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# ON LOCALLY FINITE $p$-GROUPS WITH NON-DEDEKIND NON-CYCLIC SUBGROUP NORM

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We study infinite locally finite $p$-groups whose non-cyclic norm is non-Dedekind. It is proved that such groups are finite extensions of quasicyclic subgroups. A complete description of infinite locally finite $p$-groups with non-Dedekind non-cyclic norm is given.

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Изучаются бесконечные локально конечные $p$-группы, нециклическая норма котрорых не дедекиндова. Доказано, что такие группы являются конечным расширением квазициклических групп. Получено полная характеризация бесконечных локально конечных $p$-групп с недедекиндовой нециклической нормой.

Suppose $G$ is a group and $\Sigma \neq \varnothing$ is a system of all subgroups of $G$ having some fixed group-theoretic property. The maximal subgroup of the group $G$ normalizing every subgroup of $\Sigma$ is called the $\Sigma$-norm of this group. The $\Sigma$-norm of a group is its characteristic subgroup, and it includes the center of the group and coincides with the intersection of the normalizers of all subgroups from $\Sigma$.

If the $\Sigma$-norm contains at least one subgroup of the system $\Sigma$, then all subgroups with such a property are invariant in it. Algebraists of various countries, and, especially S. N. Černikov and his followers were active researchers of such groups. So, if the $\Sigma$-norm coincides with the group $G$, then all the subgroups of $\Sigma$ are invariant in $G$. That is why it is natural to consider more general situation when the $\Sigma$-norm is a proper subgroup of $G$.

In the case when the system $\Sigma$ consists of all subgroups of the group $G$, the $\Sigma$-norm according to [1,2] is called the *norm* of the group and is denoted by $N(G)$. The norm of a group is an Abelian or Hamiltonian subgroup and is included in any other $\Sigma$-norm. So the notion of the norm of the group can be generalized by narrowing the system $\Sigma$. Among such generalizations there are: the $A$-norm (i.e. the intersection of the normalizers of all the maximal Abelian subgroups [3]), the subnormal norm (or the Wielandt subgroup), the intersection of the normalizers of all the subnormal subgroups of the group (see [4,5]), etc. see, for example, [6,7].

We continue the research started in [8], where the conception of the non-cyclic group norm had been noted. Following [8] under the non-cyclic group norm $N_G$ of a group $G$ we

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understand the $\Sigma$-norm of $G$ for the system $\Sigma$ of all non-cyclic subgroups of the group. If $N_G$ is non-cyclic, then all its non-cyclic subgroups are invariant in it. Non-Abelian groups with such features were studied in [9–11] and were called $\overline{H}$-groups (or $\overline{H_p}$-groups if they are $p$-groups).

In this paper we investigate infinite locally finite $p$-groups having non-Dedekind non-cyclic norm. It is proved that such groups are finite extensions of quasicyclic subgroups and their constructive description is obtained.

**Lemma 1.** *Let $H$ be a subgroup of a group $G$. If $N_G$ is the non-cyclic norm of $G$, then $N_G \cap H \leq N_H$.*

*Proof.* The proof of Lemma 1 is obvious. □

**Lemma 2.** *Let $H$ be an invariant non-cyclic subgroup of the group $G$ and $N_G$ is the non-cyclic norm of this group. Then $\overline{N_G} = N_G/H \leq N(\overline{G}) = N(G/H)$, where $N(\overline{G})$ is the norm of the group $\overline{G} = G/H$.*

*Proof.* Suppose $\overline{M} \leq \overline{G}$. Then the full pre-image $M$ of the group $\overline{M}$ is a non-cyclic subgroup. So $N_G \subseteq N_G(M)$, and therefore $\overline{N_G} \subseteq N_{\overline{G}}(\overline{M})$. From definition of the group norm and arbitrary choice of the subgroup $\overline{M}$ it follows that $\overline{N_G} \leq N(\overline{G})$, as required. □

**Theorem 1.** *Any infinite locally finite $p$-group $G$ having non-Dedekind non-cyclic norm $N_G$ is a finite extension of a quasicyclic subgroup $A \subset G$, and $N_G \subseteq C_G(A)$.*

*Proof.* Let a $p$-group $G$ and its non-cyclic norm $N_G$ satisfy the given condition. If $G = N_G$, then $G$ is a non-Hamiltonian $\overline{H_p}$-group and our theorem follows from Theorems 1.2–1.3 [11]. So further we may assume that $G \neq N_G$.

Suppose that the group $G$ does not satisfy the minimal condition for subgroups. Then it includes an infinite elementary Abelian subgroup $A$. Since the subgroup $N_G$ is non-Dedekind, Theorems 1.2-1.3 [11] imply $|A \cap N_G| < \infty$. Let us consider the group $G_1 = N_G A = N_G A_2$, where $A = A_1 \times A_2, A_1 = N_G \cap A, A_2 \cap N_G = E$. So if $|A_2| = \infty$, then $A_2$ is non-cyclic and therefore $A_2 \triangleleft G_1$. Hence the group $G_1 = N_G \times A_2$ is central-by-finite and by Theorem 3 [8] it is a non-Hamiltonian $\overline{H_p}$-group. It contradicts to the description of such groups (see [11]). So, $G$ is a group with minimal condition for subgroups, and by [12, Theorem 4.1] $G$ is a Černikov group.

Assume that $G$ contains a direct product $P$ of two quasicyclic subgroups. Since $[P : P \cap N_G] = \infty$ we see that $PN_G/N_G \cong P/P \cap N_G$ is a divisible Abelian group. Then by [12, Theorem 1.16] the group $G_2 = N_G P$ is central-by-finite, and by Theorem 3 [8], it is a non-Hamiltonian $\overline{H_p}$-group. However, that is impossible according to [11]. Thus, $G$ is a finite extension of the quasicyclic subgroup $A$.

Now we show that $N_G \subseteq C_G(A)$. If $|N_G| = \infty$, then [11, Theorems 1.2-1.3] implies $A \subseteq Z(N_G)$. Suppose that $|N_G| < \infty$. Since $N_G \triangleleft G$, we obtain $[G : C_G(N_G)] < \infty$ and therefore $A \subseteq C_G(N_G)$. So in every case $N_G \subseteq C_G(A)$, and this completes the proof. □

Using Theorem 1 and Theorem 3[8], we can easily prove the following results.

**Corollary 1.** *Any infinite locally finite $p$-group $G$ whose non-cyclic norm $N_G$ is infinite and non-Dedekind, is a $\overline{H_p}$-group.*

**Corollary 2.** *If the non-cyclic norm $N_G$ of an infinite locally finite $p$-group $G$ is non-Dedekind and differs from $G$, then $N_G$ is a finite group.*

**Theorem 2.** *Let $G$ be an infinite locally finite $p$-group ($p \neq 2$) with non-Abelian non-cyclic norm. Then $G$ is $\overline{H_p}$-group.*

*Proof.* Assume that $G \neq N_G$. Then $|N_G| < \infty$ by Corollary 2. On the other hand, Theorem 1 implies $G = AH$, where $A$ is a quasicyclic $p$-group and $|H| < \infty$. Using [12, Corollary 1.13] we see that $A \subseteq Z(G)$ and as a consequence $A \subseteq N_G$. Thus $|N_G| = \infty$, a contradiction. The proposition is proved. □

**Corollary 3.** *Let $G$ be an infinite locally finite $p$-group ($p \neq 2$). If $G$ has a non-invariant non-cyclic subgroup $H$, then its non-cyclic norm $N_G$ is Abelian.*

Let us recall that the low layer $\omega(G)$ of the group $G$ is the subgroup of $G$ generated by all the elements of the prime order of $G$.

Next we need the following auxiliary result.

**Lemma 3.** *If a locally finite $p$-group $G$ has the non-Dedekind non-cyclic norm $N_G$, whose law layer $\omega(N_G)$ is a central non-cyclic subgroup of $G$, then $\omega(N_G) = \omega(G)$.*

*Proof.* Suppose that this is not true and there exists an involution $x$, not belonging to $\omega(N_G) = \langle a_1 \rangle \times \langle a_2 \rangle$. Then $\langle x \rangle = \langle x, a_1 \rangle \cap \langle x, a_2 \rangle \cong G_1 = \langle x \rangle N_G$. It follows $x \in Z(G_1) \subseteq N_{G_1}$ and $|\omega(N_{G_1})| = 8$ which contradicts to [11, Lemma 1.2]. □

**Theorem 3.** *The non-cyclic norm $N_G$ of an infinite locally finite 2-group $G$ is non-Dedekind if and only if $G$ is a group of one of the following types:*

1) $G = (A \times \langle b \rangle)\lambda\langle c \rangle$, *where $A$ is a quasicyclic 2-group,* $|b| = |c| = 2, [A, \langle c \rangle] = 1, [b, c] = a_1 \in A, |a_1| = 2; N_G = G$;
2) $G = A \times H$, *where $A$ is a quasicyclic 2-group,* $H = \langle h_1, h_2 \rangle$, $|h_1| = |h_2| = 4$, ${h_1}^2 = {h_2}^2 = [h_1, h_2]$; $N_G = G$;
3) $G = (A \times \langle b \rangle)\lambda\langle c \rangle\lambda\langle d \rangle$, *$A$ is a quasicyclic 2-group,* $|b| = |c| = |d| = 2$, $[A, \langle c \rangle] = 1$, $d^{-1}ad = a^{-1}$ *for each element* $a \in A, [b, c] = [d, b] = [d, c] = a_1 \in A, |a_1| = 2; N_G = ((\langle a \rangle \times \langle b \rangle)\lambda\langle c \rangle, a \in A, |a| = 4$;
4) $G = (A \times H)\langle d \rangle$, *$A$ is a quasicyclic 2-group,* $d^2 = a_1 \in A$, $|a_1| = 2$, $d^{-1}ad = a^{-1}$ *for each element* $a \in A$, $H = \langle h_1, h_2 \rangle$, $|h_1| = |h_2| = 4$, ${h_1}^2 = {h_2}^2 = [h_1, h_2]$; $N_G = \langle h_2 \rangle\lambda\langle ah_1 \rangle$, $a \in A$, $|a| = 4$.

*Proof. Necessity.* Let $G$ be a group under consideration and $N_G$ its non-cyclic norm. If $|N_G| = \infty$ then it follows from Corollary 1 and the description of the $\overline{H_p}$-groups (see [11]) that $G = N_G$ and $G$ is a group of type 1) or 2) of the Theorem.

Suppose $|N_G| < \infty$. Then by Theorem 1 we get that $G$ is a finite extension of a quasicyclic 2-group $A$ and, moreover, $N_G \subseteq C = C_G(A)$. Since $A \not\subseteq Z(G), [G : C] = 2$ and $G = C\langle d \rangle$, where $d^2 \in C$. So the element $d$ induces a nontrivial automorphism of the order 2 on $A$ and $d^{-1}ad = a^{-1}$ for each element $a \in A$

By Lemma 1 and Corollary 2, the inclusion $N_G \subseteq C$ implies that $N_C = C$ and $C$ is a non-Hamiltonian $\overline{H_2}$-group. Using the description of such groups [11, Theorems 1.2-1.3] we conclude that $C$ is a group of the following types:

1) $G = (A \times \langle b \rangle)\lambda\langle c \rangle$, $A$ is a quasicyclic 2- group, $|b| = |c| = 2$, $[A, \langle c \rangle] = 1$, $[b, c] = a_1 \in A$, $|a_1| = 2$;

2) $G = A \times H$, $A$ is a quasicyclic 2- group, $H = \langle h_1, h_2 \rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2 = [h_1, h_2]$.

Further we examine separately each of the above two types.

Let $G$ be a group of type 1). Since $N_G$ is non–Dedekind and $N_G \subseteq C$, we get that subgroup $B = \langle b, c\rangle$ lies in $C$. Put $\overline{G} = G/A$. Clearly, $\overline{G} \cong \overline{N_G}\langle\overline{d}\rangle$, where $\overline{N_G} = \overline{B}$, $\overline{d^2} \in \overline{B}$ and $|\overline{d}| \le 4$. By Lemma 2, in consideration of $[\overline{G} : \overline{N_G}] = 2$ we obtain that in $\overline{G}$ each subgroup is normal, if it does not belong to $\overline{N_G}$. This means that $\overline{G}$ is an Abelian group and consequently $G' \subseteq A$.

Since $\omega(C) \triangleleft G$ and $[\omega^2(C), G] = 1$, we see that $[\omega(C), G] \subseteq \langle a_1\rangle$, where $a_1 \in A$, $|a_1| = 2$. So $B \triangleleft G$, $[B, G] \subseteq \langle a_1\rangle \subseteq Z(G)$ and by Proposition 1.3 [11],

$$G = BC_G(B),\ B \cap C_G(B) = \langle a_1\rangle.$$

If $|\overline{d}| = 2$, then $|d| \le 4$. Suppose $|d| = 2$. Then $[d, y] \ne 1$ for each noncentral element $y \in B$. Conversely since $\langle d, y\rangle \triangleleft G_1 = \langle d\rangle N_G$, we see that $\langle y\rangle = \langle d, y\rangle \cap N_G \triangleleft G_1$, which is impossible. So $[d, b] = [d, c] = a_1$ and $G$ is a group of type 3) of the Theorem. Assume $|d| = 4$. It is clearly that $d^2 = a_1$ and if $d \in C_G(B)$, then $|dbc| = 2$. Replacing element $d$ by $dbc$ we obtain the group of the type 3) of the Theorem. Suppose $d \notin C_G(B)$. Then there exists an involution $x \in B$, such that $[d, x] = a_1$. Consequently, $|dx| = 2$ and replacing the element $d$ by $dx$ we see that $G$ is the group of type 3) again.

Suppose $|\overline{d}| = 4$. Then $d^2 = a'y$, where $a' \in A$, $y \in B \setminus \langle a_1\rangle$. Choose an element $x \in B$ with $[x, y] \ne 1$. Then $[x, d] \in A \cap B = \langle a_1\rangle$ and $[x, d^2] = 1$, which contradicts to $[x, d^2] = [x, y] \ne 1$. Case 1) is considered.

Let $C$ be a group of type 2). Then $Z(G) \supseteq \langle a_1\rangle \times \langle h^2\rangle$, where $a_1 \in A$, $|a_1| = 2$, $h \in H$, $|h| = 4$. Let us examine the factor-group $C/A = \overline{G} \cong \overline{H}\langle\overline{d}\rangle$, $\overline{d^2} \in \overline{H}$. Since $N_G$ is non-Dedekind, we can suppose that $\overline{H} = \overline{N_G}$. By Lemma 2, $\overline{N_G} \le N(\overline{G})$, so the norm $N(\overline{G})$ is Hamiltonian and by [2] $\overline{G}$ contains no element of order 8. Thus, $|\overline{d}| \le 4$.

If $|\overline{d}| = 2$ then $\langle\overline{d}\rangle \triangleleft \overline{G}$ and $\overline{G} = \overline{H} \times \overline{d}$. Suppose $|\overline{d}| = 4$. Then $\overline{d^2} = \overline{h^2} \in \overline{H}$ and since $\overline{C'} \subseteq \langle\overline{d}\rangle \cap \overline{H} = \langle\overline{h^2}\rangle$ we obtain, that there exists an element $\overline{h} \in \overline{H}$, $|\overline{h}| = 4$ such that $[\overline{h}, \overline{d}] = 1$. It means that $|\overline{dh}| = 2$ and $\overline{H} \subseteq N_{\overline{G}}(\langle\overline{dh}\rangle)$ by the Lemma 2. So again $\overline{G} = \overline{H} \times \langle\overline{d'}\rangle$, where $\overline{d'} = \overline{dh}$, $|\overline{d'}| = 2$.

It follows from Lemma 3 and $d^2 \in A$ that $|d| = 4$ and $\langle d\rangle \cap C = \langle a_1\rangle \in A$. It is also clear that $[H, \langle d\rangle] \subseteq A$ and $[H^2, \langle d\rangle] = 1$. Suppose $K = (\langle a\rangle \times H)\langle d\rangle$, $a \in A$, $|a| = 4$. Since $[K, \langle d\rangle] \subseteq \langle a^2\rangle A$, we get $K' = \langle a^2, h^2\rangle$. By Lemma 3 and Theorem B [13] we conclude that $K$ is a semi-direct product of two quaternion groups. In this case, $G$ is a group of type 4) of the Theorem.

*Sufficiency.* If $G$ is a group of type 1) or 2) of Theorem, then it is a $\overline{H_p}$-group and thus, $G = N_G$.

Let $G$ be a group of type 3) of the Theorem. Prove that its non-cyclic norm coincides with the group $N = (\langle a\rangle \times \langle b\rangle)\lambda\langle c\rangle$, where $a \in A$, $|a| = 4$. Indeed, since $N_1 = N_G(\langle a_1, d\rangle) = (\langle a\rangle \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle d\rangle$, and $N_2 = N_G(\langle a_1, a'd\rangle) = (\langle a\rangle \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle a'd\rangle$, where $a_1, a'$
$nA$, $|a_1| = 2$,$|a'| > 4$, we get $N_G \subseteq N_1 \cap N_2 = N = (\langle a\rangle \times \langle b\rangle)\lambda\langle c\rangle$. Now, taking into account that every non-cyclic subgroup contains the element $a_1$ and $[G, N_G] \subseteq \langle a_1\rangle$, we conclude, that $N$ normalize all non-cyclic subgroup. So, $N_G = N$.

Suppose that $G$ is a group of type 4) of the Theorem. It is obvious that $N_G \subseteq N_1 \cap N_2 = N = \langle h_2\rangle\lambda\langle h_1 a\rangle$, where $N_1 = N_G(\langle h_1 d, h_2\rangle) = \langle h_1 d, h_2, h_1 a\rangle$ and $N_2 = N_G(H) = A \times H$. In light of $[G, N] \subseteq \omega(G) = \langle h^2\rangle \times \langle a^2\rangle$ it is enough to show that the subgroup $N$ normalizes all generalized quaternion groups. It is obvious for subgroups which belong to $C = A \times H$. Suppose $Q$ is a generalized quaternion group containing the element $da_i$, where $a_i \in A$, $|a_i| = 2^i$, $j \ge 0$. Since $[\langle d\rangle, N] \subseteq \langle d^2\rangle$, we get $N\langle d\rangle N_G(Q)$. If $Q$ contains the element $dh_1 a_i$, then $Q = \langle dh_1 a_i, a_1^m h_2 \rangle$, $m = 0, 1$. The inclusion $[Q, N] \subseteq \langle h_2 \rangle \subseteq Q$ imply $N \subseteq N_G(Q)$. As we have no other generalized quaternion groups which are not included in $C$, then $N$ normalizes all non-cyclic subgroups and $N = N_G$. The Theorem is proved. □

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