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I. I. Reznykov, V. I. Sushchansky

2-GENERATED SEMIGROUP OF AUTOMATIC TRANSFORMATIONS WHOSE GROWTH IS DEFINED BY FIBONACHI SERIES

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All non-free semigroups of exponential growth defined by two-state Mealy automata over a two-symbol alphabet are described in terms of generators and defining relations. It was found out, that there are up to isomorphism seven such semigroups. The proofs for these semigroups are substantially similar, therefore we investigate in details one of them, the semigroup $\langle\ f_1, f_2 \ \mid\ f_2^2 \cdot f_1 = f_2 \cdot f_1^2\ \rangle$. It is proved that its growth function is sequences like a Fibonacci series.

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В терминах образующих и определяющих соотношений описаны все несвободные полугруппы экспоненциального роста, порождаемые автоматами Мили с двумя состояниями над двухэлементным алфавитом. Оказалось, что вплоть до изоморфизма имеется семь таких полугрупп. Поскольку все доказательства для них вполне аналогичны, в работе подробно исследуется одна из них - полугруппа $\langle\ f_1, f_2 \ \mid\ f_2^2 \cdot f_1 = f_2 \cdot f_1^2\ \rangle$. Установлено, что ее функция роста является последовательностью типа Фибоначчи.

1. Introduction

The groups and semigroups of automatic transformations have been studied actively since the 60th years of the last century. These investigations are substantially stimulated by the researches where finite automata have been used for construction of the groups with various extreme properties: infinite periodic with a finite number of the generators [1], [2], [3], of intermediate growth [4], [5], just infinite [6]. The Mealy automata became a convenient method of the definition of transformation groups and semigroups, because even small automata (by the number of states and symbols in alphabet) can define complex semigroups and groups with interesting properties. In particular, there exist the Mealy automata with a small number of states considered over two- or three-symbol alphabet which define free groups and free semigroups [7], [8], [9], groups of Burnside type [2], groups of integer matrices [10] and others. As usual in such situation, a problem on complete classification of studied objects for small values of parameters has appeared. The construction of groups defined by the Mealy automata even in a case of two-state automata over the two-symbol alphabet is

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quite nontrivial. Groups defined by invertible automata of such sort were completely described in [11]. In communication [12] we have given the list of all semigroups defined by two-state Mealy automata over two-symbol alphabet.

From the point of view of current researches, the most interesting sphere is that of problems concerning automata and semigroups determined by them is connected with investigations of their growth [13], [14], so we have studied completely the growth of all semigroups from the mentioned class. It was found out that polynomial growth of such automata and semigroups is always linear, and there are as free, as well as non-free semigroups of exponential growth. The papers of one of the authors [7], [15] are devoted to the description of automata that determine free semigroups. In this article we pay attention to the studying of automata that determine non-free semigroups of exponential growth. There are exactly 32 such automata, as it is noted in [12]. The researches of these automata and semigroups determined by them are substantially similar, therefore we investigate in details one of such semigroups and discuss the others. Let a Mealy automaton A over the alphabet $\{a, b\}$ (for the definition see Sec.2) be defined by its Moore diagram (fig. 1), where $\sigma_a = \begin{pmatrix} a & b \\ a & a \end{pmatrix}$ and $\sigma_b = \begin{pmatrix} a & b \\ b & b \end{pmatrix}$.

Fig 1.: Moore diagram of automaton A

Let us denote the transformations determined by A at the states q_1, q_2 by the symbols f_1 and f_2 respectively, and let $S_A = \langle f_1, f_2 \rangle$ be the automatic transformation semigroup determined by the automaton A. The main result of this article is the following one:

Theorem. *1. The semigroup S_A has the following co-presentation:*

$$S_A = \left\langle \ f_1, f_2 \ \middle| \ f_2^2 f_1 = f_2 f_1^2 \ \right\rangle. \tag{1}$$

2. The amount $W(n)$ of those elements of the semigroup S_A that can be represented as the product of minimal possible length n of the generators f_1 and f_2 satisfies the following recurrent relation:

$$W(n+1) = W(n) + W(n-1) + 1, \tag{2}$$

where $W(1) = 2$, $W(2) = 4$, $n \geq 2$.

The corollary follows immediately from this theorem.

Corollary. *1. The subsemigroup of the semigroup S_A generated by the transformations f_1 and $f_1 f_2$ is a proper free subsemigroup of S_A.*

2. The growth function of the automaton A is defined by the equality:

$$\gamma_A(n) = \frac{1}{\sqrt{5}}\left(\left(\sqrt{5}-2\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\sqrt{5}+2\right)\left(\frac{1+\sqrt{5}}{2}\right)^n\right) - 1, \tag{3}$$

where $n \geq 1$.

3. The growth function of the semigroup S_A with respect to the system of generators f_1 and f_2 is defined by the equality:

$$\gamma_{S_A}(n) = \frac{1}{2\sqrt{5}}\left(\left(5\sqrt{5}-11\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(5\sqrt{5}+11\right)\left(\frac{1+\sqrt{5}}{2}\right)^n\right) - (n+5) \tag{4}$$

where $n \geq 1$.

2. Mealy automata and growth functions

Recall that a *Mealy automaton* is an ordered quintuple $A = (X_I, X_O, Q, \pi, \lambda)$, where X_I is the alphabet of input symbols, X_O is the alphabet of output symbols ($|X_I| < \infty$, $|X_O| < \infty$), Q is the set of internal states, $\pi\colon X_I \times Q \to Q$ and $\lambda\colon X_I \times Q \to X_O$ are its transition and output functions respectively. Further we are interested in the automata whose input and output alphabets coincide and are equal to X, $|X| = n$. An automaton A over the alphabet X is called *finite*, if $|Q| < \infty$.

It is convenient to describe finite automata by the Moore diagrams. The Moore diagram of an automaton A is an edge-labeled directed graph D_A with the set of vertices Q. Vertices q_i and q_j of the graph D_A are connected by the oriented edge in direction from q_i to q_j, if there exists $x \in X$ such thatthe equality $\pi(x, q_i) = q_j$ holds. Every edge is marked by the label $x|y$, where $y = \lambda(x, q_i)$. The modification of the Moore diagram used on fig. 1 consists in the fact that the edges of the graph D_A are marked by input symbol x, and every vertex q is labeled by the transformation σ_q of the alphabet X that corresponds to the output function at state q:

$$\sigma_q = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ \lambda(x_1, q) & \lambda(x_2, q) & \dots & \lambda(x_n, q) \end{pmatrix}.$$

The functions π and λ can be extended naturally to mappings of the set $X^* \times Q$ into the sets Q and X^* (X^* denotes the set of all (finite) words over the alphabet X) by the following equalities:

$$\begin{aligned} &\pi(\Lambda, q) = q, && \pi(wx, q) = \pi(x, \pi(w, q)), \\ &\lambda(\Lambda, q) = \Lambda, && \lambda(wx, q) = \lambda(w, q)\,\lambda(x, \pi(w, q)), \end{aligned}$$

where $\Lambda \in X^*$ is the empty word, $q \in Q$, $w \in X^*$, $x \in X$. This allows us to introduce the transformation $f_{A_q}\colon X^* \to X^*$ determined by the automaton A at the state $q \in Q$ in the following way:

$$f_{A_q}(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n, \tag{5}$$

where $y_i = \lambda(x_i, \pi(x_1 x_2 \dots x_{i-1}, q))$. The Mealy automaton A with the set of states $Q = \{q_0, q_1, \dots, q_{m-1}\}$ determines the set of transformations $P_A = \left\{f_{A_{q_1}}, f_{A_{q_2}}, \dots, f_{A_{q_m}}\right\}$ on X^*. Automata A_1 and A_2 are called *equivalent*, if $P_{A_1} = P_{A_2}$.

Lemma 1. [16] *Each class of equivalent (finite) Mealy automata over the alphabet X contains a unique up to isomorphism reduced or minimal (by the number of states) Mealy automaton.*

The minimal automaton can be found by using standard algorithm of minimization (see, for example, [16]).

There are many various operations for Mealy automata (see, for example, [17]). One of the most important operations is the multiplication of automata; it corresponds to superposition of the automatic transformations. Let $A_1 = (X, Q_1, \pi_1, \lambda_1)$ and $A_2 = (X, Q_2, \pi_2, \lambda_2)$ be two arbitrary Mealy automata. The automaton $A = \langle X, Q_1 \times Q_2, \pi, \lambda\rangle$ whose transition and output functions are defined as

$$\pi(x, (q_1, q_2)) = (\pi_1(x, q_1), \pi_2(\lambda(x, q_1), q_2)), \quad \lambda(x, (q_1, q_2)) = \lambda_2(\lambda_1(x, q_1), q_2),$$

where $x \in X$, $(q_1, q_2) \in Q_1 \times Q_2$, is called the *product* of automata A_1 and A_2.

Lemma 2. [18] *For any states $q_1 \in Q_1$, $q_2 \in Q_2$ of automata A_1, A_2 and arbitrary word $u \in X^*$ the following equality holds:*

$$f_{(A_1 \cdot A_2)_{(q_1, q_2)}}(u) = f_{(A_1)_{q_1}}\left(f_{(A_2)_{q_2}}(u)\right).$$

As the multiplication of automata is an associative operation, then for any automaton A and for any natural number n the power $A^n = \underbrace{A \cdot A \cdot \dots \cdot A}_{n}$ is defined. Usually the automaton A^n is not minimal in the class of equivalent to him automata. Let us denote a result of application of minimization algorithm to A^n by $A^{(n)}$. It follows from the definition of the product of automata that the following inequality holds: $|Q_{A^{(n)}}| \le |Q_A|^n$.

Definition 1. A function of natural argument $\gamma_A(n)$ defined by the equality

$$\gamma_A(n) = |Q_{A^{(n)}}|, \quad n \in \mathbb{N},$$

is called the *growth function* of a Mealy automaton A.

Let S be a semigroup (group) with a finite set of generators P, so $S = \langle P\rangle$. A function $\gamma_S(n)$ of natural argument n that equals at the point n to the amount of those elements of S that can be represented as a product of the length less or equal to n of elements from P is called the *growth function of the semigroup (group) S respect to the system of generators P*.

Two nondecreasing functions of natural argument γ_1 and γ_2 have the *same growth order* or *are equivalent*, if there arenatural numbers $C_1, C_2, N_0 \in \mathbb{N}$ such that the following inequalities hold simultaneously for any $n \ge N_0$: $\gamma_1(n) \le \gamma_2(C_1 n)$ and $\gamma_2(n) \le \gamma_1(C_2 n)$. Let us denote by the symbol $\sim$ the equivalence of functions. The classes of equivalence of functions are called the *growth orders* and are denoted by $[\gamma]$ (see, for example, [19]).

Lemma 3. *[19] Let S be an arbitrary finitely generated semigroup, γ_1 and γ_2 its growth functions respect to two systems of generators. Then $\gamma_1 \sim \gamma_2$.*

Definition 2. The semigroup $S_A = \left\langle f_{A_{q_1}}, \ldots, f_{A_{q_m}} \right\rangle$ is called the *semigroup of automatic transformations* which is defined by the automaton A with the set of internal states $Q = \{q_1, q_2, \ldots, q_m\}$.

For invertible Mealy automata, i.e. the automata such that the output function $\lambda(X, q)$ for every $q \in Q$ determines a permutation over the set X, the transformations $f_{A_{q_i}}$ $(1 \le i \le m)$ are bijections. In this case let us examine the *group of transformations* which is defined by the Mealy automaton. The relation between the automaton growth and the growth of the transformation semigroup determined by the aforementioned automaton, is given in the following statement

Lemma 4. [20] *The value of $\gamma_A(n)$ is equal to the amount of those elements of the semigroup S_A that can be represented as a product of length n of generators $\{f_{A_{q_1}}, f_{A_{q_2}}, \ldots, f_{A_{q_m}}\}$ for any automaton A and for any $n \in \mathbb{N}$.*

This statement can be used for calculation of the growth function of an automaton as well as for computation the growth function of transformation semigroup determined by this automaton.

3. Proofs

Before to prove the first part of the theorem formulated in the introduction, we will study in details the action of the transformations f_1 and f_2 on arbitrary words from the set X^*.

Lemma 5. *Let $u \in X^*$ be an arbitrary word of length at least 2.*

1. *The word $f_1(u)$ is built from the word u in the following way:*
   (a) *the symbol b is added to the beginning of the word u and the last character of u is erased;*
   (b) *in all maximal subwords such as a^n of the word obtained previously, each second character a is replaced by the symbol b and the other characters do not change.*
2. *The word $f_2(u)$ is defined on the word u by using the following transformations:*
   (a) *the first character of u is substituted by the word ab and last character of u is erased;*
   (b) *in all maximal subwords such as a^n of the word obtained previously, each second character a is replaced by the symbol b and the other characters don't change.*

Proof. Direct check. □

For words of length 1 we have:

$$f_1(a) = b, \quad f_1(b) = b, \quad f_2(a) = a, \quad f_2(b) = a.$$

Example. Let $u = aaabbbaa$. Then

$$f_1(u) = f_1(aaabbbaa) = \lambda(aaabbbaa, q_1) =$$
$$= \lambda(a, q_1) \cdot \lambda(a, q_2) \cdot \lambda(a, q_1) \cdot \lambda(b, q_2) \cdot \lambda(b, q_1) \cdot \lambda(b, q_1) \cdot \lambda(a, q_1) \cdot \lambda(a, q_2) = bababbba,$$

whence we get $f_1(aaa \cdot bbb \cdot aa) = b \cdot aba \cdot bbb \cdot a$. Similarly,

$$f_2(u) = f_2(aaabbbaa) = \lambda(aaabbbaa, q_2) =$$
$$= \lambda(a, q_2) \cdot \lambda(a, q_1) \cdot \lambda(a, q_2) \cdot \lambda(b, q_1) \cdot \lambda(b, q_1) \cdot \lambda(b, q_1) \cdot \lambda(a, q_1) \cdot \lambda(a, q_2) = ababbbba,$$

that is $f_2(aaa \cdot bbb \cdot aa) = ab \cdot ab \cdot bbb \cdot a$.

Let us denote by X^ω the set of all infinite (to the right) words over the alphabet X. It is obvious that the definition of automatic transformation of word that is given by the equality (5) can be extended to the set of infinite words. Let us introduce two transformations of the set X^ω for the describing action of aforementioned transformations f_1 and f_2 of this set:

1. the transformation g_1 in all maximal subwords such as a^n of infinite word replaces each second character a by the symbol b and does not change the other characters;
2. the transformation g_2 substitutes first character of every word from X^ω by the symbol b and acts on the remaining subword as g_1.

It is obvious that $g_1 \neq g_2$ and for any infinite word $u = x_1x_2x_3\ldots \in X^\omega$ we have:

$$f_1(u) = b \cdot g_1(u) = b \cdot g_1(x_1x_2x_3\ldots), \tag{6}$$

$$f_2(u) = a \cdot g_2(u) = ab \cdot g_1(x_2x_3\ldots). \tag{7}$$

Let $v \in X^*$ be a word that does not contain a subword aa and ends with the symbol b.

Lemma 6. *1. The transformations g_1 and g_2 are idempotents.*

2. For any $u \in X^\omega$ and for any $n \in \mathbf{N}$ the following equalities hold:

$$g_1^n(v \cdot u) = v \cdot g_1(u), \quad g_2^n(b \cdot v \cdot u) = b \cdot v \cdot g_1(u).$$

Proof. 1. According to the definition of g_1, for any word $u \in X^\omega$ an image $g_1(u)$ of the transformation g_1 does not contain the subword aa. Therefore

$$g_1(g_1(u)) = g_1(u), \tag{8}$$

whence it follows that $g_1^2 = g_1$, i.e. g_1 is idempotent. It follows from the definition of g_2 and (8) that $g_2(g_2(u)) = g_2(u)$, i.e. g_2 is also idempotent.

2. It is possible to restrict the action of the transformations g_1 and g_2 on the set X^*. Then obviously we receive $g_1(v) = v$, $g_2(b \cdot v) = b \cdot g_1(v) = b \cdot v$. For any word $u \in X^\omega$ $g_1(v \cdot u) = v \cdot g_1(u)$, and, therefore, for any n we obtain $g_1^n(v \cdot u) = v \cdot g_1^n(u) = v \cdot g_1(u)$. Similarly, for any word $u \in X^\omega$ at any n we have $g_2^n(b \cdot v \cdot u) = g_2^{n-1}(b \cdot v \cdot g_1(u)) = b \cdot v \cdot g_1(u)$. □

Lemma 7. *Let $v \in X^*$ be a non-empty word whose first and last character is equal to b, does not contain the subword aa, andlet $u \in X^\omega$ be an arbitrary infinite word. For the transformations g_1 and g_2 the following relations hold:*

$$g_2(g_1(v \cdot u)) = v \cdot g_1(u), \quad g_1(g_2(v \cdot u)) = v \cdot g_1(u).$$

Proof. The statement follows from Lemma 6. □

Lemma 8. *For any natural $n \geq 2$ the transformations f_1 and f_2 defined by A (fig. 1) satisfy the equality:*

$$f_2^n f_1 = f_2 f_1^n \tag{9}$$

Proof. At $n = 2$ for any word $u \in X^*$ by virtue of equalities (6), (7) and Lemmas 6, 7 we have:

$$\left(f_2^2 f_1\right)(u) = f_2^2\left(b \cdot g_1(u)\right) = f_2\left(ab \cdot g_1(u)\right) = abb \cdot g_1(u).$$

Similarly,

$$\left(f_2 f_1^2\right)(u) = f_2 f_1\left(b \cdot g_1(u)\right) = f_2\left(bb \cdot g_1(u)\right) = abb \cdot g_1(u),$$

that is $(f_2^2 f_1)(u) = (f_2 f_1^2)(u)$. Since u is an arbitrary word, we have $f_2^2 f_1 = f_2 f_1^2$.

We shall be convinced now that the relation (9) for $n > 2$ is a consequence of the relation

$$f_2^2 f_1 = f_2 f_1^2. \tag{10}$$

We have

$$f_2^n f_1 = f_2^{n-2} f_2^2 f_1 = f_2^{n-2} f_2 f_1^2 = f_2^{n-3} f_2^2 f_1 f_1 = f_2^{n-3} f_2 f_1^2 f_1 = f_2^{n-2} f_1^3 = \ldots = f_2 f_1^n,$$

and the lemma is proved. □

Proof of the theorem. First, let us prove that the semigroup S_A has the required definition by generators and determining relations:

$$S_A = \left\langle\ f_1, f_2\ \middle|\ f_2^2 f_1 = f_2 f_1^2\ \right\rangle.$$

We can write any element of the semigroup S_A as:

$$s = f_1^{p_1} f_2^{q_1} f_1^{p_2} f_2^{q_2} \ldots f_1^{p_s} f_2^{q_s},$$

where $p_1 \geq 0$, $q_s \geq 0$, $p_i, q_i \geq 1$ for other values of indices i and the sum of all degrees is non-zero. The relation $f_2 f_1^2 = f_2^2 f_1$ allows us to substitute subwords like $f_2^p f_1^q$ by subwords $f_2 f_1^{q+p-1}$ for arbitrary $p, q \geq 1$. Taking it into account that each semigroup word of length $n \geq 1$ can be resulted like:

$$\tilde{s} = \underbrace{f_2^{\varepsilon} f_1^{p_1} f_2 f_1^{p_2} f_2 \ldots f_1^{p_m} f_2^{\mu}}_{n \text{ symbols}} \tag{11}$$

where $m \geq 0$ and all $p_i \geq 1$; besides $\varepsilon = 0$, $\mu \geq 1$ when $m = 0$ and $\varepsilon \in \{0, 1\}$, $\mu \geq 0$ at $m \geq 1$. Such presentation of an element is called *canonical*. Let us remark that, since application of the determining relation (10) doesn't change the length of a semigroup word, the length of any element always coincides with the length of its canonical presentation.

As easily follows from Lemmas 6 and 7, any element $\tilde{s}$ like (11) acts on any infinite word u from X^ω in the following way:

$$\tilde{s}(u) = f_2^{\varepsilon} f_1^{p_1} f_2 f_1^{p_2} \ldots f_2 f_1^{p_m}(u) = a^{\varepsilon} b^{p_1} a b^{p_2} \ldots a b^{p_m} \cdot g_1(u), \tag{12}$$

if $\mu = 0$;

$$\tilde{s}(u) = f_2^{\varepsilon_1} f_1^{p_1} f_2 f_1^{p_2} \dots f_2 f_1^{p_m} f_2^{\mu}(u) = a^{\varepsilon_1} b^{p_1} a b^{p_2} \dots a b^{p_m} a b^{\mu-1} \cdot g_2(u) \tag{13}$$

if $\mu \geq 0$.

For the proof of the theorem it is enough to show that two semigroup elements s_1 and s_2 having various canonical presentations determine different transformations of X^ω. Let:

$$s_1 = f_2^{\varepsilon_1} f_1^{p_1} f_2 f_1^{p_2} f_2 \dots f_1^{p_m} f_2^{\mu_1}, \tag{14}$$

and

$$s_2 = f_2^{\varepsilon_2} f_1^{q_1} f_2 f_1^{q_2} f_2 \dots f_1^{q_l} f_2^{\mu_2}. \tag{15}$$

By contradiction, assume that s_1 and s_2 determine the same transformation of X^ω. Therefore, the equality $s_1(u) = s_2(u)$ holds for any word $u \in X^\omega$, and, in particular, for the words $u_1 = ab^* = abbb\dots$ and $u_2 = bab^* = babbb\dots$. Thus

$$\begin{aligned} g_1(u_1) &= ab^*, & g_2(u_1) &= b^*, \\ g_1(u_2) &= bab^*, & g_2(u_2) &= bab^*. \end{aligned} \tag{16}$$

Let us consider the possible cases for the canonical presentation (14) and (15).

1. $\mu_1 = \mu_2 = 0$. From (12) and (16) for the word u_1 we have

$$a^{\varepsilon_1} b^{p_1} a b^{p_2} \dots a b^{p_m} \cdot ab^* = a^{\varepsilon_2} b^{q_1} a b^{q_2} \dots a b^{q_l} \cdot ab^*,$$

   whence $m = l$, $\varepsilon_1 = \varepsilon_2$, $p_i = q_i$, $i = 1, \dots, m$, that contradicts to the choice of s_1 and s_2.

2. $\mu_1 \geq 1$ and $\mu_2 = 0$ (the case $\mu_1 = 0$ and $\mu_2 \geq 1$ is considered similarly). From (12), (13) and (16) for the word u_1 we get

$$a^{\varepsilon_1} b^{p_1} a b^{p_2} \dots a b^{p_m} a b^{\mu_1 - 1} \cdot b^* = a^{\varepsilon_2} b^{q_1} a b^{q_2} \dots a b^{q_l} \cdot ab^*.$$

   whence $m = l$, $\varepsilon_1 = \varepsilon_2$, $p_i = q_i$, $i = 1, \dots, m$. Similarly, for the word u_2:

$$s_1(u_2) = s_1(bab^*) = a^{\varepsilon_1} b^{p_1} a b^{p_2} \dots a b^{p_m} a b^{p_{m+1} - 1} \cdot bab^*,$$

$$s_2(u_2) = s_2(bab^*) = a^{\varepsilon_1} b^{p_1} a b^{p_2} \dots a b^{p_m} \cdot bab^*,$$

   whence $s_1(u_2) \neq s_2(u_2)$, that contradicts to the assumption of the equality of transformations determined by s_1 and s_2.

3. $\mu_1 \geq 1$ and $\mu_2 \geq 1$. From (13) and (16) for the word u_2 we have

$$a^{\varepsilon_1} b^{p_1} a \dots b^{p_m} a b^{\mu_1 - 1} \cdot bab^* = a^{\varepsilon_2} b^{q_1} a \dots b^{q_l} a b^{\mu_2 - 1} \cdot bab^*,$$

   whence $m = l$, $\varepsilon_1 = \varepsilon_2$, $\mu_1 = \mu_2$, $p_i = q_i$, $i = 1, \dots, m$, that contradicts to the choice of s_1 and s_2.

The obtained contradiction proves the first part of the theorem.

Let us show that equality (2) holds. We denote by $F(n)$ for $n \in \mathbf{N}$ the amount of sequences of length n from the symbols a and b where two symbols a do not stand together. A numerical sequence $F(n)$ is a Fibonacci sequence (see, for example, [21]) with start values $F(1) = 2$, $F(2) = 3$. Let us add to this sequence the value $F(0) = 1$.

We denote by symbol $W(n)$ the amount of different elements of the semigroup S_A of the length n. It is obvious, that $W(1) = 2$, $W(2) = 4$. Let us find the relation between sequences $W(n)$ and $F(n)$, $n \geq 1$.

Since each element from S_A decomposed to a product of length n of generators f_1 and f_2 can be represented like (11) and action of this element on ω-word is set by (12) or (13), we can set the correspondence between each semigroup element and some sequence of length n from a and b, where two symbols a do not stand together. A sequence like

$$\underbrace{a^{\varepsilon} b^{p_1} a b^{p_2} \ldots a b^{p_m} a}_{n},$$

where last character is the symbol a, corresponds to one semigroup element $\tilde{s}$, for which $\mu = 1$,

$$\tilde{s} = f_2^{\varepsilon} f_1^{p_1} f_2 f_1^{p_2} \ldots f_2 f_1^{p_m} f_2;$$

the unique semigroup element $\tilde{s} = f_1^n$ corresponds to the sequence b^n; and all other sequences

$$\underbrace{a^{\varepsilon} b^{p_1} a b^{p_2} \ldots a b^{p_m} a b^{\mu}}_{n},$$

where $\mu > 0$, have in correspondence two semigroup elements:

$$\tilde{s}_1 = f_2^{\varepsilon} f_1^{p_1} f_2 f_1^{p_2} \ldots f_2 f_1^{p_m} f_2^{\mu+1}$$

and

$$\tilde{s}_2 = f_2^{\varepsilon} f_1^{p_1} f_2 f_1^{p_2} \ldots f_2 f_1^{p_m} f_2 f_1^{\mu}.$$

The amount of sequences of length n ended with the symbol a, is equal to $F(n-2)$, because we need to add the word ba to a sequence of length $n-2$. The amount of sequences of length n that contain the character a but not on the last place, is equal to $F(n-1) - 1$, because we can add the symbol b to all the sequences of length $n-1$ excepting b^{n-1}. Thus,

$$W(n) = 2(F(n-1) - 1) + F(n-2) + 1 = 2F(n-1) + F(n-2) - 1 = F(n+1) - 1.$$

From here $F(n+1) = W(n) + 1$, and, as $F(n+1) = F(n) + F(n-1)$, equality (2) holds, and the theorem is completely proved. □

The proof of the corollary requires

Lemma 9. *The amount of different semigroup elements of length $n \geq 1$ in the semigroup S_A equals to*

$$W(n) = \frac{1}{\sqrt{5}} \left(\left(\sqrt{5} - 2\right) \left(\frac{1 - \sqrt{5}}{2} \right)^n + \left(\sqrt{5} + 2\right) \left(\frac{1 + \sqrt{5}}{2} \right)^n \right) - 1.$$

Proof. As $W(n) = F(n+1) - 1$, $n \geq 1$, and

$$F(n) = \frac{\sqrt{5}-3}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}+3}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

then

$$W(n) = \frac{\sqrt{5}-3}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \frac{\sqrt{5}+3}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - 1 =$$
$$= \frac{1}{\sqrt{5}} \left(\left(\sqrt{5}-2\right) \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\sqrt{5}+2\right) \left(\frac{1+\sqrt{5}}{2}\right)^n\right) - 1,$$

as was to be shown. □

Proof of the conclusion.

1) Let us consider the subset FS_A of the semigroup S_A consisting from the following elements:

$$\hat{s} = f_1^{p_1} (f_2 f_1) f_1^{p_2} (f_2 f_1) f_1^{p_3} \dots (f_2 f_1) f_1^{p_m} \tag{17}$$

where $p_i \geq 0$, $i = 1, \dots, m$. Since not every element of S_A can be presented as (17), we see that FS_A is a subsemigroup and $FS_A \neq S_A$. Besides, any element of the semigroup FS_A can be decomposed in a product of the transformation f_1 and $f_2 f_1$. From the proved theoremit follows that such decomposition is unique, that is the semigroup FS_A is free. Thus

$$FS_A = \langle f_1, (f_2 f_1) \rangle .$$

2) The length of any element always coincides with length of its canonical presentation (11), because the application of determining relation (10) does not change the length of a semigroup word. Using Lemma 4 we get that the value of the growth function $\gamma_A(n)$ of an automaton coincides with the amount of those elements of the semigroup S_A that can be represented like a product of minimal possible length n of the generators f_1 and f_2, that is with the number $W(n)$. Then it follows from Lemma 9 that equality (3) holds.

3) By definition, the growth function $\gamma_{S_A}(n)$ of the semigroup S_A is equal to

$$\gamma_{S_A}(n) = \sum_{l=1}^{n} W(l).$$

It follows immediately from Lemma 9 that

$$\gamma_S(n) = \sum_{l=1}^{n} \left(\frac{1}{\sqrt{5}} \left(\left(\sqrt{5}-2\right) \left(\frac{1-\sqrt{5}}{2}\right)^l + \left(\sqrt{5}+2\right) \left(\frac{1+\sqrt{5}}{2}\right)^l \right) - 1 \right) =$$
$$= \frac{\sqrt{5}-2}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right) \cdot \frac{\left(\frac{1-\sqrt{5}}{2}\right)^n - 1}{\frac{1-\sqrt{5}}{2} - 1} + \frac{\sqrt{5}+2}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right) \cdot \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - 1}{\frac{1+\sqrt{5}}{2} - 1} - n =$$
$$= \frac{1}{2\sqrt{5}} \left(\left(5\sqrt{5}-11\right) \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(5\sqrt{5}+11\right) \left(\frac{1+\sqrt{5}}{2}\right)^n \right) - (n+5)$$

as was to be shown. □

4. Final remarks

It is convenient to introduce a relation of equivalence on the set of automata so that equivalent automata would have the same semigroup. It is enough to investigate representative automata from the classes of equivalence for constructing semigroups and finding growth functions. Such relation of equivalence is introduced when the automata considered equivalent differ in numeration of states and/or by permutation of alphabet symbols. The set of 32 automata that determine non-free semigroups of exponential growth order divides into 8 classes of equivalence.

It is obvious that equivalent automata determine the same transformation semigroup of the set of words over the base alphabet, and, consequently, have the same growth function. On the other hand, non-equivalent automata also may determine the same semigroup. We came to the conclusion that there are only seven pairwise nonisomorphic semigroups determined by abovementioned 32 automata. These semigroups have the following representations by generators and determining relations:

$$S_1 = S_A = \left\langle\ f_1, f_2 \ \middle|\ f_2 \cdot f_1^2 = f_2^2 \cdot f_1 \ \right\rangle,$$

$$S_2 = \left\langle\ f_1, f_2 \ \middle|\ f_1 \cdot f_2 \cdot f_1 = f_2^2 \cdot f_1 \ \right\rangle,$$

$$S_3 = \left\langle\ f_1, f_2 \ \middle|\ f_2^2 \cdot f_1 = f_1^3 \ \right\rangle,$$

$$S_4 = \left\langle\ f_1, f_2 \ \middle|\ f_2^2 \cdot f_1^{2k} \cdot f_2 = f_2 \cdot f_1^{2k+1} \cdot f_2,\ k \geq 0 \ \right\rangle,$$

$$S_5 = \left\langle\ f_1, f_2 \ \middle|\ f_1 \cdot f_2 \cdot f_1 = f_2^2 \cdot f_1, f_1 \cdot f_2^2 = f_1^2 \cdot f_2 \ \right\rangle,$$

$$S_6 = \left\langle\ f_1, f_2 \ \middle|\ f_2 \cdot f_1 \cdot f_2 = f_1^2 \cdot f_2, f_2 \cdot f_1^3 = f_2^2 \cdot f_1^2 \ \right\rangle,$$

$$S_7 = \left\langle\ f_1, f_2 \ \middle|\ f_2 \cdot f_1 \cdot f_2 = f_2^3 \ \right\rangle.$$

The semigroups S_1, S_2, S_3, S_4 have the same growth function defined by formula (4). It follows from the fact that the amount $W(n)$ of elements of each of these semigroups that can be represented as a product of minimal possible length n of the generators f_1 and f_2 satisfies the recurrent relation (2). The number $W(n)$ of semigroups S_5, S_6, S_7 satisfies some recurrent relations that are slightly different from (2). Therefore, the construction of the growth function formulas for these semigroups is quite analogous. All semigroups S_i, $i = 1, \ldots, 7$, include free two-generated semigroups.

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Kyiv National University

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