there exists a neighborhood $V \in \mathcal{B}$ such that $V^2 \subset U$. Then $V \ov \subset \overline{U}$. Hence $\ov \subset \text{int} \overline{U} = U$. \[ \square \]

**Proposition 1.6.** Every $\pi$-semiregular paratopological group is quasiregular.

**Proof.** Let $\mathcal{B}$ be a base at the unit of a $\pi$-semiregular paratopological group $G$. For every neighborhood $U \in \mathcal{B}$ there exist a point $x \in U$ and a neighborhood $V \in \mathcal{B}$ such that $x \text{int} V \subset U$. Choose a neighborhood $W \in \mathcal{B}$ such that $W^2 \subset V$. Then $xW \subset x\ov \subset x \text{int} V$ and hence $G$ is quasiregular. \[ \square \]
Proposition 1.7. Let $G$ be a paratopological group such that $\text{int } U^{-1} \neq \varnothing$ for every nonempty open set $U \subset G$. Then $G$ is quasiregular.

Proof. Let $B$ be a base at the unit of such a group $G$. Let $U \in B$ be an arbitrary neighborhood. Choose a neighborhood $V \in B$ such that $V^2 \subset U$. There exist a neighborhood $W \in B$ and a point $x \in V$ such that $W^{-1}x \subset V$. Then $\overline{V}x \subset VW^{-1}x \subset U$ and hence the group $G$ is quasiregular. \hfill \Box

Example 1.8. Guran posed the following question: Let $G$ be a Baire regular paratopological group and $U$ be nonempty open subset of $G$. Is $\text{int } U^{-1} \neq \varnothing$? We construct a zero-dimensional Hausdorff first countable Baire paratopological group $G$ and a nonempty open set $U \subset G$ such that $\text{int } U^{-1} = \varnothing$. Let $G$ be a Tychonov product $\mathbb{R}^\omega$. Define a base at the zero $B = \{U_n : n \in \mathbb{N}\}$, where $U_n = \{(x_i) \in G : x_i = 0 \text{ for } i \leq n \text{ and } x_i \geq 0 \text{ for } i > n\}$. Since every set $U_n$ is clopen, $G$ is a zero-dimensional Hausdorff. It is clear that $\text{int } U_{-1}^{-1} = \varnothing$. We claim that $G$ is Baire. Since every second category homogeneous space is Baire [3, Prop. 1.27] we show that $G$ is of second category. Let $\{G_n\}$ be a decreasing sequence of dense open subsets of $G$. By induction we can construct sequences $\{x^n\} \subset G$ and $\{k_n\} \subset \mathbb{N}$ such that:
1) $x^{n+1} + U_{k_{n+1}} \subset G_n \cap (x^n + U_{k_n})$; 2) $k_{n+1} > k_n$.
Since $x^n \in x^n + U_{k_n}$ for every natural $n$ and every $m \geq n$, the first $k_n$ coordinates of the points $x^n$ and $x^m$ are the same. Hence there exists a unique point $x \in G$ such that for every natural $n$ the first $k_n$ coordinates of the points $x^n$ and $x$ are the same. By the construction $x_i \geq x_i^n$ for every natural $i$ and $n$. Hence $x \in x^n + U_{k_n} \subset G_{n+1}$ for every natural $n \geq 1$ and therefore the space $G$ is of second category.

Example 1.9. Let $(G, \tau)$ be a semitopological group. Define the topology $\tau_\tau$ on the group $G$ as follows. Let $B$ be a base of the unit of the topology $\tau$. Put $B_\tau = \{\text{int } \overline{U} : U \in B\}$. Verify Pontrjagin conditions for the topology $B_\tau$.

1. Trivial.

3. Let $U \in B$ and $x \in \text{int } \overline{U}$. There exists a neighborhood $V \in B$ such that $xV \subset \text{int } \overline{U}$. Then $xV \subset \overline{U}$. Hence $x \in \text{int } \overline{V} \subset \text{int } \overline{U}$.

4. Trivial.

Thus $\tau_\tau$ is a semitopology on the group $G$. Moreover, if $\tau$ is a paratopology then $\tau_\tau$ is a paratopology as well. Indeed, in this case we must verify Pontrjagin condition 2. Let $U \in B$. There exists a neighborhood $V \in B$ such that $V^2 \subset U$. Then $\overline{V}^2 \subset \overline{U}$ and since $\text{int } \overline{U}$ is open, we see that $(\text{int } \overline{V})^2 \subset \text{int } \overline{U}$.

Clearly, $\tau_\tau$ does not depend on the choice of the basis $B$ of a semitopology $\tau$. Moreover, $\tau_\tau$ is a CB-semitopology. Indeed, it suffices to verify that for every neighborhood $U \in B$ we have $\text{int } \overline{U}^\tau \subset \overline{U}$, because then $\text{int}_\tau \text{int } \overline{U}^\tau \subset \text{int } \overline{U}^\tau \subset \text{int } \overline{U}$. Let $y \in \text{int } \overline{U}^\tau$. Then for every neighborhood $V \in B$ we have $y \in \text{int } V \cap \text{int } \overline{U} \neq \varnothing$ and thus $yV \cap \text{int } \overline{U} \neq \varnothing$. Hence $y \in \text{int } \overline{V} \subset \overline{U}$. Remark that if $\tau$ is Hausdorff then $\tau_\tau$ is $T_1$ and hence regular.

Engelking in [1, Ex. 1.7.8] considers a similar construction. A subset $U$ of a topological space $(G, \tau)$ is called canonically open if $U = \text{int } \overline{U}$. The interior of a closed set is canonically open. Stone and Katetov construct the topology $\tau_\tau$ on $G$ generated by the base consisting of all canonically open sets of the space $(G, \tau)$. This definition coincides with our definition for the semitopological groups.

Let $f$ be a function from a space $X$ into a space $Y$. Then $f$ will be called $\delta$-open if for every nowhere dense subset $N$ of $Y$, $f^{-1}(N)$ is nowhere dense. If a function $f : X \rightarrow Y$ is continuous and the image of each nonempty open subset of $X$ is somewhere dense in $Y$ then
$f$ is $\delta$-open [3, Prop. 4.4]. If $f$ is a continuous $\delta$-open function from a Baire space $X$ onto a space $Y$ then $Y$ is a Baire space. Let $U$ be an open subset of a semitopological group $(G, \tau)$. Then $\overline{U^\alpha} \supset \overline{U} \supset \text{int}\overline{U}$ hence $U$ is somewhere dense in the topology $\tau$, and the identity map from $(G, \tau)$ to $(G, \tau_\tau)$ is $\delta$-open. Hence if $(G, \tau)$ is Baire then $(G, \tau_\tau)$ is Baire as well.

**Proposition 1.10.** Every Hausdorff semitopological group is Urysohn.

**Proof.** Let $(G, \tau)$ be such a group. Example 1.9 implies that $(G, \tau_\tau)$ is regular. Since $(G, \tau_\tau)$ is Urysohn and $\tau_\tau \subset \tau$, we see that $(G, \tau)$ is Urysohn.

**Example 1.11.** Put $G = \mathbb{Z}/2\mathbb{Z}$ with trivial topology. Then $G$ is functionally $T_3$ not $T_0$ paratopological group.

**Example 1.12.** Put $G = (\mathbb{R}, +)$ and $\tau = \{x + [0; \infty) : x \in \mathbb{R}\}$. Then $G$ is $T_0$ not $T_1$ paratopological group.

**Example 1.13.** Put $G = (\mathbb{R}, +)$ and $\tau = \{\{x\} \cup [y; \infty) : x, y \in \mathbb{R}\}$. Then $G$ is $T_1$ not $T_2$ paratopological group.

**Example 1.14.** Put $G = \mathbb{T} \times \mathbb{Z}_2$ where $\mathbb{T}$ is the unit circle and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. For every $\varepsilon > 0$ put $U(\varepsilon) = \{(0, 0)\} \cup (0; \varepsilon) \times \mathbb{Z}_2$. It easy to check that $\mathcal{B} = \{U(\varepsilon) : \varepsilon > 0\}$ is a base at the zero of a paratopology on the group $G$. Then $G$ is $T_1$ quasiregular not $T_2$.

**Example 1.15.** Put $G = \mathbb{R}^2$. Define a base $\mathcal{B}$ at the unit of the group $G$ putting $\mathcal{B} = \{(0, 0)\} \cup \{(x, y) : x \in \mathbb{R}^2 : 0 < x, y < 1/n\} : n \in \mathbb{N}\}$. Then $G$ is functionally $T_2$ quasiregular not $T_3$ paratopological group.

**Example 1.16.** The $p$-arrow. Let $p$ be a natural number. Put $(\mathbb{Z}, \tau) = (\mathbb{Z}, \tau_{\mathbb{N}} \land \tau_p)$, where $\tau_p$ has a base at the unit $\mathcal{B}_p = \{p^n\mathbb{Z} : n \in \mathbb{N}\}$ and $\mathbb{Z}_{\mathbb{N}} = \mathbb{N} \cup \{0\}$. Then $G$ is functionally $T_2$ not $\pi$-semiregular paratopological group.

A subgroup $(H, \tau|H)$ of a semitopological group $(G, \tau)$ is a subgroup of the group $G$ endowed with the induced from $(G, \tau)$ topology $\tau|H$. Clearly, any subgroup of a semitopological (resp. paratopological, quasi-topological, topological) group is again a semitopological (resp. paratopological, quasi-topological, topological) group.

**Remark 1.17.** The closure of a subgroup of a paratopological group is a not a paratopological group in general. Indeed, let $G$ be the group from Example 1.12 and $H = \{0\}$. Then $\overline{H} = (-\infty; 0]$ is not a subgroup.

**Proposition 1.18.** Every element of a Hausdorff paratopological group is contained in a closed abelian subgroup.

**Proof.** Let $G$ be such a group and $x \in X$. We construct a transfinite sequence $\{G_\alpha\}$ of closed abelian subsemigroups of $G$ by the following. Put $G_0 = \overline{\{x\}}$. Let $G_\alpha$ be already defined for all ordinals $\alpha < \beta$. If $\beta$ is a limit ordinal we put $G_\beta = \bigcup\{G_\alpha : \alpha < \beta\}$. If $\beta = \alpha + 1$ then put $G_\beta = \overline{G_\alpha}$. There exists an ordinal $\alpha$ such that $G_\alpha = G_\alpha^1$. Then $G_\alpha$ is a closed abelian group containing the element $x$.

The product of a family of semitopological groups $\{(G_\alpha, \tau_\alpha) : \alpha \in A\}$ is the product $\prod G_\alpha$ endowed with the Hausdorff product topology $\prod \tau_\alpha$. The box product of a family of semitopological groups $\{(G_\alpha, \tau_\alpha) : \alpha \in A\}$ is the product $\prod G_\alpha$ endowed with the box product topology $\tau_\alpha$. It is easy to see that both the product and the box product of a family of semitopological (resp. paratopological, quasi-topological and topological) groups is again a semitopological (resp. paratopological, quasi-topological and topological) group.
Example 1.19. Reznichenko posed the following question. Can every Hausdorff paratopological group \((G, \tau)\) be embedded into the product \((G_1, \tau_1) \times (G_2, \tau_2)\) of paratopological groups where the topology \(\tau^\oplus\) is discrete and \((G_2, \tau_2)\) is a topological group? This example shows that the answer is negative. Put \(G = (\mathbb{R}, +)\). Let \(\rho_1\) denote the standard topology of \(\mathbb{R}\), \(\rho_2\) denotes the topology induced on \(\mathbb{R}\) by dense wrapping around the torus. Let \(B_i\) be a base at the zero of the topology \(\rho_i\) for \(i = 1, 2\). Define a base \(\mathcal{B}\) at the zero of a topology \(\tau\) as follows: \(\mathcal{B} = \{U_i \cap (-\infty; 0] \cup (U_2 \cap [0; \infty)) : U_i \in B_i\}\). Then \(\tau\) is the supremum of the following two paratopologies: the topology which is on \(\mathbb{R}\) wrapped dense around the torus and the topology with a base at the unit \(\{-\varepsilon; \infty\}, \varepsilon > 0\). Therefore \(\tau\) is a paratopology on the group \(\mathbb{R}\). It is easy to see that \(\tau^\oplus = \rho_1\). Suppose that there exists an embedding defined above. Let \(\pi: G \to G_1\) be the standard projection. Proposition 1.2 implies that the map \(\pi: (G, \tau^\oplus) \to (G_1, \tau_1^\oplus)\) is continuous. Since the topology \(\tau_1^\oplus\) is discrete and \((G, \tau^\oplus)\) is connected then the map \(\pi\) is trivial. Therefore \((G, \tau)\) is a subgroup of a topological group \((G_2, \tau_2)\) and hence is a topological group, a contradiction. Therefore the embedding defined above for the group \((G, \tau)\) does not exist.

2. Cardinal invariants

A function \(\varphi\) defined in the class \(\mathcal{C}\) of paratopological groups is called a cardinal function if it assigns to each member \(G \in \mathcal{C}\) an infinite cardinal number \(\varphi(G)\). Now we shall list the cardinal functions to be examined in what follows. Remark that in the definitions below \(\min^*\{\}\) = \(\omega \cdot \min\{\}\) and \(\sup^*\{\}\) = \(\omega \cdot \sup\{\}\).

Precompactness. Let \(\lambda\) be an ordinal. A semitopological group is left (right) \(\lambda\)-precompact if for every open set \(U\) there exists a set \(A \subset G\) such that \(|A| \leq \lambda\) and \(AU = G\) (\(UA = G\)). A semitopological group is left (right) precompact if for every open set \(U\) there exists a finite set \(A \subset G\) such that \(AU = G\) (\(UA = G\)). A subset \(K\) of a semitopological group is left (right) precompact if for every open set \(U \subset G\) there exists a finite set \(A \subset K\) such that \(AU \supset K\) (\(UA \supset K\)). A subset \(K\) of a semitopological group is left (right) hereditarily precompact if every set \(L \subset K\) is left (right) precompact. \(pr_\lambda(G) = \min^*\{\lambda \in \text{Card} : G\}\) is left \(\lambda\)-precompact. Similarly \(pr_\lambda(G)\) is defined.

Cellularity: \(c(G) = \sup^*\{|U| : U\} \) is a disjoint family of open subsets of \(G\).

Character: \(\chi(G) = \min^*\{|\mathcal{B}| : \mathcal{B}\} \) is a neighborhood base at unit of \(G\).

Density: \(d(G) = \min^*\{|S| : S \subset G, \mathbb{S} = G\}\).

Network weight: \(nw(G) = \min^*\{|U| : U\} \) is a network for \(G\).

A family \(\mathcal{B}\) of open sets is called a \(\pi\)-base at a point \(x\) if for every open set \(U \ni x\) there exists a set \(V \in \mathcal{B}\) such that \(V \subset U\).

\(\pi\)-character: \(\pi_\chi(G) = \min^*\{|\mathcal{B}| : \mathcal{B}\} \) is a \(\pi\)-base at unit of \(G\).

Pseudocharacter: \(\psi(G) = \min^*\{|U| : U\} \) is a family of open sets and \(\bigcap U = \{e\}\). Let \(X\) be a subset of \(G\). Then \(\psi(X, G) = \min^*\{|U| : U\} \) is a family of open sets and \(\bigcap U = X\). A set \(D_G = \{(x, x^{-1}) : x \in G\}\) is called a diagonal of the space \(G\). Define the cardinal function \(\Delta(G) = \psi(D_G, G^2)\).

Spread: \(s(G) = \sup^*\{|S| : S \subset G, S\} \) is discrete as a subspace.

Weakly Lindelöf degree: \(wl(G) = \min^*\{\lambda \in \text{Card} : \text{in every open cover } \mathcal{V} \text{ there exists a subfamily } \mathcal{U} \subset \mathcal{V} \text{ such that } \mathcal{U} = G \text{ and } |\mathcal{U}| \leq \lambda\}\).

A family \(\mathcal{B}\) of open sets is called a \(\pi\)-base of \(G\) if for every open set \(U \subset G\) there exists a set \(V \in \mathcal{B}\) such that \(V \subset U\).

\(\pi\)-weight: \(\pi w(G) = \min^*\{|\mathcal{B}| : \mathcal{B}\} \) is a \(\pi\)-base of \(G\).
Weight: \( w(G) = \min^* \{|U| : U \text{ is an open base of } G \} \).

**Proposition 2.1.** Every left precompact paratopological group is right precompact. Every left \( \omega \)-precompact Baire paratopological group is right \( \omega \)-precompact.

**Proof.** Let \( G \) be a left precompact paratopological group. Let \( U \subseteq G \) be an arbitrary open set. There exists a finite set \( A \subseteq G \) such that \( AU = G \). Then \( U^{-1} A^{-1} = G \) and hence \( \emptyset \neq \text{int } \overline{U} \subseteq \text{int}(U^{-1})^2 \). Hence there exists a finite set \( B \subseteq G \) such that \( B(U^{-1})^2 = G \). Then \( U^2 B^{-1} = G \) and hence the group \( G \) is left precompact. The proof of the second claim is similar. \( \square \)

**Question 2.2.** (Guran) Is every left \( \omega \)-precompact paratopological group a right \( \omega \)-precompact?

**Proposition 2.3.** Let \((G, \tau)\) be a paratopological S\(\in\)N-group. Then

\[
wl(G, \tau) \leq \chi(G, \tau) l(G, \tau_g).
\]

**Proof.** Let \((G, \tau)\) be such a group. If \( U, V \) are open in \( G \) then \( UV^{-1} \) is open in \((G, \tau_g)\) by Remark 2.4. Let \( \mathcal{U} \) be an arbitrary open cover of \((G, \tau)\) and \( \mathcal{B} \) be a base at the unit of \((G, \tau)\) such that \( |\mathcal{B}| = \chi(G, \tau) \). Then for every neighborhood \( V \in \mathcal{B} \) the family \( \{ U^{-1} V : U \in \mathcal{U} \} \) is an open cover of \((G, \tau_g)\). Hence there exists a family \( \mathcal{U}_V \subseteq U \) such that \( |\mathcal{U}_V| \leq l(G, \tau_g) \) and \( G = \bigcup \{ U : U \in \mathcal{U}_V \} V^{-1} \). Put \( \mathcal{U}' = \bigcup \{ \mathcal{U}_V : V \in \mathcal{B} \} \). Then \( |\mathcal{U}'| \leq \chi(G, \tau) l(G, \tau_g) \) and \( \bigcup \{ U : U \in \mathcal{U}' \} \) is dense in \((G, \tau)\). \( \square \)

**Proposition 2.4.** (Guran [2]) Let \( G \) be a paratopological group such that \( \text{int } U^{-1} \neq \emptyset \) for every nonempty open set \( U \subseteq G \). Then \( pr_1(G) \leq wl(G) \).

**Example 2.5.** Guran posed the following question: Let \( G \) be a paratopological group and \( c(G) \leq \omega \) (respectively \( wl(G) \leq \omega \)). Is \( G \) \( \omega \)-precompact? We construct a zero-dimensional Hausdorff first countable paratopological group \( G \) such that \( c(G) \leq \omega \) and \( wl(G) \leq \omega \) but \( G \) is not \( \omega \)-precompact. We shall proceed similarly as in Example 1.8. Let \( G \) be a Tychonov product \( \mathbb{R}^\omega \). Define a base at the zero \( \mathcal{B} = \{ U_n : n \in \mathbb{N} \} \), where \( U_n = \{ (x_i) : 0 \leq x_i < 1/n \text{ for } i \leq n \text{ and } x_i \geq 0 \text{ for } i > n \} \). Since every set \( U_n \) is clopen then \( G \) is a zero-dimensional Hausdorff. Suppose that there exists a countable set \( \{ a_n \} \subseteq G \) such that \( \{ a_n \} + U_1 = G \). There exists a point \( a \in G \) such that the \( n \)-th coordinate of the point \( a \) is smaller than that of the point \( a_n \) for every natural \( n \). Clearly, \( a \notin \{ a_n \} + U_1 \). Hence \( pr_1(G) > \omega \).

Suppose that there exists an uncountable family \( \mathcal{U} \) of disjoint open subsets of \( G \). Without loss of generality we may suppose that each element has a form \( x + U_\alpha \) for some \( x \in G \) and some natural \( \alpha \). Let \( U_\alpha \) be all elements of \( \mathcal{U} \) which has a form \( x + U_\alpha \) for some \( x \in G \). The set of such \( x \) will be denoted as \( X_\alpha \). There exists \( n \) such that \( U_\alpha \) and hence \( X_\alpha \) is uncountable. Hence there exist elements \( x, y \in X_\alpha \) such that for every coordinate \( 1 \leq i \leq n \) we have \( |x_i - y_i| < 1/n \). But then the point \( z \) defined as \( z_i = \max(x_i, y_i) \) for every \( i \) is contained in \( (x + U_\alpha) \cap (y + U_\alpha) \), a contradiction. Hence \( c(G) = \omega \). Since \((G, \tau_g)\) is the Tychonoff product \( \mathbb{R}^\omega \) with the standard topology, by Proposition 2.3, \( wl(G) = \omega \).

**Proposition 2.6.** Let \( G \) be a semitopological group. Then \( pr_1(G) \leq d(G) \).

**Proof.** Let \( X \) be a dense subset of the group \( G \). Then for every element \( g \in G \) and every neighborhood \( U \ni e \) of the unit there exists an element \( x \in U g^{-1} \cap X \). Then \( g \in x^{-1} U \) and hence \( G = X^{-1} U \). \( \square \)
Proposition 2.7. Let $G$ be a paratopological group. Then $\pi w(G) = \pi \chi(G) \text{pr}_I(G)$.

Proof. Since $\pi w(G) \geq \pi \chi(G) \text{pr}_I(G)$, it suffices to show that $\pi w(G) \leq \pi \chi(G) \text{pr}_I(G)$. Let $B$ be a $\pi$-base at the unit of $G$ such that $|B| \leq \pi \chi(G)$. For every set $V \in B$ there exists a set $A_V$ such that $|A_V| \leq \text{pr}_I(G)$ and $V^{-1}A_V = G$. Put $A = \{V, A_V : V \in B\}$. We show that the family $A$ is a $\pi$-base of the group $G$. If $U$ is an open subset of the group $G$ then $e \in \imath^{-1}U$. Select a set $V \in B$ such that $V^2 \subset U \cup \imath^{-1}U$. Since $V^{-1}A_V = G$, there exists a point $a \in A_V$ such that $V^{-1}a \supset u$. Then $V a \in V V U \subset \imath^{-1}u \cup \imath^{-1}u = U$. □

A topological space is pseudocompact if every discrete family of open subsets of the space is finite. A Tychonov space is pseudocompact if and only if every real-valued continuous function on the space is bounded. Every countably compact space is pseudocompact.

Proposition 2.8. (Reznichenko, personal communication) If $G$ is a pseudocompact paratopological group then $c(G) \leq \omega$.

Question 2.9. (Guran) Let $G$ be a topological group and for every neighborhood $U \ni e$ there exists a countable set $F \subset G$ such that $UFU = G$. Is $G$ $\omega$-precompact?

Proposition 2.10. Let $G$ be a paratopological group, $H$ be a subgroup of $G$. Then $\text{pr}_I(G) \leq \text{pr}_I(H) \text{pr}_I(G/H)$.

Proof. Consider an arbitrary neighborhood $U$ of the unit. Then there exist sets $A, B \subset G$ such that $|A| \leq \text{pr}_I(H)$, $|B| \leq \text{pr}_I(G/H)$, $AU \supset H$ and $BHU = G$. Then $BAU^2 = G$. □

Proposition 2.11. If $G$ is a $T_1$ paratopological group then $w(G) = \chi(G)l(G^2)$.

Proof. Let $B$ be a base at the unit of the topology $\tau$ on the group $G$. Let $m : G \times G \to G$ be the multiplication on the group $G$. Then $D = m^{-1}(e)$ is a closed subset of the space $G \times G$ and hence $l(D) \leq l(G^2)$. Let $\tau^a$ be a group topology on the group $G$ with a base $B^a = \{U \cap U^{-1} : U \in B\}$. Let $\pi : G \times G \to G, (x, y) \mapsto x$ be the projection onto the first coordinate. Then $\pi[D : D \to (G, \tau^a)]$ is a homeomorphism. Therefore $w(D, \tau^a) \leq w(G, \tau^a) \leq \chi(G, \tau^a) \text{pr}_I(G, \tau^a) \leq \chi(G, \tau^a)l(D) \leq \chi(G)l(G^2)$. By Proposition 2.3 from [6] $w(G) \leq nw(G) \chi(G) \leq \chi(G)l(G^2) \leq w(G)$. □

Proposition 2.12. (Y. Q. Chen) If $G$ is a Hausdorff semitopological group then $\Delta(G) \leq \pi \chi(G)$.

Proof. Let $B$ be a $\pi$-base at the unit. Since $G$ is Hausdorff, $\bigcap\{UU^{-1} : U \in B\} = \{e\}$. For every set $U \in B$ put $D_U = \bigcup\{Ug \times Ug : g \in G\}$. Then $xy^{-1} \in UU^{-1}$ for every point $(x, y) \in D_U$. Indeed, there exists an element $g \in G$ such that $x, y \in Ug$. Thus $x = u_x g$ and $y = u_y g$ for some points $u_x, u_y \in U$. Then $xy^{-1} = u_x g (u_y g)^{-1} = u_x u_y^{-1} \in UU^{-1}$. Hence $\bigcap\{D_U : U \in B\} = \{(x, x) : x \in G\}$. □

Example 2.13. Spread of a paratopological group has a bad behavior with respect to multiplication. Let $G = (\mathbb{R}, \tau_s)$ be the Sorgenfrey arrow. Then $s(G) = \omega$ but $s(G^2) = 2\omega$.

Example 2.14. Let $G$ be a free abelian group over an infinite set $X$. For every element $x \in X$ let $U(x)$ is the semigroup generated by the set $X \setminus \{x\}$. Put $B = \{U(x) : x \in X\}$. Then $B$ is a subbase at the zero of a paratopology $\tau$ on the group $G$.

It is easy to show that every neighborhood of the unit of every topological group contains a $G_\delta$-subgroup. In our case $U(x)$ contains only trivial subgroups. Hence if $X$ is uncountable
then in every neighborhood \( U(x) \) there are no \( G_\delta \)-subgroups. The group \( G \) is Hausdorff zero-dimensional and all the sets \( U(x) \) are clopen. The sets \( U(x) - x \) are pairwise disjoint and hence \( c(G) = |X| \). Also \( \text{pr}_1(G) = |X| \). Indeed, suppose that \( \text{pr}_1(G) = \lambda < |X| \). Choose an arbitrary element \( x \in X \). Let \( A + U(x) = G \) and \( |A| = \lambda \). Let \( B \) be the set of all elements of \( X \) contained in the words of the set \( A \). Then \( |B| \leq \omega \lambda < |X| \) and \( -y \notin A + U(x) \) for every element \( y \in X \setminus B \).

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