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I. E. Chyzhykov*

ON MINIMUM MODULUS OF AN ENTIRE FUNCTION OF ZERO ORDER

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For a non-negative function $\psi \in C^2[1,+\infty)$ we define $\psi_j(r) = \frac{d^j\psi(r)}{(d\ln r)^j}$, $j \in \{1,2\}$. Suppose that $\psi_2(r) \to +\infty$ and $\psi_2(2r) \sim \psi_2(r)$ $(r \to +\infty)$. Then there exist an entire function $g(z)$ of zero order with the following properties: a) $n(r,0,g) \sim \psi_1(r)$, $\ln M(r,g) \sim \psi(r)$ $(r \to +\infty)$; b) $\ln M(r,g) - \ln \mu(r,g) \geq (1+o(1))\frac{\pi^2}{2}\psi_2(r)$ $(r \to +\infty)$, where $M(r,g) = \max\{|g(z)| : |z| = r\}$, $\mu(r,g) = \min\{|g(z)| : |z| = r\}$, $n(r,0,g)$ is the zero counting functions of g.

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Для неотрицательной функции $\psi \in C^2[1,+\infty)$ определим $\psi_j(r) = \frac{d^j\psi(r)}{(d\ln r)^j}$, $j \in \{1,2\}$. Предположим, что $\psi_2(r) \to +\infty$ and $\psi_2(2r) \sim \psi_2(r)$ $(r \to +\infty)$. Тогда существует целая функция $g(z)$ нулевого порядка со следующими свойствами: a) $n(r,0,g) \sim \psi_1(r)$, $\ln M(r,g) \sim \psi(r)$ $(r \to +\infty)$; b) $\ln M(r,g) - \ln \mu(r,g) \geq (1+o(1))\frac{\pi^2}{2}\psi_2(r)$ $(r \to +\infty)$, где $M(r,g) = \max\{|g(z)| : |z| = r\}$, $\mu(r,g) = \min\{|g(z)| : |z| = r\}$, $n(r,0,g)$ — считающая функция нулей g.

1. Introduction and result. Let f be an entire function, $M(r,f) = \max\{|f(z)| : |z| = r\}$, $\mu(r,f) = \min\{|f(z)| : |z| = r\}$, $r > 0$.

The well-known $\cos\pi\rho$-theorem (see e.g. [1, Chap.6]) implies that if $\rho[f] = \varlimsup\limits_{r\to+\infty} \frac{\ln\ln M(r,f)}{\ln r} = 0$, then $\ln\mu(r_n,f) \sim \ln M(r_n,f)$ on a sequence $r_n \to +\infty$ $(n \to +\infty)$. However, the last relation can be improved.

Let $u(z)$ be a subharmonic in $\mathbb{C}$ function. We denote $A(r,u) = \inf\{u(z) : |z| = r\}$, $B(r,u) = \max\{u(z) : |z| = r\}$, $r > 0$. Let Ψ^* be the class of twice continuously differentiable functions $\psi\colon [1,+\infty) \to \mathbb{R}_+$ such that $\psi_2(2r) \sim \psi_2(r)$, $\psi_1(r) \uparrow +\infty$ as $r \uparrow +\infty$, here $\psi_j(r) = \frac{d^j\psi(r)}{(d\ln r)^j}$, $j \in \{1,2\}$. It is clear that $\psi_1(r) \sim \psi_1(2r)$ and $\psi(r) \sim \psi(2r)$ as $r \to +\infty$ for $\psi \in \Psi^*$.

In [2, Th.1] the author has proved the following theorem.

Theorem A. *Let $u(z)$ be a subharmonic function in $\mathbb{C}$. If there exists $\psi \in \Psi^*$ such that $\varliminf\limits_{r\to+\infty} B(r,u)/\psi(r) \leq 1$, then $\forall\varepsilon > 0$ the inequality*

$$A(r,u) \geq B(r,u) - (1+\varepsilon)\frac{\pi^2}{2}\psi_2(r) \tag{1}$$

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holds outside a set E_ε of values r satisfying

$$\varliminf_{r\to+\infty} \frac{1}{\psi_1(r)} \int_{E_\varepsilon\cap[1,r]} d\psi_1(t) \le \frac{1}{1+\varepsilon}. \tag{2}$$

Remark 1. In [2] the existence of a function ψ from a slightly wider class Ψ is required.

In 1962 P. Barry proved [3] that inequality (1) holds on a sequence of values $r_n \to +\infty$ $(n \to +\infty)$ provided that $B(r,u) \le \psi(r)$ $(r \ge r_0)$ with a function ψ satisfying all conditions of the definition of the class Ψ^* except $\psi_1(r) \uparrow +\infty$ $(r \to +\infty)$. P. Fenton proved [4] Theorem A in the case when it is possible to choose $\psi(r) = \sigma(\ln r)^{p+1}$, $p > 1$, $0 < \sigma < +\infty$.

Putting $u(z) = \ln|f(z)|$, where $f(z)$ is an entire function of zero lower order satisfying $\varliminf_{r\to+\infty} \frac{\ln M(r,f)}{\psi(r)} \le 1$ with $\psi \in \Psi^*$, from Theorem A we obtain

$$(\forall\varepsilon > 0):\ \ln\mu(r,f) \ge \ln M(r,f) - (1+\varepsilon)\frac{\pi^2}{2}\psi_2(r), \quad r \notin E_\varepsilon, \tag{3}$$

where E_ε satisfies (2).

However, it turned out that inequality (3) could be improved when $\psi(r) = O(\ln^2 r)$ $(r \to +\infty)$.

In the case $\psi(r) = \sigma(\log^+ r)^2$ $(r \ge 0)$ A. A. Gol'dberg [5] and P. Fenton [6, 7] showed that

$$\varlimsup_{r\to+\infty} \frac{\mu(r,f)}{M(r,f)} \ge C_1(\sigma) = \Big(\prod_{n=1}^{+\infty} \frac{1-q^{2n-1}}{1+q^{2n-1}}\Big)^2,$$

where $q = e^{-\frac{1}{4\sigma}}$, $C_1(0) = 1$, for every entire function [7] such that $\varliminf_{r\to+\infty} \frac{B(r,\ln|f|)}{\ln^2 r} = \sigma < +\infty$ (for every meromorphic function [5] such that $\varliminf_{r\to+\infty} N(r,0,\infty,f)\ln^{-2} r \le \sigma < +\infty$), here $N(r,0,\infty,f) = N(r,0,f) + N(r,\infty,f)$ is Nevanlinna's counting function of zeros and poles.

Moreover, in 1982 P. Barry [8] established a sharp on a sequences lower estimate of $\mu(r,f)/M(r,f)$ for entire functions f satisfying $\ln M(r,f) = O(\ln^{p+1} r)$, $r \to +\infty$, $0 < p < 1$. Those estimates are different from those that can be obtained from (1).

Nevertheless, in this paper it will be proved that inequality (3) is unimprovable in the class of entire functions that grow faster that $\ln^2 r$ in some sense.

Theorem 1. *Suppose that $\psi \in \Psi^*$ satisfy $\psi_2(r) \to +\infty$ $(r \to +\infty)$. Then there exists an entire function $g(z)$ of zero order with the following properties:*

a) $n(r,0,g) \sim \psi_1(r)$, $\ln M(r,g) \sim \psi(r)$ $(r \to +\infty)$;

b) $\ln M(r,g) - \ln\mu(r,g) \ge (1+o(1))\frac{\pi^2}{2}\psi_2(r)$ $(r \to +\infty)$.

Remark 2. Clearly, the last inequality holds in some neighbourhood of values r_n such that there is a zero of $g(z)$ laying on the circle $\{z : |z| = r_n\}$. In [2, Th.2] the author constructed an entire function of zero order $f(z) = \prod_{n=1}^{+\infty}\Big(1 + \frac{z}{x_n}\Big)$ such that

$$\ln M(r,f) \sim N(r,0,f) \sim \frac{\sigma}{p+1}\ln^{p+1} r, \quad r \to +\infty,$$

$p, \sigma \in (0, +\infty)$ and extremal asymptotic equalities between $\mu(r, f)$ and $M(r, f)$ attain on the sequence $r_n = \sqrt{x_n x_{n+1}}$ for any $p > 0$.

To prove Theorem 1 we use a recent result of Yu. Lyubarskii and Eu. Malinnikova [9] on approximation of a subharmonic function by the logarithm of modulus of an entire function.

2. Proof of Theorem 1. Without loss of generality we may assume that $\psi(1) = 0$. Thus $\psi_1(r) = \int_1^r \frac{\psi_2(t)}{t} dt$, where $\psi_2(t)$ is slowly varying and positive function on $[1, +\infty)$ ($\psi_1(t)$ is an increasing function). Therefore, for any $\varepsilon \in (0, 1)$

$$\psi_1(r) \geq \int_{\varepsilon r}^{r} \frac{\psi_2(t)}{t} dt = (1 + o(1))\psi_2(r) \int_{\varepsilon r}^{r} \frac{dt}{t} \geq \frac{1}{2}\psi_2(r) \ln \frac{1}{\varepsilon}, \quad r \to +\infty.$$

Hence, $\psi_2(r) = o(\psi_1(r))$ $(r \to +\infty)$, and similarly, one can deduce

$$\psi_1(r) = o(\psi(r)), \quad r \to +\infty. \tag{4}$$

We define a subharmonic in $\mathbb{C}$ function v by the equality

$$v(z) = \int_0^{+\infty} \ln\Big|1 - \frac{z}{t}\Big| \, d\psi_1(t).$$

Since the Riesz measure of a subharmonic function is unique [10, Th.2], [11, Sec. 3.0], this representation implies that the Riesz measure μ_v of the function v is supported on the ray $[1, +\infty)$ and $\mu_v(\{\zeta : |\zeta| \leq t\}) = \psi_1(t)$.

Clearly,

$$B(r, v) = v(-r) = \int_0^{+\infty} \ln\Big(1 + \frac{r}{t}\Big) \, d\psi_1(t), \quad A(r, v) = v(r) = \int_0^{+\infty} \ln\Big|1 - \frac{r}{t}\Big| \, d\psi_1(t).$$

Therefore,

$$\begin{gathered} B(r, v) - A(r, v) = \int_0^{+\infty} \ln\Big|\frac{r + t}{r - t}\Big| \, d\psi_1(t) = \\ = \int_0^{+\infty} \ln\Big|\frac{r + t}{r - t}\Big| \frac{\psi_2(t)}{t} \, dt = \int_0^{+\infty} \ln\Big|\frac{1 + \tau}{1 - \tau}\Big| \frac{\psi_2(r\tau)}{\tau} \, d\tau. \end{gathered} \tag{5}$$

Since $\int_0^{+\infty} \ln\Big|\frac{1+\tau}{1-\tau}\Big| \frac{d\tau}{\tau} = \frac{\pi^2}{2}$, for every $\varepsilon > 0$ there exists $b \in (0, 1)$ such that $\int_b^{1/b} \ln\Big|\frac{1+\tau}{1-\tau}\Big| \frac{d\tau}{\tau} > \frac{\pi^2}{2} - \varepsilon$. Using the fact that $\psi_2(r)$slowly varying we get

$$\begin{gathered} \int_0^{+\infty} \ln\Big|\frac{1 + \tau}{1 - \tau}\Big| \frac{\psi_2(rt)}{\tau} \, d\tau > \int_b^{1/b} \ln\Big|\frac{1 + \tau}{1 - \tau}\Big| \frac{\psi_2(rt)}{\tau} \, d\tau = \\ = (1 + o(1))\psi_2(r) \int_0^{+\infty} \ln\Big|\frac{1 + \tau}{1 - \tau}\Big| \frac{d\tau}{\tau} \geq \Big(\frac{\pi^2}{2} - 2\varepsilon\Big)\psi_2(r), \quad r \to +\infty. \end{gathered}$$

From the last inequality, (5), and arbitrarity of $\varepsilon > 0$ we deduce

$$B(r, v) - A(r, v) \geq \Big(\frac{\pi^2}{2} + o(1)\Big)\psi_2(r), \quad r \to +\infty. \tag{6}$$

Define the increasing sequence (r_n) by the equalities $\psi_1(r_n) = 2n$, $n \in \mathbb{N}$, $r_0 = 1$. Then, by the Lagrange theorem $2 = \psi_1(r_{n+1}) - \psi_1(r_n) = \psi_1'(\xi_n)(r_{n+1} - r_n)$ for some $\xi_n \in [r_n, r_{n+1}]$. Hence

$$0 \le \frac{r_{n+1} - r_n}{r_{n+1}} \le \frac{r_{n+1} - r_n}{\xi_n} = \frac{2}{\psi_1'(\xi_n)\xi_n} = \frac{2}{\psi_2(\xi_n)} = o(1), \quad n \to +\infty.$$

Thus, $1 - \frac{r_n}{r_{n+1}} \to 0$, i.e. $r_n \sim r_{n+1}$ $(n \to +\infty)$.

In order to use the approximation result from [9] we need to constructa so called *partition of slow variation* for μ_v.

Put $Q_m = \left\{z : r_{m-1} \le |z| \le r_m, |\arg z| \le \frac{1}{\psi_2(r_m)}\right\}$, $m \in \mathbb{N}$, and let $\mu^{(m)} = \mu_v\big|_{Q_m}$ be the restriction of the measure on Q_m, actually $\mu^{(m)} = \mu\big|_{[r_{m-1}, r_m]}$.

Let us estimate the diameter of Q_m. Since $\psi_2(r_m) \to +\infty$ $(m \to +\infty)$, for any $\zeta \in Q_m$ using the definition of Q_m, the asymptotic equality $r_m \sim r_{m-1}$ $(m \to +\infty)$, and slow variation of ψ_2 we have

$$|\operatorname{Im} \zeta| = |\zeta| \operatorname{tg}(\arg \zeta) = (1 + o(1)) r_m |\arg \zeta| \le (1 + o(1)) \frac{r_m}{\psi_2(r_m)}, \quad m \to +\infty.$$

On the other hand,

$$\max\{|\mathrm{Re}\, \zeta_1 - \mathrm{Re}\, \zeta_2| : \zeta_1, \zeta_2 \in Q_m\} = (1 + o(1))(r_m - r_{m-1}) =$$
$$= (1 + o(1)) r_m \frac{2}{\psi_2(\xi_m)} = (1 + o(1)) \frac{2r_m}{\psi_2(r_m)}, \quad m \to +\infty.$$

Therefore,

$$(1 + o(1)) \frac{2r_m}{\psi_2(r_m)} \le \operatorname{diam} Q_m \le (1 + o(1)) 2\sqrt{2} \frac{r_m}{\psi_2(r_m)}, \quad m \to +\infty. \tag{7}$$

The partition $\big((Q_m), (\mu^{(m)})\big)_{m \in \mathbb{N}}$ of the measure μ has the following properties

i) $\operatorname{supp} \mu^{(m)} \subset Q_m$, $\mu^{(m)}(Q_m) = 2$;

ii) $\mu - \sum_m \mu^{(m)} \equiv 0$;

iii) every point $z \in \mathbb{C}$ belongs to at most two different Q_ms;

iv) for every $m \in \mathbb{N}$ the set $\ln Q_m = \{\zeta = \xi + i\eta : \zeta = \ln z, z \in Q_m\}$ is a rectangle with sides parallel to the coordinate axes, the ratio of the length of the sides lies between two positive constants independent of m;

v) $\frac{1}{2} b(\operatorname{dist}(0, Q_m)) \le \operatorname{diam} Q_m = 2b(\operatorname{dist}(0, Q_m))$ $(m \to +\infty)$, where $b(t) = \frac{2t}{\psi_2(t)}$.

Properties i)–iii) follow immediately from the definitions of r_m and Q_m, v) follows from (7). It remains to prove iv). From the definition of Q_m we obtain

$$\ln Q_m = \left\{\xi + i\eta : \ln r_{m-1} \le \xi \le \ln r_m, |\eta| \le \frac{1}{\psi_2(r_m)}\right\}.$$

But, by (7),

$$\ln r_m - \ln r_{m-1} = \ln\Big(1 + \frac{r_m - r_{m-1}}{r_{m-1}}\Big) \sim \frac{r_m - r_{m-1}}{r_m} \sim \frac{2}{\psi_2(r_m)}, \quad m \to +\infty,$$

and iv) follows. Thus, by Definition 1 [9, Sec. 3] the measure μ_v admits a partition of slow variation $\big((Q_m), (\mu^{(m)})\big)_{m \in \mathbb{N}}$.

We need one more definition [9, Def. 2, Sec. 3].

Definition. Given a function $b(t) > 0$ such that $b(t_1) \asymp b(t_2)$ for $\frac{1}{2} < \frac{t_1}{t_2} < 2$ and $b(t) = o(t)$ $(t \to +\infty)$, we say that a measure μ is *locally regular* with respect to b if it satisfies

$$\int_0^{b(|z|)} \frac{\mu(\{\zeta : |\zeta - z| < s\})}{s} ds = O(1), \quad z \in \mathbb{C},\ |z| > r_0.$$

Theorem B. (Theorem 3′ [9]) *Let u be a subharmonic function on $\mathbb{C}$, μ be its Riesz measure, and $\mu(\mathbb{C}) = \infty$. Let μ admit a partition of slow variation and μ be locally regular with respect to the corresponding function b. Then there exists an entire function f such that, for each $\varepsilon > 0$,*

$$u(z) - \ln|f(z)| = O(1), \quad z \notin E_\varepsilon,\ z \to \infty,$$

with $E_\varepsilon = \{z : \operatorname{dist}(z, Z_f) \le \varepsilon b(|z|)\}$, where Z_f is the zero set of f. In addition, for some $C > 0$

$$\ln|f(z)| < u(z) + C \qquad \text{for all } \ z \in \mathbb{C}.$$

Remark 3. From the proof of Theorem 3′ it follows that zeros (a_m) of the function $f(z)$ can be ordered such that

$$\operatorname{dist}(a_j, Q_m) \le C_2 \operatorname{diam} Q_m, \quad j \in \{2m-1, 2m\}.$$

Let us prove that μ_v is locally regular with respect to $b(t) = 2/\psi_2(t)$. For $|z| = r$ and $0 \le s \le b(r) = o(r)$ $(r \to +\infty)$ we have

$$\mu_v(\{\zeta : |\zeta - z| < s\}) \le \psi_1(r+s) - \psi_1(r-s) = \frac{\psi_2((1+o(1))r)}{(1+o(1))r} 2s = (1+o(1)) \frac{2s\psi_2(r)}{r}.$$

Hence,

$$\int_0^{b(|z|)} \frac{\mu(\{\zeta : |\zeta - z| < s\})}{s}\, ds \le (2+o(1)) \frac{b(r)}{r} \psi_2(r) = 4 + o(1), \quad r \to +\infty.$$

Consequently, μ_v satisfy the conditions of Theorem B.

By Theorem B there exists an entire function $g(z)$ such that for every $\varepsilon > 0$

$$v(z) - \ln|g(z)| = O(1), \quad z \notin F_\varepsilon,\ z \to \infty$$

where F_ε is the ε-neighbourhood of the zero set of g, and

$$\ln|g(z)| < v(z) + C, \quad z \in \mathbb{C}, \tag{8}$$

for some positive constant C.

Note that according to Remark 3 zeros c_j of the function $g(z)$ satisfy

$$\operatorname{dist}(c_j, Q_m) \le C_2 \operatorname{diam} Q_m, \quad j \in \{2m-1, 2m\}. \tag{9}$$

Since $r_m \sim r_{m+1}$, $\operatorname{diam} Q_m = o(r_m)$ $(m \to +\infty)$, for all $\zeta \in Q_m$ we have $\zeta \sim r_m$ $(m \to +\infty)$. By (9) $c_{2m-1} \sim c_{2m} \sim r_m$ $(m \to +\infty)$, and consequently $\arg c_j \to 0$ $(j \to +\infty)$.

Therefore, $F_\varepsilon \subset \{z : |z| \le R_\varepsilon\} \cup \{z : |\arg z| < \varepsilon\}$ for some $R_\varepsilon > 0$. This implies $B(r,v) = \ln M(r,g) + O(1)$ $(r \to +\infty)$. In fact, for $r > R_\varepsilon$ we have $-r \notin F_\varepsilon$, and so

$$\ln M(r,g) \ge \ln |g(-r)| = v(-r) + O(1) = B(r,v) + O(1).$$

On the other hand, if $\ln M(r,g) = \ln |g(re^{i\theta_r})|$, then, by (8),

$$\ln |g(re^{i\theta_r})| \le v(re^{i\theta_r}) + C \le B(r,v) + C.$$

Hence, $B(r,v) = \ln M(r,g) + O(1)$.

From (8) it follows that $\ln \mu(r,g) \le A(r,v) + C$. Thus, using also (6) we obtain

$$\ln M(r,g) - \ln \mu(r,g) \ge B(r,v) + O(1) - A(r,v) = (1+o(1))\frac{\pi^2}{2}\psi_2(r), \quad r \to +\infty.$$

Hence, assertion b) of Theorem 1 is proved.

Estimates (9) imply that $n(r,0,g) \sim \psi_1(r)$ $(r \to +\infty)$. By [11, Sec. 4.2] and (4),

$$\ln M(r,g) \le r\int_r^{+\infty} \frac{N(t,0,g)}{t^2}dt = (1+o(1))r\int_r^{+\infty} \frac{\psi(t)}{t^2}dt =$$

$$= (1+o(1))\Big(\psi(r) + r\int_r^{+\infty} \frac{\psi_1(t)}{t^2}dt\Big) = (1+o(1))\Big(\psi(r) + o\Big(r\int_r^{+\infty} \frac{\psi(t)}{t^2}dt\Big)\Big).$$

This yields $r\int_r^{+\infty} \psi(t)t^{-2}\,dt \sim \psi(r)$ $(r \to +\infty)$, and consequently $\ln M(r,g) \le (1+o(1))\psi(r)$ $(r \to +\infty)$. Since $\psi(r) \sim N(r,0,g) \le \ln M(r,g)$, we obtain $\ln M(r,g) \sim \psi(r)$ $(r \to +\infty)$. The theorem is proved.

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Faculty of Mechanics and Mathematics, Lviv National University

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