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IDEALS OF ALGEBRAS OF ANALYTIC FUNCTIONS ON BANACH SPACES

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It is proved that if X is not a symmetrically regular Banach space then there exist finite codimensional primary ideals on the algebra of entire functions of bounded type on X, $H_b(X)$ and on the algebra of uniformly continuous bounded functions on the unit ball $\mathcal{B}$, $H^\infty_{uc}(\mathcal{B})$.

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Доказано, что для не симметрически регулярного банахового пространства X существуют примарные идеалы конечной коразмерности на алгебре $H_b(X)$ целых функций ограниченного типа на X и на алгебре равномерно непрерывных ограниченых функций $H^\infty_{uc}\mathcal{B}$ на единичном шаре $\mathcal{B} \in X$.

Let X be a complex Banach space and $\mathcal{B}$ its unit ball. We call $H^\infty_{uc}(\mathcal{B})$ the algebra of bounded analytic functions on $\mathcal{B}$, uniformly continuous on the closure $\bar{\mathcal{B}}$. It is well-known that the algebra $H^\infty_{uc}(\mathcal{B})$ endowed with norm $\|f\| = \sup_{x\in\mathcal{B}} |f(x)|$ is a Banach algebra. The algebra $H_b(X)$ of entire functions on X that are bounded on the bounded sets can be defined as the projective limit of algebras $H^\infty_{uc}(r\mathcal{B})$, where r is a real positive number. The purpose of this paper is investigation of finite dimensional homomorphisms of algebras $H_b(X)$ and $H^\infty_{uc}(r\mathcal{B})$, when the space X is not symmetrically regular. For the symmetrically regular algebras the structure of set of complex homomorphisms on $H_b(X)$ was investigated in [1].

For background information on holomorphic functions in infinite dimensions, we refer to [2] or [3].

Given a continuous n-linear mapping $B\colon X \times \cdots \times X \to \mathbb{C}$, B can be extended to a continuous, n-linear mapping $\widetilde{B}:\ X'' \times \cdots \times X'' \to \mathbb{C}$ by

$$\widetilde{B}(x''_1, \ldots, x''_n) = \lim_{\alpha_1} \ldots \lim_{\alpha_n} B(x_{\alpha_1}, \ldots, x_{\alpha_n}), \qquad (1)$$

where for each k, (x_{α_k}) is a net in X weak-star convergingto x''_k. It is known that $\|\widetilde{B}\| = \|B\|$. It is essential for us that if B is symmetric, it does not necessary follow that $\widetilde{B}$ is symmetric.

A Banach space X is called *regular* if

$$\widetilde{B}(x''_1, \ldots, x''_n) = \lim_{\alpha_{\sigma(1)}} \ldots \lim_{\alpha_{\sigma(n)}} B(x_{\alpha_1}, \ldots, x_{\alpha_n}),$$

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for every permutation σ on the set $\{1,\dots,n\}$ and *symmetrically regular* if $\widetilde{B}$ is symmetric for every symmetric bilinear map B on $X\times X$.

Let B be a symmetric n-linear form and $\widetilde{B}$ its extension to the bidual space. Let us consider an n-linear form $A[B]$ on X'' defined by

$$A[B](z_1,\dots,z_n)=\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\widetilde{B}(z_{\sigma(1)},\dots,z_{\sigma(n)}),$$

where S_n is the group of permutations on the set $\{1,\dots,n\}$. Let us denote by $\mathcal{L}_a({}^nX)$ the set of continuous antisymmetric n-linear forms on X^n. Thus the map $A\colon B\mapsto A[B]$ is a linear continuous operator from $\mathcal{L}_s({}^nX)$ to $\mathcal{L}_a({}^nX'')$. It is easy to see that this operator is trivial if and only if X is symmetrically regular. For any $G\in\mathcal{L}({}^nX'')$, we shall denote by $as(G)$ the antisymmetrization operator:

$$as(G)=\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma G(z_{\sigma(1)},\dots,z_{\sigma(n)}).$$

Therefore, $A[B]=as(\widetilde{B})$. Let $D_1\in\mathcal{L}_a({}^nX'')$ and $D_2\in\mathcal{L}_a({}^mX'')$. Put $D_1\wedge D_2:=as(D_1D_2)$. Using the simple induction, it is easy to see that

$$D_1\wedge D_2=(-1)^{nm}D_2\wedge D_1.$$

Let us denote by $\mathcal{L}_a(X'')$ the direct sum of spaces $\mathcal{L}_a({}^nX'')$, $n=0,\dots,\infty$, endowed with the direct sum topology. We shall assume that $\mathcal{L}_a({}^0X'')$ is the field $\mathbb{C}$ and $\mathcal{L}_a({}^1X'')=X'''$.

It is clear that the operation $\wedge$ is associative and well-defined on $\mathcal{L}_a(X'')$.

Proposition 1. *The space $\mathcal{L}_a(X'')$ with operation $\wedge$ is a locally multiplicatively convex algebra.*

Proof. Since the direct sum of Banach spaces is a locally convex space, it is enough to check that $\|D_1\wedge D_2\|\le\|D_1\|\|D_2\|$.

For any $z_1,\dots,z_n\in\mathcal{B}(X'')$ we can write

$$|D_1\wedge D_2(z_1,\dots,z_n)|\le\sup_{\sigma\in S_n}|D_1D_2(z_{\sigma_1},\dots,z_{\sigma_n})|\le\|D_1\|\|D_2\|.$$

□

Proposition 2. *The map $A\colon B\mapsto as(\widehat{B})$ is a continuous homomorphism from algebra $\mathcal{P}(X)$ to $\mathcal{L}_a(X'')$.*

Proof. For arbitrary $B_1\in\mathcal{L}_s({}^nX)$, $B_2\in\mathcal{L}_s({}^mX)$ we have

$$A[B_1B_2]=as(\widehat{B_1B_2})=as(as(\widehat{B}_1)as(\widehat{B}_2))=A[B_1]\wedge A[B_2].$$

Since, $\|\widetilde{B}\|=\|B\|$ and $\|\widetilde{B}\|\ge\|as(\widetilde{B})\|$, the operator A is continuous. □

Let us denote by $\mathcal{A}$ the image of the operator A.

Corollary 1. *The image $\mathcal{A}$ of the operator A is a commutative subalgebra in $\mathcal{L}_a({}^nX'')$.*

Proof. Let $h_1, \dots, h_m$ be some linearly independent vectors in X and $D \in \mathcal{L}_a({}^nX'')$. Put

$$\Phi_{h_1,\dots,h_m}(D) := \sum_{1 \le i_1 < \dots < i_n \le m} D(t_{i_1}h_{i_1}, \dots, t_{i_n}h_{i_n})$$

if $n > 0$ and $\Phi_{h_1,\dots,h_m}(1) = 1$ and extend it by linearity to the space $\mathcal{L}_a(X'')$. □

Theorem 1. *The map $\Phi_{h_1,\dots,h_m}$ is a continuous homomorphism from $\mathcal{L}_a(X'')$ into the algebra Ω_n of antisymmetric forms on $\mathbb{C}^n$.*

Proof. Evidently, $\|\Phi_{h_1,\dots,h_m}(D)\| \le \|D\|$ and $\|\Phi_{h_1,\dots,h_m}(1)\| = 1$, thus we have $\|\Phi_{h_1,\dots,h_m}\| = 1$. Also

$$\Phi_{h_1,\dots,h_m}(D_1 \wedge D_2) = \Phi_{h_1,\dots,h_m} as(D_1 D_2) =$$

$$as \sum_{1 \le i_1 < \dots < i_n \le m} D(t_{i_1}h_{i_1}, \dots, t_{i_n}h_{i_n}) = \Phi_{h_1,\dots,h_m}(D_1) \wedge \Phi_{h_1,\dots,h_m}(D_2).$$

Let us recall that an ideal J is called a *primary ideal* if it contained in a unique maximal ideal. □

Theorem 2. *If X is not a symmetrically regular Banach space then there exists a finite codimensional primary ideal on $H_b(X)$ and on $H^\infty_{uc}(\mathcal{B})$.*

Proof. Note first that $\mathcal{P}(X)$ is a dense subspace in $H_b(X)$ and $H_b(X)$ is a dense subspace in $H^\infty_{uc}(\mathcal{B})$. So every continuous homomorphism on $\mathcal{P}(X)$ can be extended to a continuous homomorphism on $H_b(X)$ and on $H^\infty_{uc}(\mathcal{B})$. Let us prove the theorem for $H_b(X)$, the proof for $H^\infty_{uc}(\mathcal{B})$ is similar. Let $\Psi_{h_1,\dots,h_m}$ denote the extension of homomorphism $A \circ \Phi_{h_1,\dots,h_m}$ to $H_b(X)$. Since $\Psi_{h_1,\dots,h_m}$ is a finite-dimensional homomorphism, the zero set $Z := \ker \Psi_{h_1,\dots,h_m}$ is a finite-codimensional ideal in $H_b(X)$. It is clear that Z contained in the zero set of the point evaluation functional δ_0 at the origin. Let I be a maximal ideal containing Z. Since Z is a finite codimensional subspace, $H_b(X) = Z \oplus V_n$, where V_n is a finite dimensional subalgebra. Thus V_n is isomorphic to Ω_n and the restriction of I to V_n is a maximal ideal on V_n. Since every nonconstant form from Ω_n is nilpotent, the point evaluation functional at the origin is a unique complex homomorphism on Ω_n. Thus I coincides with $\ker \delta_0$ on V_n and therefore on X. Hence $\ker \delta_0$ is a unique maximal ideal containing Z. □

Note that every finite-codimensional ideal of the algebra of entire functions on $\mathbb{C}^n$ coincides with the intersection of a finite number of maximal ideals.

REFERENCES

1. Aron R.M., Galindo P., Garcia D., Maestre M., *Regularity and algebras of analytic function in infinite dimensions*, Trans. Amer. Math. Soc., **348** (1996), 543-559.
2. Dineen S., *Complex Analysis on Infinite Dimensional Spaces*, Springer Monographs in Mathematic, 1999.
3. Mujica J., *Complex Analysis in Banach spaces*, North-Holland, Amsterdam, 1986.

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