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THE EXACT SOLUTIONS OF SOME MULTICOMPONENT INTEGRABLE MODELS

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The exact solutions of multicomponent generalization of nonlinear Yajima-Oikawa model and their (2+1)-dimensional extensions are constructed in an explicit form. A vector Melnikov-like system has been integrated too.

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Точные решения мультикомпонентного обобщения нелинейной модели Яджимы-Ойкавы построены в явной форме. Также проинтегрирована векторная система типа Мельникова.

Introduction.

The hierarchy of Kadomtsev-Petviashvili equations can be given as an infinite sequence of the Sato-Wilson operator equations [1–2]

$$\alpha_n W_{t_n} = -(W\mathcal{D}^n W^{-1})_- W, \quad n \in \mathbb{N}, \alpha_n \in \mathbb{C} \tag{1}$$

where

$$W = 1 + w_1\mathcal{D}^{-1} + w_2\mathcal{D}^{-2} + \dots \tag{2}$$

is a microdifferential operator (MDO) with coefficients $w_i, i \in \mathbb{N}$, depending on the variables $\mathbf{t} = (t_1, t_2, \dots)$, $t_1 := x$ and $\mathcal{D} := \frac{\partial}{\partial x}$, $\mathcal{D}\mathcal{D}^{-1} = 1$. The differential and integral parts of the microdifferential operator $W\mathcal{D}^n W^{-1}$ are denoted by $(W\mathcal{D}^n W^{-1})_+$ and $(W\mathcal{D}^n W^{-1})_-$ respectively and we have

$$(W\mathcal{D}^n W^{-1})_- := W\mathcal{D}^n W^{-1} - (W\mathcal{D}^n W^{-1})_+.$$

In the MDO algebra ζ the operation of multiplication is induced by the generalized Leibnitz rule

$$\mathcal{D}^n f := \sum_{j=0}^{\infty} \binom{n}{j} f^{(j)}\mathcal{D}^{n-j}, n \in \mathbb{Z} \ ; \quad \mathcal{D}^m(f) := \frac{\partial^m f}{\partial x^m} = f^{(m)}, m \in \mathbf{Z}_+, \tag{3}$$

where

$$\mathcal{D}^n\mathcal{D}^m := \mathcal{D}^m\mathcal{D}^n := \mathcal{D}^{n+m}; \quad n, m \in \mathbb{Z},$$

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and f is the operator of multiplication by a function $f(\mathbf{t})$ which belongs to the same functional space $\mathcal{A}$ that the coefficients of microdifferential operators $L \in \zeta$ do:

$$L \in \zeta = \left\{ \sum_{i=-\infty}^{n(L)} a_i \mathcal{D}^i : a_i = a_i(\mathbf{t}) \in \mathcal{A};\, i, n(L) \in \mathbb{Z} \right\},$$

With the aid of the MDO L defined by the formula

$$L := W\mathcal{D}W^{-1} = \mathcal{D} + U\mathcal{D}^{-1} + U_2\mathcal{D}^{-2} + \dots,$$

system (1) can be rewritten in the form of the Lax representation

$$\alpha_n L_{t_n} = [B_n, L] := B_n L - L B_n, \tag{4}$$

where

$$B_n = (L^n)_+ = (W\mathcal{D}^n W^{-1})_+, \quad n \in \mathbb{N}.$$

Nonlocal reduced hierarchy of Kadomtsev-Petviashvili is the system of operator equations (4) with the additional restriction, so-called *k-constraint* of the form [3–8] (see also [9])

$$L^k := (L^r)^k = B_k + \sum_{i=1}^{l} q_i \mathcal{D}^{-1} r_i^{\top},$$

where "$\top$" denotes the transposition according with dynamics of system (4) , if the field-variables q_i, r_i satisfy the system of the following equations

$$\begin{cases} \alpha_n q_{it_n} = B_n(q_i), \\ \alpha_n r_{it_n} = -B_n^{\tau}(r_i). \end{cases} \tag{5}$$

In formula (5) and in the sequel the symbol "τ" denotes the transposition of a differential operator.

The equations from k-reduced hierarchy of Kadomtsev-Petviashvili admit the Lax representation

$$[B_k + \mathbf{q}\mathcal{D}^{-1}\mathbb{R}^{\top}, \alpha_n \partial_{t_n} - B_n] = 0, n \in \mathbb{N}. \tag{6}$$

Equations (6) have real physical applications in the case of additional constraints, for example, reductions of complex conjugation (see below). Integration of equations (6) with reductions is the goal of this paper.

The exact solutions of vector generalization of Yajima-Oikawa model.
Consider the case $k = 2, n = 2$.

$$\begin{cases} \alpha_2 \mathbf{q}_{t_2} = \mathbf{q}_{xx} + 2U\mathbf{q}, \\ \alpha_2 U_{t_2} = (\mathbf{q}\mathbb{R}^{\top})_x, \\ \alpha_2 \mathbb{R}_{t_2} = -\mathbb{R}_{xx} - 2U\mathbb{R}. \end{cases} \tag{7}$$

In formulas (6)–(7) and everywhere below $\mathbf{q} = (q_1, \dots, q_l)$, $\mathbf{r} = (r_1, \dots\ r_l)$ and

$$\mathbf{q}\mathcal{D}^{-1}\mathbb{R}^{\top} = \sum_{i=1}^{l} q_i \mathcal{D}^{-1} r_i.$$

Introduce the additional reductions of complex conjugation $\alpha_2 = i, t_2 = t, \mathbb{R} = i\bar{\mathbf{q}}M^\top$, where $M \in \mathrm{Mat}_{l\times l}(\mathbb{C}), M = M^*$.

System (7) can be represented as:

$$\begin{cases} i\mathbf{q}_t = \mathbf{q}_{xx} + 2U\mathbf{q}, \\ U_t = (\mathbf{q}M\mathbf{q}^*)_x. \end{cases} \tag{8}$$

System (8) is a vector (l-components) generalization of Yajima-Oikawa model [10,5,6,9]. If the matrix M is diagonal ($M = \mathrm{diag}(\mu_1, \dots, \mu_l), \mu_j \in \mathbb{R}, j \in \{1, \dots, l\}$), we can rewrite the second equation in system (8) in the following form:

$$U_t = \sum_{i=1}^{l} \mu_i(|q_i|^2)_x.$$

The operators of system (8) in the Lax representation ($[L, A] = 0$) are of the form:

$$L = \mathcal{D}^2 + 2U + i\mathbf{q}M\mathcal{D}^{-1}\mathbf{q}^*, \quad A = i\partial_t - \mathcal{D}^2 - 2U.$$

Proposition 1. [2,11] *Let $B = B_+$ be a differential operator; $\mathbf{f}\mathcal{D}^{-1}\mathbf{g}$, $\tilde{\mathbf{f}}\mathcal{D}^{-1}\tilde{\mathbf{g}} \in \zeta$. Then the following relations hold:*

$$\begin{aligned} B\mathbf{f}\mathcal{D}^{-1}\mathbf{g}^\top &= (B\mathbf{f}\mathcal{D}^{-1}\mathbf{g}^\top)_+ + B(\mathbf{f})\mathcal{D}^{-1}\mathbf{g}^\top, \\ \mathbf{f}\mathcal{D}^{-1}\mathbf{g}^\top B &= (\mathbf{f}\mathcal{D}^{-1}\mathbf{g}^\top B)_+ + \mathbf{f}\mathcal{D}^{-1}(B^\tau\mathbf{g})^\top), \\ \mathbf{f}\mathcal{D}^{-1}\mathbf{g}^\top\tilde{\mathbf{f}}\mathcal{D}^{-1}\tilde{\mathbf{g}}^\top &= \mathbf{f}\Big(\int \mathbf{g}^\top\tilde{\mathbf{f}}\Big)\mathcal{D}^{-1}\tilde{\mathbf{g}}^\top - \mathbf{f}\mathcal{D}^{-1}\Big(\int \mathbf{g}^\top\tilde{\mathbf{f}}\Big)\tilde{\mathbf{g}}^\top. \end{aligned} \tag{9}$$

In formulas (9) by the symbol $(\int \mathbf{g}^\top\tilde{\mathbf{f}})$ we denote an arbitrary fixed primitive of $(\mathbf{g}^\top\tilde{\mathbf{f}})(x, t_2)$ as a function of x, that is

$$\frac{\partial}{\partial x}\Big(\int \mathbf{g}^\top\tilde{\mathbf{f}}\Big) = \mathbf{g}^\top\tilde{\mathbf{f}}.$$

Let $\varphi := (\varphi_1, \varphi_2, \dots, \varphi_K) = \varphi(x, t_2)$, $\psi := (\psi_1, \psi_2, \dots, \psi_K) = \psi(x, t_2)$ be smooth complex functions of real variables $x, t_2 \in \mathbb{R}$, $C = (C_{mn}) = \mathrm{const} \in \mathrm{Mat}_{K\times K}(\mathbb{C})$, and also:

1) the improper integral

$$\int_{-\infty}^{x} \psi^\top\varphi\, ds := \int_{-\infty}^{x} \psi^\top(s, t_2)\varphi(s, t_2)\, ds$$

converges absolutely $\forall(x, t_2) \in \mathbb{R} \times \mathbb{R}_+$ and admits differentiation by parameter $t_2 \in \mathbb{R}_+$,

2) the matrix-function $\Omega(x, t_2) := C + \int_{-\infty}^{x} \psi^\top\varphi\, ds$ is nondegenerate in $(x, t_2) \in \sigma \subset \mathbb{R} \times \mathbb{R}_+$.

Define the functions $\Phi = \Phi(x, t_2), \Psi = \Psi(x, t_2)$ and MDO W (2) by the following way

$$\Phi = \varphi\Omega^{-1}, \Psi^\top = \Omega^{-1}\psi^\top, W = 1 - \Phi\mathcal{D}^{-1}\psi^\top. \tag{10}$$

Lemma. *The components Φ_i, Ψ_i of the vector-functions Φ, Ψ (17) can be given as:*

$$\Phi_i = (\varphi\Omega^{-1})_i = (-1)^{K+i}\frac{\begin{vmatrix} \Omega_{(i)} \\ \varphi \end{vmatrix}}{|\Omega|}, \quad 1 \le i \le K, \tag{11}$$

$$\Psi_i = (\psi\Omega^{\top -1})_i = (-1)^{K+i}\frac{\begin{vmatrix}\Omega^{\top}_{(i)}\\ \psi\end{vmatrix}}{|\Omega|}, \quad 1 \le i \le K \tag{12}$$

Here $\Omega_{(i)}$ is obtained from Ω by deletion of the i-line.

Proof. In order to prove (11),(12) we use the well-known algebraic equality for framed determinant:

$$\det\begin{pmatrix}\Omega & \psi^{\top}\\ \varphi & \alpha\end{pmatrix} := \begin{vmatrix}\Omega & \psi^{\top}\\ \varphi & \alpha\end{vmatrix} = \alpha\det\Omega - \varphi\Omega^{C}\psi^{\top},$$

where Ω^C is the matrix of cofactors. We have

$$\Phi_i = (\varphi\Omega^{-1})_i = \varphi\Omega^{-1}e_i^{\top} = (-1)^{K+i}\frac{\begin{vmatrix}\Omega_{(i)}\\ \varphi\end{vmatrix}}{|\Omega|}.$$

Here $e_i = (e_{i_1}, \dots, e_{i_K}), e_{i_i} = 1, e_{i_j} = 0$, for $i, j \in \{1, \dots, K\}, i \ne j$.

By a similar reasoning, formula (12) can be proved. □

Theorem 1. [11] *MDO W has an inverse operator W^{-1} and:*

$$W^{-1} = 1 + \varphi\mathcal{D}^{-1}\Psi^{\top}.$$

Proposition 2. *For MDO W (10) the following equalities are true:*

$$W\mathcal{D}^2W^{-1} = (I - \Phi\mathcal{D}^{-1}\psi^{\top})\mathcal{D}^2(I + \varphi\mathcal{D}^{-1}\Psi^{\top}) =$$

$$= \mathcal{D}^2 + 2(\varphi\Omega^{-1}\psi^{\top})_x - \Phi\mathcal{D}^{-1}\left(\psi^{\top}_{xx} - \int_{-\infty}^{x}\psi^{\top}_{ss}\varphi\,ds\,\Psi^{\top}\right) + \left(\varphi_{xx} - \Phi\int_{-\infty}^{x}\psi^{\top}\varphi_{ss}\,ds\right)\mathcal{D}^{-1}\Psi^{\top},$$

$$W(i\partial_t - \mathcal{D}^2)W^{-1} =$$

$$= i\partial_t - \mathcal{D}^2 - 2(\varphi\Omega^{-1}\psi^{\top})_x + \Phi\mathcal{D}^{-1}\left\{(i\psi^{\top}_t + \psi^{\top}_{xx}) - \int_{-\infty}^{x}(i\psi^{\top}_t + \psi^{\top}_{ss})\varphi\,ds\,\Psi^{\top}\right\} +$$

$$+ \left\{(i\varphi_t - \varphi_{xx}) - \Phi\int_{-\infty}^{x}\psi^{\top}(i\varphi_t - \varphi_{ss})\,ds\right\}\mathcal{D}^{-1}\Psi^{\top}.$$

The proof of Proposition 2 is based on formulas (3), (9).

Consider operators

$$L_0 = \mathcal{D}^2, \quad A_0 = i\partial_t - \mathcal{D}^2, \quad \hat{L} = WL_0W^{-1}, \quad \hat{A} = WA_0W^{-1}.$$

Theorem 2. *Let:*

a) φ *be a solution of the equation* $i\varphi_t = \varphi_{xx}$;

b) $\varphi_{xx} = \varphi\Lambda$, *where* $\Lambda = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \dots, \lambda_K^2) = \mathrm{const} \in \mathrm{Mat}_{K\times K}(\mathbb{C})$;

c) $\psi = \bar{\varphi}$;

d) $C = C^*$.

Then

1) $\Psi = \bar{\Phi}$;

2) $\hat{L} = \mathcal{D}^2 + 2(\varphi\Omega^{-1}\varphi^*)_x + \Phi J\mathcal{D}^{-1}\bar{\Phi}^\top$, *where* $J = C\Lambda - \Lambda^* C$;

3) $\hat{A} = i\partial_t - \mathcal{D}^2 - 2(\varphi\Omega^{-1}\varphi^*)_x$.

Proof. 1) From definitions (10) and condition d we have:

$$\bar{\Phi} = \overline{\varphi(C + \int_{-\infty}^x \varphi^*\varphi\, ds)^{-1}} = \bar{\varphi}(\bar{C} + \int_{-\infty}^x \varphi^\top\bar{\varphi}\, ds)^{-1} = \psi(C^\top + \int_{-\infty}^x \varphi^\top\psi\, ds)^{-1} = \Psi.$$

2) From Proposition 2, condition b and the properties: $\Phi\int_{-\infty}^x \psi^\top\varphi\, ds = \varphi - \Phi C$ and $\int_{-\infty}^x \psi^\top\varphi\, ds\Psi^\top = \psi^\top - C\Psi^\top$, it follows that

$$\hat{L} = \mathcal{D}^2 + 2(\varphi\Omega^{-1}\varphi^*)_x - \Phi\mathcal{D}^{-1}\Lambda^*\psi^\top + \Phi\mathcal{D}^{-1}\Lambda^*\int_{-\infty}^x \psi^\top\varphi\, ds\Psi^\top +$$

$$+\varphi\Lambda\mathcal{D}^{-1}\Psi^\top - \Phi\int_{-\infty}^x \psi^\top\varphi\, ds\Lambda\mathcal{D}^{-1}\Psi^\top =$$

$$= \mathcal{D}^2 + 2(\varphi\Omega^{-1}\varphi^*)_x + \Phi(C\Lambda - \Lambda^*C)\mathcal{D}^{-1}\Psi^\top.$$

3) The validity of this item follows from Proposition 2 and conditions a,c. □

Proposition 3. *The matrix* $J = (J_{mn}), m, n \in \{1, \ldots, K\}$ *has the following properties:*

1) $J = -J^*$;

2) $J_{mn} = C_{mn}(\lambda_m^2 - \overline{\lambda_n^2})$;

3) If the matrix C *is diagonal, then*

$$J = \mathrm{diag}(2ic_1\mathrm{Im}\lambda_1^2, 2ic_2\mathrm{Im}\lambda_2^2, \ldots, 2ic_K\mathrm{Im}\lambda_K^2).$$

Proof. 1)

$$J^* = (C\Lambda - \Lambda^*C)^* = \Lambda^*C^* - C^*\Lambda = \Lambda^*C - C\Lambda = -J.$$

The proof of the 2), 3) is based on formulas of operations with matrixes. □

Corollaries.

Corollary 1. *Let* K *be an arbitrary natural number,* $l = 1$*, and suppose that the matrix* J *can be represented in the form* $J = iMI^\top I$*, where* $I = (1, \ldots, 1)$ *is* K*-dimensional vector. Then the functions*

$$\mathbf{q} = \sum_{i=1}^K \Phi_i = \frac{\begin{vmatrix} & 1 \\ \Omega & \vdots \\ & 1 \\ \varphi & 0 \end{vmatrix}}{|\Omega|}, \tag{13}$$

$$U = (\varphi\Omega^{-1}\varphi^*)_x = \left(\begin{vmatrix} \Omega & \varphi^* \\ \varphi & 0 \end{vmatrix} |\Omega|^{-1}\right)_x = \frac{\partial^2}{\partial x^2}\ln|\det\Omega| \tag{14}$$

are solutions of system (8) considered as a scalar one.

In the proof of the given corollary we represent J in the form $J = iMI^\top I$, that is, using Proposition 3 we consider $C_{mn} = \frac{iM}{\lambda_m^2 - \bar{\lambda}_n^2}$ for $m, n \in \{1, \dots, K\}$. Then

$$\Phi J \mathcal{D}^{-1} \bar{\Phi}^\top = \Phi i M I^\top \mathcal{D}^{-1} I \bar{\Phi}^\top = iM \sum_{i=1}^{K} \Phi_i \mathcal{D}^{-1} \sum_{i=1}^{K} \bar{\Phi}_i,$$

and hence formula (13) follows from these equalities and formula (11) (see the Lemma above).

Formulas (13)–(14) determine the solution of classical (1-component) equation by Yajima-Oikawa [10] which depends on K functional parameters $\varphi = (\varphi_1, \dots, \varphi_K)$ (solutions of the linear system $i\varphi_t = \varphi_{xx}, \varphi_{xx} = \varphi\Lambda$). In particular, if $\varphi_j = \hat{c}_j e^{\lambda_j x - i\lambda_j^2 t}$, $\hat{c}_j, \lambda_j \in \mathbb{C}$, $\lambda_j \neq \lambda_k$ for $k \neq l$, $\mathrm{Re}\,\lambda_j > 0, j \in \{1, \dots, K\}$, formulas (13)–(14) determine a K-soliton solution of system (8) in the form which is analogous to a K-soliton solution of nonlinear Shrödinger equation (see [12]).

Consider the scalar case ($K = l = 1$). Then $\varphi = \hat{c}e^{\lambda x - i\lambda^2 t}$, $M = \mu$, and under conditions $\mathrm{Re}\lambda > 0$, $C = \frac{\mu}{4\mathrm{Re}\lambda \mathrm{Im}\lambda}$ solutions have the form:

$$\mathbf{q} = \frac{2\hat{c}\mathrm{Re}\lambda e^{\lambda x - i\lambda^2 t}}{\frac{\mu}{2\mathrm{Im}\lambda} + |\hat{c}|^2 e^{2\mathrm{Re}\lambda x + 4\mathrm{Re}\lambda\mathrm{Im}\lambda t}}, \quad U = \frac{8|\hat{c}|^2 \mu \mathrm{Re}^2\lambda \mathrm{Im}\lambda e^{2\mathrm{Re}\lambda x + 4\mathrm{Re}\lambda\mathrm{Im}\lambda t}}{(\mu + 2\mathrm{Im}\lambda |\hat{c}|^2 e^{2\mathrm{Re}\lambda x + 4\mathrm{Re}\lambda\mathrm{Im}\lambda t})^2}.$$

If $\mu = 2\mathrm{Im}\lambda$ we obtain the functions $\mathbf{q}$ and U in the following form:

$$\mathbf{q} = \frac{A\exp\{i(vx + (v^2 - u^2)t + \varphi_0)\}}{\mathrm{ch}\{u(x + 2vt + x_0)\}}, \quad U = \frac{A^2}{\mathrm{ch}^2\{u(x + 2vt + x_0)\}},$$

where

$$u = \mathrm{Re}\lambda, \quad v = \mathrm{Im}\lambda, \quad A = 2\mathrm{Re}\lambda, \quad \varphi_0 = \arg\hat{c}, \quad x_0 = \frac{\ln|\hat{c}|}{\mathrm{Re}\lambda}.$$

The solutions $q(x,t)$ and $U(x,t)$ are smooth functions localized along the direction $x(t) = -x_0 - 2vt$.

In particular, if $\lambda = 1 + i, \hat{c} = 1$ then previous formulas can be rewritten as:

$$\mathbf{q} = \frac{2e^{x+2t+ix}}{1 + 2e^{2(x+2t)}}, \quad U = \frac{4e^{2(x+2t)}}{(1 + e^{2(x+2t)})^2}.$$

One can see that in the case $\mu = 2\mathrm{Im}\lambda$ we have $|\mathbf{q}|^2 = U$ and $|\mathbf{q}|^2, U \to 0$ as $|x+2t| \to +\infty$.

Corollary 2. *Let $K \geq l$, C be a diagonal matrix and the product $C_{kk}\mathrm{Im}\lambda_k^2 \neq 0$ for l values of the index $k \in \{1, \dots, K\}$ (without loss of generality we can suppose that this condition is satisfied for $k \in \{1, \dots, l\}$).*

Then the functions $\mathbf{q} = (q_1, \dots, q_l)$ and U, where

$$q_j = (-1)^{j+K} \frac{\begin{vmatrix} \Omega_{(j)} \\ \varphi \end{vmatrix}}{|\Omega|}, \tag{15}$$

$$U = \left(\begin{vmatrix} \Omega & \varphi^* \\ \varphi & 0 \end{vmatrix} |\Omega|^{-1} \right)_x,$$

are solutions of system (8) for diagonal matrix M and $C_{jj} = \frac{\mu_j}{4\mathrm{Re}\lambda_j \mathrm{Im}\lambda_j}, j \in \{1, \dots, K\}$.

The proof of Corollary 2 is based on the application of formula (11) and Proposition 3. Let us consider the case:

$$K = l = 2, \quad C = \mathrm{diag}(c_1, c_2), \quad \varphi_j = \hat{c}_j e^{\lambda_j x - i\lambda_j^2 t}, \quad c_j = \frac{\mu_j}{4\mathrm{Re}\,\lambda_j \mathrm{Im}\,\lambda_j}, \quad j \in \{1,2\}.$$

We denote: $u_j = \mathrm{Re}\lambda_j, v_j = \mathrm{Im}\lambda_j$, $j \in \{1,2\}$. Under the conditions $\mathrm{Re}\lambda_1 > 0$ and $\mathrm{Re}\lambda_2 > 0$ we obtain:

$$q_1 = \frac{\hat{c}_1 e^{u_1 x + 2u_1 v_1 t + i(v_1 x + (v_1^2 - u_1^2)t)}}{|\Omega|}\left(\frac{\mu_2}{4u_2 v_2} + \frac{u_1 - u_2 + i(v_1 - v_2)}{2u_2(u_1 + u_2 + i(v_1 - v_2))} e^{2u_2 x + 4u_2 v_2 t}\right),$$

$$q_2 = \frac{\hat{c}_2 e^{u_2 x + 2u_2 v_2 t + i(v_2 x + (v_2^2 - u_2^2)t)}}{|\Omega|}\left(\frac{\mu_1}{4u_1 v_1} + \frac{u_2 - u_1 + i(v_2 - v_1)}{2u_1(u_1 + u_2 + i(v_2 - v_1))} e^{2u_1 x + 4u_1 v_1 t}\right),$$

$$U = \left(\frac{P}{|\Omega|}\right)_x,$$

where

$$|\Omega| = \frac{|\hat{c}_1|^2 |\hat{c}_2|^2 ((u_1 - u_2)^2 + (v_1 - v_2)^2)}{4u_1 u_2 ((u_1 + u_2)^2 + (v_1 - v_2)^2)} e^{2(u_1 + u_2)x + 4(u_1 v_1 + u_2 v_2)t} + $$
$$+ \frac{\mu_2 |\hat{c}_1|^2}{8u_1 u_2 v_2} e^{2u_1 x + 4u_1 v_1 t} + \frac{\mu_1 |\hat{c}_2|^2}{8u_1 u_2 v_1} e^{2u_2 x + 4u_2 v_2 t} + \frac{\mu_1 \mu_2}{16 u_1 u_2 v_1 v_2},$$

$$P = \left(\frac{u_1 + u_2}{2u_1 u_2} - \frac{2(u_1 + u_2)}{(u_1 + u_2)^2 + (v_1 - v_2)^2}\right) |\hat{c}_1|^2 |\hat{c}_2|^2 e^{2(u_1 + u_2)x + 4(u_1 v_1 + u_2 v_2)t} +$$
$$+ \frac{\mu_2 |\hat{c}_1|^2}{8u_1 u_2 v_2} e^{2u_1 x + 4u_1 v_1 t} + \frac{\mu_1 |\hat{c}_2|^2}{8u_1 u_2 v_1} e^{2u_2 x + 4u_2 v_2 t}.$$

For example, if $\lambda_1 = 1 + i, \lambda_2 = 1 - i, \hat{c}_1 = \hat{c}_2 = 1$:

$$q_1 = \frac{4(1+i)e^{(3+i)x - 2t} - 4\mu_2 e^{(1+i)x + 2t}}{2e^{4x} - 2\mu_2 e^{2x + 4t} + 2\mu_1 e^{2x - 4t} - \mu_1 \mu_2},$$

$$q_2 = \frac{4(1-i)e^{(3-i)x + 2t} + 4\mu_1 e^{(1-i)x - 2t}}{2e^{4x} - 2\mu_2 e^{2x + 4t} + 2\mu_1 e^{2x - 4t} - \mu_1 \mu_2},$$

$$U = \frac{24 e^{6x}(\mu_1 e^{4t} - \mu_2 e^{-4t}) - 32\mu_1 \mu_2 e^{4x} + 4\mu_1 \mu_2 e^{2x}(\mu_2 e^{4t} - \mu_1 e^{-4t})}{(2e^{4x} - 2\mu_2 e^{2x + 4t} + 2\mu_1 e^{2x - 4t} - \mu_1 \mu_2)^2}.$$

Corollary 3. *Let K be an arbitrary natural number, $l \le K$, the matrix J can be represented in the form $J = \mathrm{diag}(A_1, \ldots, A_l)$, where A_j is the square matrix block of dimension k_j which is given in the form $A_j = i\mu_j I_j^\top I_j$, $j =\in \{1, \ldots, l\}$, where I_j is the vector-string of dimension k_j whose components are equal to 1.*
Then the functions

$$q_1 = \sum_{j=1}^{k_1} \Phi_j,\ q_2 = \sum_{j=1}^{k_2} \Phi_{k_1 + j},\ \ldots, q_l = \sum_{j=1}^{k_l} \Phi_{k_{l-1} + j}, \tag{16}$$

$$U = \left(\begin{vmatrix} \Omega & \varphi^* \\ \varphi & 0 \end{vmatrix} |\Omega|^{-1}\right)_x$$

where Φ_i are defined by formula (11), $k_1 + k_2 + \cdots + k_l = K$, are solutions of an l-components case of system (8) with the diagonal matrix M.

The further proof is similar to that of the previous corollary.

The exact solutions of (2+1)-dimension generalization of Yajima-Oikawa model. Consider the system ((2+1)-dimension generalization of system (7) [9]):

$$\begin{cases} \alpha_2 \mathbf{q}_{t_2} = \mathbf{q}_{xx} + 2U\mathbf{q}, \\ \alpha_2 U_{t_2} = \alpha_1 U_{t_1} + (\mathbf{q}\mathbb{R}^\top)_x, \\ \alpha_2 \mathbb{R}_{t_2} = -\mathbb{R}_{xx} - 2U\mathbb{R}. \end{cases} \tag{17}$$

Introduce the additional reductions of complex conjugation $\alpha_1 = \alpha_2 = i, t_1 = y, t_2 = t, \mathbb{R} = i\bar{\mathbf{q}}M^\top$, where $M = \operatorname{diag}(\mu_1, \ldots, \mu_l) \in \operatorname{Mat}_{l\times l}(\mathbb{R}), \mu_j \in \mathbb{R}$.

System (17) can be represented as

$$\begin{cases} i\mathbf{q}_t = \mathbf{q}_{xx} + 2U\mathbf{q}, \\ U_t = U_y + (\mathbf{q}M\mathbf{q}^*)_x = U_y + \sum_{i=1}^{l} \mu_i (|q_i|^2)_x, \end{cases} \tag{18}$$

and we have (2+1)-dimension generalization of multicomponent Yajima-Oikawa model (8). The operators of system (39) in the Lax representation are of the form:

$$L = i\partial_y - \mathcal{D}^2 - 2U - i\mathbf{q}M\mathcal{D}^{-1}\mathbf{q}^*, \quad A = i\partial_t - \mathcal{D}^2 - 2U.$$

Remark. For MDO W (10) the equality is true (from Proposition 2):

$$W(i\partial_y - \mathcal{D}^2)W^{-1} = \tag{19}$$

$$= i\partial_y - \mathcal{D}^2 - 2(\varphi\Omega^{-1}\psi^\top)_x + \Phi\mathcal{D}^{-1}\left\{(i\psi_y^\top + \psi_{xx}^\top) - \int_{-\infty}^{x}(i\psi_y^\top + \psi_{ss}^\top)\varphi\, ds\,\Psi^\top\right\} +$$

$$+\left\{(i\varphi_y - \varphi_{xx}) - \Phi\int_{-\infty}^{x}\psi^\top(i\varphi_y - \varphi_{ss})\, ds\right\}\mathcal{D}^{-1}\Psi^\top.$$

Consider operators $L_0 = i\partial_y - \mathcal{D}^2, A_0 = i\partial_t - \mathcal{D}^2, \hat{L} = WL_0W^{-1}, \hat{A} = WA_0W^{-1}$.

Theorem 3. *Let:*

a) φ be the solution of the equation $i\varphi_t = \varphi_{xx}$;

b) $i\varphi_y - \varphi_{xx} = -\varphi\Lambda$, where $\Lambda = \operatorname{diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_K^2) = \operatorname{const} \in \operatorname{Mat}_{k\times k}(\mathbb{C})$;

c) $\psi = \bar{\varphi}$.

Then

1) $\hat{L} = i\partial_y - \mathcal{D}^2 - 2(\varphi\Omega^{-1}\varphi^*)_x - \Phi J\mathcal{D}^{-1}\bar{\Phi}^\top$, where $J = C\Lambda - \Lambda^* C$;

2) $\hat{A} = i\partial_t - \mathcal{D}^2 - 2(\varphi\Omega^{-1}\varphi^*)_x$.

Proof. 1) From formula (19), condition b and the properties: $\Phi\int_{-\infty}^{x}\psi^\top\varphi\, ds = \varphi - \Phi C$ and $\int_{-\infty}^{x}\psi^\top\varphi\, ds\Psi^\top = \psi^\top - C\Psi^\top$, it follows that

$$\hat{L} = i\partial_y - \mathcal{D}^2 - 2(\varphi\Omega^{-1}\varphi^*)_x + \Phi\mathcal{D}^{-1}\Lambda^*\psi^\top - \Phi\mathcal{D}^{-1}\Lambda^*\int_{-\infty}^{x}\psi^\top\varphi\, ds\Psi^\top -$$

$$-\varphi\Lambda\mathcal{D}^{-1}\Psi^\top + \Phi\int_{-\infty}^{x}\psi^\top\varphi\, ds\Lambda\mathcal{D}^{-1}\Psi^\top =$$

$$= i\partial_y - \mathcal{D}^2 - 2(\varphi\Omega^{-1}\varphi^*)_x - \Phi(C\Lambda - \Lambda^* C)\mathcal{D}^{-1}\Psi^\top.$$

2) The validity of this item follows from Proposition 2 and condition a. □

For system (18), Corollaries 1–3 of the previous part are valid and formulas (13)–(16) hold.

Let us consider some simplest cases.

1. Consider the scalar case ($K = l = 1$). Let

$$\varphi = \hat{c}e^{2\lambda x - 3i\lambda^2 y - 4i\lambda^2 t}, \quad C = \frac{\mu}{4\mathrm{Re}\lambda\mathrm{Im}\lambda},$$

under the condition $\mathrm{Re}\lambda > 0$ the solutions will have the form:

$$\mathbf{q} = \frac{4\hat{c}\mathrm{Re}\lambda\mathrm{Im}\lambda e^{2\lambda x - 3i\lambda^2 y - 4i\lambda^2 t}}{\mu + |\hat{c}|^2\mathrm{Im}\lambda e^{4\mathrm{Re}\lambda x + 4\mathrm{Re}\lambda\mathrm{Im}\lambda(3y+4t)}},$$

$$U = \frac{16|\hat{c}|^2\mu\mathrm{Re}^2\lambda\mathrm{Im}\lambda e^{4\mathrm{Re}\lambda x + 4\mathrm{Re}\lambda\mathrm{Im}\lambda(3y+4t)}}{(\mu + |\hat{c}|^2\mathrm{Im}\lambda e^{4\mathrm{Re}\lambda x + 4\mathrm{Re}\lambda\mathrm{Im}\lambda(3y+4t)})^2}.$$

Let $\mu = \mathrm{Im}\lambda$. We obtain the functions $\mathbf{q}$ and U in the following form:

$$\mathbf{q} = \frac{A\exp\{i(2vx + (v^2 - u^2)(3y + 4t) + \varphi_0)\}}{\mathrm{ch}\{2u(x + v(3y + 4t) + x_0)\}}, \quad U = \frac{A^2}{\mathrm{ch}^2\{2u(x + v(3y + 4t) + x_0)\}},$$

where $u = \mathrm{Re}\lambda, v = \mathrm{Im}\lambda, A = 4\mathrm{Re}\lambda$, $\varphi_0 = \arg\hat{c}, x_0 = \frac{\ln|\hat{c}|}{\mathrm{Re}\lambda}$. Solution $q(x, y, t)$ is a smooth function localized along surface $x(y, t) = -x_0 - v(3y + 4t)$.

In particular, if $\lambda = 1 + i, \hat{c} = 1$ then the previous formulas can be rewritten as:

$$\mathbf{q} = \frac{4e^{2(1+i)x+6y+8t}}{1 + e^{4(x+3y+4t)}}, \quad U = \frac{16e^{4(x+3y+4t)}}{(1 + e^{4(x+3y+4t)})^2}.$$

One can see that in the case $\mu = \mathrm{Im}\lambda$ we have $|\mathbf{q}|^2 = U$ and $|\mathbf{q}|^2, U \to 0$ as $|x+3y+4t| \to +\infty$.

2. Case

$$K = l = 2, \quad C = \mathrm{diag}(c_1, c_2), \quad \varphi_j = \hat{c}_j e^{2\lambda_j x - 3i\lambda_j^2 y - 4i\lambda_j^2 t}, \quad c_j = \frac{\mu_j}{4\mathrm{Re}\lambda_j\mathrm{Im}\lambda_j}, \quad j \in \{1, 2\}.$$

We denote: $u_j = \mathrm{Re}\lambda_j, v_j = \mathrm{Im}\lambda_j$, $j \in \{1, 2\}$.

If $\mathrm{Re}\lambda_1 > 0$ and $\mathrm{Re}\lambda_2 > 0$ than we have the following formulas

$$q_1 = \frac{\hat{c}_1 e^{2u_1 x + 2u_1 v_1(3y+4t) + (2v_1 x + (u_1^2 - v_1^2)(3y+4t))i}}{|\Omega|} \times$$

$$\times \left(\frac{\mu_2}{4u_2 v_2} + \frac{u_1 - u_2 + i(v_1 - v_2)}{4u_2(u_1 + u_2 + i(v_1 - v_2))} e^{4u_2 x + 4u_2 v_2(3y+4t)}\right),$$

$$q_2 = \frac{\hat{c}_2 e^{2u_2 x + 2u_2 v_1(3y+4t) + (2v_2 x + (u_2^2 - v_2^2)(3y+4t))i}}{|\Omega|} \times$$

$$\times \left(\frac{\mu_1}{4u_1 v_1} + \frac{u_2 - u_1 + i(v_2 - v_1)}{4u_1(u_1 + u_2 + i(v_2 - v_1))} e^{4u_1 x + 4u_1 v_1(3y+4t)}\right),$$

$$U = \left(\frac{P}{|\Omega|}\right)_x,$$

where

$$|\Omega| = \frac{|\hat{c}_1|^2|\hat{c}_2|^2((u_1-u_2)^2+(v_1-v_2)^2)}{16u_1u_2((u_1+u_2)^2+(v_1-v_2)^2)}e^{4(u_1+u_2)x+4(u_1v_1+u_2v_2)(3y+4t)}+$$

$$+\frac{\mu_2|\hat{c}_1|^2}{16u_1u_2v_2}e^{4u_1x+4u_1v_1(3y+4t)}+\frac{\mu_1|\hat{c}_2|^2}{16u_1u_2v_1}e^{4u_2x+4u_2v_2(3y+4t)}+\frac{\mu_1\mu_2}{16u_1u_2v_1v_2},$$

$$P = \left(\frac{u_1+u_2}{4u_1u_2} - \frac{u_1+u_2}{(u_1+u_2)^2+(v_1-v_2)^2}\right)|\hat{c}_1|^2|\hat{c}_2|^2e^{4(u_1+u_2)x+4(u_1v_1+u_2v_2)(3y+4t)}+$$

$$+\frac{\mu_2|\hat{c}_1|^2}{4u_1u_2v_2}e^{4u_1x+4u_1v_1(3y+4t)}+\frac{\mu_1|\hat{c}_2|^2}{4u_1u_2v_1}e^{4u_2x+4u_2v_2(3y+4t)}.$$

The exact solutions of higher equation from hierarchy of Yajima-Oikawa. Let us consider the following operators

$$L = \mathcal{D}^2 + 2U + i\mathbf{q}M\mathcal{D}^{-1}\mathbf{q}^*$$

$$A = \partial_t - \mathcal{D}^3 - 3U\mathcal{D} - \frac{3}{2}U_x - \frac{3}{2}i\mathbf{q}M\mathbf{q}^*$$

The result of equation $[L, A] = 0$ will be the system:

$$\begin{cases}\mathbf{q}_t = \mathbf{q}_{xxx} + 3U\mathbf{q}_x + \frac{3}{2}U_x\mathbf{q} + \frac{3}{2}i\mu\mathbf{q}\mathbf{q}^*\mathbf{q} \\ U_t = \frac{1}{4}U_{xxx} + 3UU_x + \frac{3}{4}i\mu(\mathbf{q}_{xx}\mathbf{q}^* - \mathbf{q}\mathbf{q}^*_{xx})\end{cases}. \qquad (20)$$

This system is a vector generalization of the Melnikov model [13].

Proposition 4.

$$W(\partial_t - \mathcal{D}^3)W^{-1} =$$

$$= \partial_t - \mathcal{D}^3 - 3(\varphi\Omega^{-1}\psi^\top)_x\mathcal{D} - \frac{3}{2}(\varphi_{xx}\Omega^{-1}\psi^\top - \varphi\Omega^{-1}\psi^\top_{xx} + \varphi\Omega^{-1}\psi^\top_x -$$

$$-\varphi\Omega^{-1}\psi^\top\varphi_x\Omega^{-1}\psi^\top) + \Phi\mathcal{D}^{-1}\left\{(\psi^\top_t - \psi^\top_{xxx}) - \int_{-\infty}^{x}(\psi^\top_t - \psi^\top_{sss})\varphi\, ds\,\Psi^\top\right\} +$$

$$+\left\{(\varphi_t - \varphi_{xxx}) - \Phi\int_{-\infty}^{x}\psi^\top(\varphi_t - \varphi_{sss})\, ds\right\}\mathcal{D}^{-1}\Psi^\top.$$

Consider the operators $L_0 = \mathcal{D}^2, A_0 = \partial_t - \mathcal{D}^3, \hat{L} = WL_0W^{-1}, \hat{A} = WA_0W^{-1}$.

Theorem 4. *Let:*

a) φ *be the solution of the equation* $\varphi_t = \varphi_{xxx}$;

b) $\varphi_{xx} = \varphi\Lambda$, *where* $\Lambda = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_K^2) = \mathrm{const} \in \mathrm{Mat}_{K\times K}(\mathbb{C})$;

c) $\psi = \bar{\varphi}$;

d) $C = C^*$.

Then

1) $\hat{L} = \mathcal{D}^2 + 2(\varphi\Omega^{-1}\varphi^*)_x + \Phi J\mathcal{D}^{-1}\bar{\Phi}^\top$, *where* $J = C\Lambda - \Lambda^*C$;

2) $\hat{A} = \partial_t - \mathcal{D}^3 - 3(\varphi\Omega^{-1}\varphi^*)_x\mathcal{D} - \frac{3}{2}(\varphi_{xx}\Omega^{-1}\varphi^* - \varphi\Omega^{-1}\varphi^*_{xx} + \varphi\Omega^{-1}\varphi^*_x - \varphi\Omega^{-1}\varphi^*\varphi_x\Omega^{-1}\varphi^*)$.

Proof. 2) The validity of this item follows from Proposition 4 and conditions a,c. □

For system (20), Corollaries 1-3 are true (see above).

Consider the scalar case ($K = l = 1$). Let $\varphi = \hat{c}e^{\lambda x+\lambda^3 t}$ and $C = \frac{\mu}{4\mathrm{Re}\lambda\mathrm{Im}\lambda}$, under the condition $\mathrm{Re}\lambda > 0$ solutions will be like these:

$$\mathbf{q} = \frac{4\hat{c}\mathrm{Re}\lambda\mathrm{Im}\lambda e^{\lambda x+\lambda^3 t}}{\mu + 2|\hat{c}|^2\mathrm{Im}\lambda e^{2\mathrm{Re}\lambda x+(\mathrm{Re}^3\lambda-3\mathrm{Re}\lambda\mathrm{Im}^2\lambda)t}}, \quad U = \frac{8|\hat{c}|^2\mu\mathrm{Re}^2\lambda\mathrm{Im}\lambda e^{2\mathrm{Re}\lambda x+(\mathrm{Re}^3\lambda-3\mathrm{Re}\lambda\mathrm{Im}^2\lambda)t}}{(\mu + 2|\hat{c}|^2\mathrm{Im}\lambda e^{2\mathrm{Re}\lambda x+(\mathrm{Re}^3\lambda-3\mathrm{Re}\lambda\mathrm{Im}^2\lambda)t})^2}.$$

In particular, if $\lambda = 1 + i, \hat{c} = 1, \mu = 2$, then the previous formulas can be rewritten as:

$$\mathbf{q} = \frac{2e^{(1+i)x-2(1-i)t}}{1+e^{2(x-2t)}}, \quad U = \frac{4e^{2(x-2t)}}{(1+e^{2(x-2t)})^2}.$$

One can see that $|\mathbf{q}|^2, U \to 0$ as $|x-2t| \to +\infty$ and $|\mathbf{q}|^2 = U$ under the condition $\mu = 2\mathrm{Im}\lambda$.

A method of finding of the exact solution for non-linear evolutional equations (8), (20) can be generalized for the matrix case of hierarchy (6). The result of this investigation will be published.

REFERENCES

1. Date E., Jimbo M., Kashiwara M., Miwa T. Nonlinear ntegrable systems: classical theory and quantum theory. Ed. Jimbo M. and Miwa T. – Singapore: World Scientific, 1983. – P. 39–119.
2. Dickey L.A. *Soliton equations and Hamiltonian systems* // Advanced Series in Mathematical Physics. – 1991. – V.12. – 310 p.
3. Sidorenko Yu., Strampp W. *Symmetry constraints of the KP-hierarchy* // Inverse Problems. – 1991. – V.7 – P.L-37–L-43.
4. Konopelchenko B., Sidorenko Yu., Strampp W. *(1+1)-dimensional integrable systems as symmetry constraints of (2+1)-dimensional systems* // Phys. Lett. A. – 1991. – V.151. – P.17–21.
5. Sidorenko Yu. *KP-hierachy and (1+1)-dimensional multicomponent integrable systems* // Ukr. math. journ. – 1993. – V.25, №1. – P.91–104.
6. Sidorenko Yu., Strampp W. *Multicomponents integrable reductions in Kadomtsev-Petviashvilli hierarchy* // J. Math. Phys. – 1993. – V.34, №.4. – P.1429–1446.
7. Oevel W., Strampp W. *Constrained KP-hierarchy and bi-Hamiltonian structures* // Commun. Math. Phys. – 1993. – V.157. – P.51–81.
8. Oevel W., Sidorenko Yu., Strampp W. *Hamiltonian structures of the Melnicov system and its reductions* // Inverse Problems. – 1993. – V.9. – P.737–747.
9. Samoilenko A.M., Samoilenko V.H., Sidorenko Yu.M. *Hierarchy of equations Kadomtsev-Petviashvili with nonlocal constraints: Multidimensional generalizations and exact solutions of reduced systems* // / Ukr. math. jour. – 1999. – V.51, №1. – P.78–97.
10. Yajima N., Oikawa M. *Formation and interaction of Sonic-Langmur solitons: inverse scattering method* // Progress Theoret. Phys. – 1976. – V.56, №6. – P.1719–1739.
11. Сидоренко Ю.М. *Метод інтегрування рівнянь Лакса з нелокальними редукціями* // Доп. НАН України. – 1999. – №9. – P.33–36.
12. Takhtadzhan L.A., Faddeev L.D. Hamiltonian method in the theory of solitons. Springer, Berlin. – 1987.

13. Melnikov V.K. *On equations integrable by the inverse scattering method.* – Preprint No. P2-85-958. Joint institute for nuclear research, Dubna. – 1985.

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