УДК 517.518.34

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ON NOTION OF BASES IN A HILBERT SPACE

Communicated by O. B. Skaskiv

O. Karabyn. *On notion of bases in a Hilbert space*, Matematychni Studii, **17** (2002) 105–108.

This paper is intended to study some nonstandard aspects of the bases theory. Our framework is Nelson's Internal Set Theory. The main notion is that of nst-equivalence of bases. We give conditions for nearstandardness of bases, determine the shadow of vector in terms of coordinates. Supplementing arguments of Krein we obtain results characterizing Bari's bases.

О. Карабин. *О понятии базисов в гильбертовом пространстве* // Математичні Студії. – 2002. – Т.17, №1. – С.105–108.

Рассматриваются некоторые нестандартные аспекты теории базисов. Основа работы — внутренняя теория множеств Нельсона. Введено понятие nst-эквивалентности базисов. Приведены условия околостандартности базисов, определён способ покоординатного нахождения тени вектора. Дополняя результаты Крейна, получены результаты, характеризующие базисы Бари.

1. ***nst*-equivalent bases.** In what follows $\mathbb{H}$ denotes a standard separable complex Hilbert space. Bases $(\varphi_i)_{i\in\mathbb{N}}$, $(\tilde\varphi_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$ are said to be *equivalent* iff $(\forall i\in\mathbb{N})$: $\tilde\varphi_i = U\varphi_i$ for some $U\in\mathcal{B}(\mathbb{H})$ such that $\ker U=\{0\}$ and $U^{-1}\in\mathcal{B}(\mathbb{H})$. If this holds U is called the *equivalence* of (φ_i) and $(\tilde\varphi_i)$. Equivalent bases (φ_i), $(\tilde\varphi_i)$ are said to be *nst-equivalent* iff its equivalence U is uniformly nearstandard operator, i. e. there exists a standard operator ${}^\circ U\in\mathcal{B}(\mathbb{H})$ (${}^\circ U$-shadow of U) such that $\|U-{}^\circ U\|\approx 0$ (read "is infinitesimal" for "≈ 0"). Note that the relation defined above is in fact an equivalence. For instance, its transitivity is a corollary of $\|U_1-{}^\circ U_1\|\approx 0\wedge\|U_2-{}^\circ U_2\|\approx 0 \implies \|U_1U_2-{}^\circ U_1{}^\circ U_2\|\approx 0$.

Observe that bases (ψ_i),$(\tilde\psi_i)$, associated (biorthogonal) to nst-equivalent bases (φ_i), $(\tilde\varphi_i)$ are also nst-equivalent. Namely, if U is the equivalence of (φ_i), $(\tilde\varphi_i)$, then U^* is the one for $(\tilde\psi_i)$ and (ψ_i).

Another remark: let the basis (φ_i) be nst-equivalent to an orthonormal basis (e_i). Then the basis (φ_i) is *nst-almost normed* i. e.

$$(\forall i\in\mathbb{N}):\ 0\ll\|\varphi_i\|\ll\infty, \tag{1.1}$$

(read "is positive and not infinitesimal" for "$\gg 0$" and "is not infinite" for "$\ll\infty$". Indeed, it is easy to see that $\|U\|^{-1}\le\|\varphi_i\|\le\|U^{-1}\|$.

2000 *Mathematics Subject Classification:* 47S20.

2. Nearstandard bases. Let $x \in \mathbb{H}$ be such that $\|x\| \ll \infty$. Then there exists a unique standard vector ${}^{\circ}x \in \mathbb{H}$ such that $({}^{\circ}x|y) \approx (x|y)$ for any standard $y \in \mathbb{H}$. This ${}^{\circ}x$ is said to be the *shadow* of x (in a weak sense). For a given sequence $(\varphi_i) \subset \mathbb{H}$ satisfying $\|\varphi_i\| \ll \infty$ for standard $i \in \mathbb{N}$, there exists, by the standardization principle, a unique standard sequence $(\overset{\circ}{\varphi}_i)_{i\in\mathbb{N}}$, such that $\overset{\circ}{\varphi}_i = {}^{\circ}\varphi_i$ for standard $i \in \mathbb{N}$. This sequence $(\overset{\circ}{\varphi}_i)$ will be called the *shadow* of the sequence (φ_i). *Warning*: the shadow of a basis is optionally a basis.

Let (φ_i) be a basis of $\mathbb{H}$ for which $\|\varphi_i\| \ll \infty$ for standard $i \in \mathbb{N}$. Suppose that its shadow $(\overset{\circ}{\varphi}_i)$ is also a basis and (φ_i), $(\overset{\circ}{\varphi}_i)$ are equivalent with equivalence U such that $\|U - I\| \approx 0$. Then (φ_i) is said to be a *nearstandard* basis.

It is easy to see that a basis (ψ_i) associated to a nearstandard basis (φ_i) is nearstandard too. Indeed, $\|U - I\| \approx 0$ implies $\|(U^*)^{-1} - I\| \approx 0$.

2.1. Proposition. *Let (φ_i) and $(\tilde{\varphi}_i)$ be nst-equivalent bases of $\mathbb{H}$. Then (φ_i) is nearstandard iff $(\tilde{\varphi}_i)$ so is. The shadow ${}^{\circ}U$ of an equivalence U of φ_i and $\tilde{\varphi}_i$ is the equivalence of $(\tilde{\varphi}_i)$ and $(\overset{\circ}{\tilde{\varphi}}_i)$.*

Proof. Assume that $\overset{\circ}{\varphi}_i = V\varphi_i$, where $\|V - I\| \approx 0$ and $\tilde{\varphi}_i = U\varphi_i$ where U is uniformly nearstandard. Define $\forall i \in \mathbb{N} \quad \hat{\varphi}_i = ({}^{\circ}U)\overset{\circ}{\varphi}_i$. Then $(\hat{\varphi}_i)$ is a standard basis of $\mathbb{H}$ which is equivalent to $(\overset{\circ}{\varphi}_i)$ with the standard equivalence ${}^{\circ}U$. Set $V_1 := ({}^{\circ}U)VU^{-1}$. Then ${}^{\circ}V_1 = ({}^{\circ}U)I{}^{\circ}(U^{-1}) = I$ and $\|V_1 - I\| \approx 0$. It is easy to check that $\hat{\varphi}_i = V_1\tilde{\varphi}_i$. Therefore $(\tilde{\varphi}_i)$ is nearstandard and $(\hat{\varphi}_i) = (\overset{\circ}{\tilde{\varphi}}_i)$. □

It is also easy to prove the following

2.2. Proposition. *The shadow of a nearstandard orthonormal basis is an orthonormal basis.*

Let (φ_i) be a nearstandard basis of $\mathbb{H}$. Consider an arbitrary vector $x \in \mathbb{H}$ and denote by (c_i) the sequence of coordinates of x, $x = \sum_{i\in\mathbb{N}} c_i\varphi_i$. Suppose that $\|x\| \ll \infty$, then $|c_i| \ll \infty$ for standard $i \in \mathbb{N}$ and there exists a unique standard ${}^{\circ}c_i \in \mathbb{C}$ such that $c_i \approx {}^{\circ}c_i$ By the standardization principle of IST there exists a unique standard sequence $(\overset{\circ}{c}_i)$ in $\mathbb{C}$ such that $\overset{\circ}{c}_i = {}^{\circ}c_i$ for standard $i \in \mathbb{N}$.

2.3. Proposition. *In the above assumption*

$$ {}^{\circ}x = \sum_{i\in\mathbb{N}} \overset{\circ}{c}_i \overset{\circ}{\varphi}_i, \tag{2.1} $$

where $(\overset{\circ}{\varphi}_i)$ is the shadow of the basis (φ_i).

Proof. As it was noted the assosiated basis (ψ_i) is nearstandard too. Hence $(\forall i \in \mathbb{N})$: $\|\overset{\circ}{\psi}_i - \psi_i\| \approx 0$. Whence $\|\psi_i\| \ll \infty$ for standard $i \in \mathbb{N}$. Since $\|x\| \ll \infty$, we have $|(x|\psi_i)| \ll \infty$ for standard $i \in \mathbb{N}$. Therefore indeed $|c_i| \ll \infty$ and the sequence $(\overset{\circ}{c}_i)$ is well defined above. Observe that ${}^{\circ}x = \sum\limits_{i\in\mathbb{N}} ({}^{\circ}x|\overset{\circ}{\psi}_i)\overset{\circ}{\varphi}_i$, but for standard $i \in \mathbb{N}\, ({}^{\circ}x|\overset{\circ}{\psi}_i) = {}^{\circ}(x|\overset{\circ}{\psi}_i) = {}^{\circ}(x|\psi_i) = \overset{\circ}{c}_i$. □

3. Bari's bases. Sequences (φ_i), $(\tilde{\varphi}_i)$ in $\mathbb{H}$ are said to be square close iff

$$ \sum_{i\in\mathbb{N}} \|\varphi_i - \tilde{\varphi}_i\|^2 < \infty. \tag{3.1} $$

A basis (φ_i) of $\mathbb{H}$ which is square close to some orthonormal basis is (in terminology of M. Krein) Bari's basis. The following result is due to N. K. Bari [6].

3.1. Theorem. *A σ-linearly independent sequence (φ_i) in $\mathbb{H}$ which is square close to an orthonormal basis is a Riesz basis.*

Explain that a sequence (φ_i) in $\mathbb{H}$ is said to be a σ-linearly independent iff

$$(\forall c \in \ell_2): \sum_{i \in \mathbb{N}} c_i \varphi_i = 0 \implies c = 0. \tag{3.2}$$

In his paper [1] M. Krein relates with a sequence (φ_i) in a Hilbert space $\mathbb{H}$ a sequence (V_i), where V_i is the volume of the parallelepiped constructed on vectors $\varphi_1, ..., \varphi_i$, and a measure

$$\mathcal{D}(\varphi) := \sum_{i,j \in \mathbb{N}} |(\varphi_i|\varphi_j) - \delta_{i,j}|^2 \tag{3.3}$$

of deviation of the matrix $((\varphi_i|\varphi_j))_{i,j\in\mathbb{N}}$ from the unit matrix $(\delta_{i,j})_{i,j\in\mathbb{N}}$. In terms of (V_i) and $\mathcal{D}(\varphi)$ he obtains profound results characteristic for Bari's bases. Slightly supplementing arguments of Krein we get the following.

Definition. Sequences (φ_i) and $(\tilde{\varphi}_i)$ in $\mathbb{H}$ are said to be *finitely square close (infinitely square close)* iff

$$\sum_{i \in \mathbb{N}} \|\varphi_i - \tilde{\varphi}_i\|^2 \ll \infty \quad \Big(\sum_{i \in \mathbb{N}} \|\varphi_i - \tilde{\varphi}_i\|^2 \approx 0\Big). \tag{3.4}$$

3.3. Theorem. *Let (φ_i) be a sequence of unit vectors in $\mathbb{H}$ which is infinitely square close to an orthonormal basis. Then*

$$(\forall n \in \mathbb{N}): \ V_n \approx 1. \tag{3.5}$$

Conversely, let (φ_i) be a complete sequence of unit vectors in $\mathbb{H}$, for which (3.5) holds. Then (φ_i) is a basis in $\mathbb{H}$ infinitely square close to an orthonormal basis.

3.4. Corollary. *Let (φ_i) be a standard sequence of unit vectors in $\mathbb{H}$ which is infinitely square close to an orthonormal basis of $\mathbb{H}$. Then (φ_i) is an orthonormal basis.*

Indeed, now the sequence (V_i) is standard. Therefore from (3.5) and the transfer principle it follows that $(\forall n \in \mathbb{N}): V_n = 1$.

3.5. Theorem. *Let (φ_i) be a σ-linearly independent sequence which is finitely square close to an orthonormal basis. Then*

$$(\forall n \in \mathbb{N}): \ V_n \gg 0. \tag{3.6}$$

Conversely, let (φ_i) be a complete σ-linearly independent sequence of unit vectors in $\mathbb{H}$ for which (3.6) holds. Then (φ_i) is a basis of $\mathbb{H}$ which is finitely square close to an orthonormal basis.

3.6. Theorem. *Let (φ_i) be a sequence in $\mathbb{H}$ which is infinitely square close (finitely square close) to an orthonormal basis, then*

$$\mathcal{D}(\varphi) \approx 0 \quad (\mathcal{D}(\varphi) \ll \infty). \tag{3.7}$$

Conversely, let (φ_i) be a complete σ-linearly independent sequence in $\mathbb{H}$. If (3.7) holds then (φ_i) is infinitely (finitely) square close to an orthonormal basis of $\mathbb{H}$.

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Received 15.12.1999