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ON THE TATE-SHAFAREVICH GROUPS OF FINITE MODULES OVER n-DIMENSIONAL PSEUDOLOCAL FIELDS

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Let k be an n-dimensional local field over a pseudofinite residue field k_0, p be a prime number, $p \neq \operatorname{char} k_0$, and X be a complete, smooth, absolutely irreducible curve over k with successive good reductions. Suppose that $1 \leq n \leq 3$, and that k contains the group μ_p, of p-th roots of unity. Let K be the function field on X. It is proved that the Tate-Shafarevich groups $Ш^{n+2}(\mu_p)$ and $Ш^1(\mu_p)$ are dual one another.

В. И. Андрийчук. *О группах Тэйта-Шафаревича конечных модулей над n-мерными псевдолокальными полями* // Математичні Студії. – 2002. – Т.17, №1. – С.109–112.

Пусть k — n-мерное общее локальное поле с псевдоконечным полем вычетов k_0, p — простое число, $p \neq \operatorname{char} k_0$, X — полная, гладкая, абсолютно неприводимая кривая над k с последовательными невырожденными редукциями. Пусть $1 \leq n \leq 3$, и поле k содержит все корни p-ой степени из единицы. Доказано, что группы Тэйта-Шафаревича $Ш^{n+2}(\mu_p)$ и $Ш^1(\mu_p)$ двойственны.

By an n-dimensional pseudolocal field k we mean a chain of fields $k_0, \dots, k_n = k$, where k_i is complete discretely valued field with residue field $k_{i-1}, 1 \leq i \leq n$, and the field k_0 is pseudofinite [1].

Recall that for a field K, endowed with a set of valuation V^K, and for a $\operatorname{Gal}(K_{\text{sep}}/K)$ - module M we denote by $Ш^i(M)$ the kernel of the natural localisation homomorphism

$$Ш^i(M) = \operatorname{Ker}(H^i(K, M) \to \prod_{v \in V^K} H^i(K_v, M)),$$

where K_v is the completion of the field K at $v \in V^K$.

Let now k be an n-dimensional pseudolocal field, p be a prime number, different from the characteristic of the field k_0, X be a complete, smooth, absolutely irreducible algebraic curve with successive good reductions (see [2, p.240]). K denotes the function field on X. We will denote by $\hat{A}$ the character group of a group A. We assume that all the p-th roots of unity are in k_0. Suppose that $n \leq 3$. Let $H^r(X, \mu_p)$ denote the r-th etale cohomology group. In [3] the following result is proved.

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Theorem 1. *With the notations above for any $n \le 3$ there exists a nondegenerate pairing*

$$H^{n+2}(X, \mu_p) \times H^1(X, \mu_p) \to \mu_p,$$

so the groups $H^{n+2}(X, \mu_p)$ and $H^1(X, \mu_p)$ are dual.

Note that in the case of n-dimensional local fields (that is in the case of finite k_0 in the above definition) both Theorems 1 and 2 below were proved by J.-C. Douai [4] and we essentially follow his arguments.

Theorem 2. *The groups $Ш^{n+2}(\mu_p)$ and $Ш^1(\mathbb{Z}/p\mathbb{Z})$ are finite and there is a nondegenerate pairing $Ш^{n+2}(\mu_p) \times Ш^1(\mathbb{Z}/p\mathbb{Z}) \to \mathbb{Z}/p\mathbb{Z}$, so they are dual to each another.*

Proof. The duality asserted in Theorem 2 will follow from the duality of the groups $H^1(X, \mu_p)$ and $H^{n+2}(X, \mu_p)$ proved in Theorem 1. The proof, essentially, is based on the same reasonings which are used by J.-C. Douai [4] in the case of n-dimensional local fields. Namely, consider the exact localization sequence

$$\cdots \to \oplus_{v \in X^0} H_v^{n+2}(X, \mu_p) \to H^{n+2}(X, \mu_p) \to H^{n+2}(K, \mu_p)$$
$$\to \oplus_{v \in X^0} H_v^{n+3}(X, \mu_p) \to H^{n+3}(X, \mu_p) \to \cdots, \tag{1}$$

where v runs over the set X^0 of all closed points of X. Note that we identify the points $v \in X^0$ with the corresponding valuations of the field K.

In exact sequence (1)

$$H_v^{n+2}(X, \mu_p) \cong H^{n+1}(K_v, \mu_p) \cong H^n(k(v), \mu_p),$$
$$H_v^{n+3}(X, \mu_p) \cong H^{n+2}(K_v, \mu_p) \cong H^{n+1}(k(v), \mu_p) \cong \mathbb{Z}/p\mathbb{Z}$$

according to [3, Lemma 1]. Applying [3, Proposition 1] to the field $k(v)$, we get that the group $H^n(k(v), \mu_p)$ is dual with the group $H^1(k(v), \mu_p)$, which is isomorphic to the group $k(v)/k(v)^{*p}$.

Let $\bar{X} = X \times_k k_{\text{sep}}$, k_{sep} being a separable closure of k. There is the Hochschild-Serre spectral sequence [5, p.134]

$$E_2^{r,s} = H^r(k, H^s(\bar{X}, \mu_p)) \Longrightarrow H^{r+s}(X, \mu_p).$$

The Hochschild-Serre spectral sequence implies, taking into account that $H^r(\bar{X}, \mu_p) = 0$ for $r > 2$ and [3, Lemma 1], that $H^{n+3}(X, \mu_p) = \mu_p$. So exact sequence (1) can be written as

$$\cdots \to \oplus_{v \in X^0} (\widehat{k(v)/k(v)^{*p}}) \to H^{n+2}(X, \mu_p) \to$$
$$\to H^{n+2}(K, \mu_p) \to \oplus_{v \in X^0} H^{n+2}(K_v, \mu_p) \to \mu_p \to 0.$$

Using this and the definition of the group $Ш^{n+2}(\mu_p)$, we get the exact sequence

$$\oplus_{v \in X^0} (\widehat{k(v)/k(v)^{*p}}) \longrightarrow H^{n+2}(X, \mu_p) \longrightarrow Ш^{n+2}(\mu_p) \longrightarrow 0.$$

Taking here the dual groups and using Theorem 1, we obtain the following exact sequence

$$0 \longrightarrow \widehat{Ш^{n+2}(\mu_p)} \longrightarrow H^1(X, \mu_p) \longrightarrow \prod_{v \in X^0} (k(v)/k(v)^{*p}) \tag{2}$$

In order to complete the proof of Theorem 2, one needs to show that the kernel of the homomorphism $H^1(X,\mu_p) \to \prod_{v\in X^0}(k(v)/k(v)^{*p})$ in exact sequence (2) is isomorphic to $Ш^1(\mu_p) = Ш^1(\mathbb{Z}/p\mathbb{Z})$ (let us recall that $\mu_p \subset k_0 \subset K$). To see this we use as in [4] that the Leray spectral sequence for the inclusion $\operatorname{Spec} K \to X$ gives the following exact sequence

$$0 \to H^1(X,\mu_p) \longrightarrow H^1(K,\mu_p) \longrightarrow \oplus_{v\in X^0}(\mathbb{Z}/p\mathbb{Z})_v. \tag{3}$$

But

$$H^1(K_v,\mu_p) \cong K_v^*/K_v^{*p} \cong U_v/U_v^{*p} \oplus \mathbb{Z}/p\mathbb{Z} \cong$$
$$\cong k(v)^*/k(v)^{*p} \oplus U_v^1/(U_v^1)^p \oplus \mathbb{Z}/p\mathbb{Z} \cong k(v)^*/k(v)^{*p} \oplus \mathbb{Z}/p\mathbb{Z},$$

since $U_v^1 = (U_v^1)^p$ (here U_v is the group of units of the ring of integer of K_v, $U_v^1 = \{x \in U_v | v(1-x) > 0\}$). From this we get that the group $Ш^1(\mu_p)$ can be inserted in the exact sequense

$$0 \to Ш^1(\mu_p) \to H^1(K,\mu_p) \to \prod_{v\in X^0}((k(v)^*/k(v)^{*p}) \oplus \mathbb{Z}/p\mathbb{Z}). \tag{4}$$

Comparing exact sequences (3) and (4), we get

$$Ш^1(\mu_p) \cong \operatorname{Ker}(H^1(X,\mu_p) \longrightarrow \prod_{v\in X^0}(k(v)^*/k(v)^{*p})). \tag{5}$$

Taking into account isomorphism (5), it follows from exact sequence (2) that the groups $Ш^{n+2}(\mu_p)$ and $Ш^1(\mathbb{Z}/n\mathbb{Z})$ are dual finite groups (the finiteness follows from the finiteness of the group $H^1(X,\mu_p)$), and, besides, duality of these groups is induced by duality $H^{n+2}(X,\mu_p)$ and $H^1(X,\mu_p)$, which is proved in Theorem 1. This completes the proof of Theorem 2. □

Corollary. *With the assumptions and notations of Theorem 2 there are the following two exact sequences:*

$$\oplus_{v\in X^0}\widehat{H^1(K_v,\mu_p)} \longrightarrow \widehat{H^1(K,\mu_p)} \longrightarrow H^{n+2}(K,\mu_p) \longrightarrow$$
$$\longrightarrow \prod_{v\in X^0} H^{n+2}(K_v,\mu_p) \longrightarrow \mu_p \longrightarrow 0, \tag{6}$$
$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \oplus_{v\in X^0}H^0(K_v,\mu_p) \longrightarrow$$
$$\longrightarrow \widehat{H^{n+2}(K,\mu_p)} \longrightarrow H^1(K,\mu_p) \xrightarrow{\alpha} \prod_{v\in X^0} H^1(K_v,\mu_p). \tag{7}$$

Proof. The kernel of the homomorphism

$$H^{n+2}(K,\mu_p) \to \prod_{v\in X^0} H^{n+2}(K_v,\mu_p)$$

is the group

$$Ш^{n+2}(\mu_p) \cong \widehat{Ш^1(\mu_p)} = \operatorname{Coker}(\oplus_{v\in X^0}H^1(K_v,\mu_p) \to H^1(K,\mu_p)).$$

This implies exactness of sequence (6). Exactness of (7) follows now from [3, Proposition 1] taking the dual groups in (6). □

Note that both Theorems 1 and 2 were proved under some restrictive assumptions, namely: the curve X were assumed to have good reductions, $\mu_p \in k_0$, and $n \le 3$.

Problem. *An interesting question is whether or not the statements of Theorems 1 and 2 remain true for arbitrary curve X, dimension n and arbitrary finite* Gal (K_{sep}/K)*-module M instead of μ_p.*

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