УДК 517.53

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ON THE SLOW GROWTH OF POWER SERIES CONVERGENT IN THE UNIT DISK

P. V. Filevych. *On the slow growth of power series convergent in the unit disk*, Matematychni Studii, **16** (2001) 217–221.

We construct two power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ convergent in the unit disk with the maximum modulus $M_f(r)=\max\{|f(z)|:|z|=r\}$ and the maximum term $\mu_f(r)=\max\{|a_n|r^n: n\geq 0\}$ such that 1) $a_n\neq O(1)$ $(n\to\infty)$, $\ln\mu_f(r)$ is a slowly growing function and $\ln M_f(r)$ is not slowly growing; 2) $a_n=O(1)$ $(n\to\infty)$, and $\ln M_f(r)$ is not a slowly growing function are constructed.

П. В. Филевич. *О медленном росте сходящихся в единичном круге степенных рядов* // Математичні Студії. – 2001. – Т.16, №2. – С.217–221.

Построены примеры двух аналитических в единичном круге степенных рядов $f(z)=\sum_{n=0}^{\infty} a_n z^n$ с максимумом модуля $M_f(r)=\max\{|f(z)|:|z|=r\}$ и максимальным членом $\mu_f(r)=\max\{|a_n|r^n: n\geq 0\}$ таких, что: 1) $a_n\neq O(1)$ $(n\to\infty)$, $\ln\mu_f(r)$ — медленно возрастающая функция, а $\ln M_f(r)$ не является медленно возрастающей; 2) $a_n=O(1)$ $(n\to\infty)$, и $\ln M_f(r)$ не является медленно возрастающей функцией.

1°. For an analytic function

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

in the disk $\{z:|z|<1\}$ let $M_f(r)=\max\{|f(z)|:|z|=r\}$, $\mu_f(r)=\max\{|a_n|r^n: n\geq 0\}$ be the maximal term and $\nu_f(r)=\max\{n\geq 0: |a_n|r^n=\mu_f(r)\}$ be the central index, $0<r<1$.

A continuous and increasing on $(0;1)$ function l is called slowly growing iff $l((x+1)/2)\sim l(x)$ as $x\uparrow 1$.

From the inequalities

$$\ln\mu_f(r)\leq \ln M_f(r)\leq \ln\mu_f\left(\frac{r+1}{2}\right)+\ln\frac{2}{1-r},\qquad 0<r<1,$$

it follows that under the condition $-\ln(1-r)=o(\ln M_f(r))$ $(r\uparrow 1)$, $\ln M_f(r)$ is slowly growing iff $\ln\mu_f(r)$ is slowly growing. In [1] it is shown that there exists an analytic in the

2000 *Mathematics Subject Classification:* 30B10.

unit disk function f such that $\ln M_f(r)$ is slowly growing but $\ln \mu_f(r)$ is not slowly growing, and the following conjectures are stated.

Conjecture 1. *There exists an analytic function in* $\{z : |z| < 1\}$ *such that* $a_n = O(1)$, $(n \to \infty)$, *and* $\ln M_f(r)$ *is not slowly growing.*

Conjecture 2. *There exists no analytic function in* $\{z : |z| < 1\}$ *such that* $a_n \neq O(1)$ $(n \to \infty)$, $\ln \mu_f(r)$ *is slowly growing, and* $\ln M_f(r)$ *is not slowly growing.*

Here we prove that Conjecture 1 is true and Conjecture 2 is false.

2°. First we prove Conjecture 1. Let

$$h(x) = \frac{1}{2(1-x)} \ln \frac{1}{1-x}, \qquad 0 < x < 1.$$

Since h^{-1} is an increasing function to 1 on $(0; +\infty)$, it follows that there exists a sequence (p_k) of positive integers such that

$$p_k = \frac{1}{2(1-r_k)} \ln \frac{1}{1-r_k} \leq \frac{1}{4} \ln \frac{1}{1-r_{k+1}}, \qquad k \geq 0, \tag{2}$$

where $r_k = h^{-1}(p_k)$. Obviously, $2p_k \leq p_{k+1}$.

Put $a_n = 1$ if $n \in [p_k; 2p_k - 1]$ and $k \geq 0$, and $a_n = 0$ otherwise. Consider power series (1) convergent in the unit disk with the coefficients a_n. Note that

$$f(z) = \sum_{k=0}^{\infty} \sum_{n=p_k}^{2p_k-1} z^n. \tag{3}$$

We show that $\ln M_f(r)$ is not slowly growing. Applying the relation

$$x = e^{(1+o(1))(x-1)}, \qquad x \to 1, \tag{4}$$

we obtain

$$r_k^{p_k} = e^{(1+o(1))(r_k-1)p_k} = (1-r_k)^{1/2+o(1)}, \qquad k \to \infty. \tag{5}$$

Since $x < e^{x-1}$, $x < 1$, we have

$$(2r_k - 1)^{p_k} < e^{(2r_k-2)p_k} = 1 - r_k, \qquad k \geq 0. \tag{6}$$

In view of (3) and (5),

$$M_f(r_k) \geq \sum_{n=p_k}^{2p_k-1} r_k^n = r_k^{p_k} \frac{1-r_k^{p_k}}{1-r_k} = \frac{1}{(1-r_k)^{1/2+o(1)}}, \qquad k \to \infty. \tag{7}$$

On the other hand, from (3), (2), and (6) it follows

$$M_f(2r_k - 1) \leq \sum_{n=0}^{2p_{k-1}-1} (2r_k-1)^n + \sum_{n=p_k}^{\infty} (2r_k-1)^n <$$
$$< 2p_{k-1} + (2r_k-1)^{p_k} \frac{1}{1-(2r_k-1)} < \ln \frac{1}{1-r_k} + \frac{1}{2}, \qquad k \geq 1. \tag{8}$$

From (7) and (8) we see that the relation $\ln M_f(r_k) \sim \ln M_f(2r_k - 1)$ $(k \to \infty)$, is not fulfilled. Therefore, the function $\ln M_f(r)$ is not slowly growing. Conjecture 1 is proved.

3°. Now we show that *there exists an analytic in the unit disk function f such that $a_n \neq O(1)$ $(n \to \infty)$, $\ln \mu_f(r)$ is slowly growing and $\ln M_f(r)$ is not slowly growing.*

In order to prove this assertion, we need the following

Lemma. *Let $R \in (0; +\infty]$, $c_k > 0$, n_k be a nonnegative integer, $a_k \in \mathbb{C}$ for every $k \geq 0$. If $c_k \uparrow R$ $(k \to \infty)$, $n_k \uparrow \infty$ $(k \to \infty)$, $a_0 = \ldots = a_{n_0 - 1} = 0$, $a_{n_0} \neq 0$ and*

$$|a_{n_{k+1}}| = |a_{n_0}| \prod_{j=0}^{k} \frac{1}{c_j^{n_{j+1} - n_j}} \quad, k \geq 0, \quad |a_n| c_k^n \leq |a_{n_k}| c_k^{n_k}, \quad n \in (n_k; n_{k+1}),\ k \geq 0,$$

then: (i) the radius of convergence of series (1) with the coefficients a_n is $R_f = R$; (ii) $\nu_f(r) = n_{k+1}$ for $c_k \leq r < c_{k+1}$ $(k \geq 0)$ and $\nu_f(r) = n_0$ for $0 < r < c_0$.

Proof. For every $n \in [n_k; n_{k+1})$ and $k \geq 0$ we put $b_n = |a_{n_k}| c_k^{n_k - n}$. Then for $n \in [n_k; n_{k+1})$ and $k \geq 0$

$$\frac{b_n}{b_{n+1}} = c_k \uparrow R, \qquad k \to \infty.$$

This yields that the radius of convergence of the series $g(z) = \sum_{n=n_0}^{\infty} b_n z^n$ is equal to R. Therefore, $R_f \geq R$, because $|a_n| \leq b_n$. On the other hand,

$$\frac{1}{R_f} = \overline{\lim_{n\to\infty}} |a_n|^{1/n} \geq \overline{\lim_{k\to\infty}} |a_{n_{k+1}}|^{1/n_{k+1}} \geq \overline{\lim_{k\to\infty}} \left(\frac{|a_{n_0}|}{c_k^{n_{k+1} - n_0}} \right)^{1/n_{k+1}} = \overline{\lim_{k\to\infty}} \frac{1}{c_k} = \frac{1}{R},$$

i.e. $R_f \leq R$. Thus $R_f = R$.

Now we prove (ii). Let $c_{-1} \in (0; c_0)$ and $r \in [c_k; c_{k+1})$ $(k \geq -1)$. If $n \in (n_m; n_{m+1}]$ and $m \geq k+1$, then

$$|a_{n_m}| = |a_{n_{k+1}}| \prod_{j=k+1}^{m-1} \frac{1}{c_j^{n_{j+1} - n_j}} \leq |a_{n_{k+1}}| \prod_{j=k+1}^{m-1} \frac{1}{c_{k+1}^{n_{j+1} - n_j}} = |a_{n_{k+1}}| c_{k+1}^{n_{k+1} - n_m}. \tag{9}$$

Hence,

$$|a_n| r^n = |a_n| c_m^n \left(\frac{r}{c_m} \right)^n \leq |a_{n_m}| c_m^{n_m} \left(\frac{r}{c_m} \right)^n \leq \frac{|a_{n_{k+1}}|}{c_{k+1}^{n_m - n_{k+1}}} c_m^{n_m} \left(\frac{r}{c_m} \right)^n \leq$$
$$\leq |a_{n_{k+1}}| r^{n_{k+1}} \left(\frac{r}{c_{k+1}} \right)^{n - n_{k+1}} < |a_{n_{k+1}}| r^{n_{k+1}},$$

i.e. $\nu_f(r) \leq n_{k+1}$ and $\nu_f(r) = n_0$ if $r \in (0; c_0)$. If $n \in [n_m; n_{m+1})$ and $m \leq k$ $(k \geq 0)$, then

$$|a_{n_m}| = |a_{n_{k+1}}| \prod_{j=m}^{k} c_j^{n_{j+1} - n_j} \leq |a_{n_{k+1}}| c_m^{n_{m+1} - n_m} c_k^{n_{k+1} - n_{m+1}}.$$

Hence, we have

$$|a_n|r^n = |a_n|c_m^n\left(\frac{r}{c_m}\right)^n \le |a_{n_m}|c_m^{n_m}\left(\frac{r}{c_m}\right)^n \le |a_{n_{k+1}}|c_m^{n_{m+1}-n_m}c_k^{n_{k+1}-n_{m+1}}c_m^{n_m}\left(\frac{r}{c_m}\right)^n \le$$
$$\le |a_{n_{k+1}}|r^{n_{k+1}}\left(\frac{c_k}{r}\right)^{n_{k+1}-n} \le |a_{n_{k+1}}|r^{n_{k+1}}.$$

Therefore, $\nu_f(r) \ge n_{k+1}$, consequently $\nu_f(r) = n_{k+1}$. The lemma is proved. □

Let us pass to the proof of the assertion presented above.

Let (n_k) be a sequence of even positive integers such that

$$n_k \le \ln\ln n_{k+1}, \qquad k \ge 0. \tag{10}$$

We set $m_k = n_k/2$. It is clear that $n_k \le m_{k+1}$ $(k \ge 0)$. Put $a_{n_0} = 1$ and

$$c_k = 1 - \frac{1}{n_{k+1}}, \qquad a_{n_{k+1}} = \prod_{j=0}^{k} \frac{1}{c_j^{n_{j+1}-n_j}} \qquad (k \ge 0).$$

Put also $a_n = a_{n_k}c_k^{n_k-n}$ if $n \in [m_{k+1}; n_{k+1})$ for some $k \ge 0$ and $a_n = 0$ in the other cases.

By the lemma series (1) with the coefficients a_n defines an analytic function in the disk $\{z : |z| < 1\}$. First of all we prove that $a_n \ne O(1)$ $(n \to \infty)$. Since

$$c_j^{n_{j+1}-n_j} < e^{(c_j-1)(n_{j+1}-n_j)} = \frac{1}{e^{(2m_{j+1}-n_j)/n_{j+1}}} \le \frac{1}{e^{m_{j+1}/n_{j+1}}} = \frac{1}{\sqrt{e}},$$

we have

$$a_{n_k} = \prod_{j=0}^{k-1} \frac{1}{c_j^{n_{j+1}-n_j}} > e^{k/2},$$

i. e. $a_n \ne O(1)$, $n \to \infty$. In particular, this yields $\ln\mu_f(r) \nearrow +\infty$ $(r \uparrow 1)$.

Now we prove that $\ln\mu_f(r)$ is slowly growing. From the lemma we obtain $\nu_f(r) = n_{k+1} = 1/(1-c_k) \le 1/(1-r)$ for $r \in [c_k; c_{k+1})$. Thus $\nu_f(r)(1-r)/\ln\mu_f(r) \to 0$ $(r \uparrow 1)$. Therefore [1], $\ln\mu_f(r)$ is slowly growing.

We will show that $\ln M_f(r)$ is not slowly growing. We note that

$$f(z) = z^{n_0} + \sum_{k=0}^{\infty}\sum_{n=m_{k+1}}^{n_{k+1}} a_{n_k}c_k^{n_k-n}z^n. \tag{11}$$

Set

$$x_k = 1 - \frac{2\ln n_{k+1}}{n_{k+1}}, \qquad y_k = \frac{x_k+1}{2} = 1 - \frac{\ln n_{k+1}}{n_{k+1}}.$$

Using (4) we have

$$\left(\frac{y_k}{c_k}\right)^{m_{k+1}} = \left(\frac{n_{k+1}-\ln n_{k+1}}{n_{k+1}-1}\right)^{n_{k+1}/2} = \exp\left\{(1+o(1))\left(\frac{n_{k+1}-\ln n_{k+1}}{n_{k+1}-1} - 1\right)\frac{n_{k+1}}{2}\right\} =$$
$$= \exp\left\{-\left(\frac{1}{2}+o(1)\right)\ln n_{k+1}\right\} = \left(\frac{1}{n_{k+1}}\right)^{1/2+o(1)}, \qquad k \to \infty, \tag{12}$$

and similarly

$$\left(\frac{x_k}{c_k}\right)^{m_{k+1}} = \left(\frac{n_{k+1} - 2\ln n_{k+1}}{n_{k+1} - 1}\right)^{n_{k+1}/2} = \left(\frac{1}{n_{k+1}}\right)^{1+o(1)}, \qquad k \to \infty. \tag{13}$$

From (11) and (12),

$$M_f(y_k) \geq \sum_{n=m_{k+1}}^{n_{k+1}-1} a_{n_k} c_k^{n_k - n} y_k^n = a_{n_k} c_k^{n_k} \sum_{n=m_{k+1}}^{n_{k+1}-1} \left(\frac{y_k}{c_k}\right)^n = \mu_f(c_k) \sum_{n=m_{k+1}}^{n_{k+1}-1} \left(\frac{y_k}{c_k}\right)^n >$$

$$> \sum_{n=m_{k+1}}^{n_{k+1}-1} \left(\frac{y_k}{c_k}\right)^n = \frac{c_k}{c_k - y_k}\left(\frac{y_k}{c_k}\right)^{m_{k+1}}\left(1 - \left(\frac{y_k}{c_k}\right)^{m_{k+1}}\right) =$$

$$= (1 + o(1))\frac{n_{k+1}}{\ln n_{k+1}}\left(\frac{1}{n_{k+1}}\right)^{1/2+o(1)} = n_{k+1}^{1/2+o(1)}, \qquad k \to \infty. \tag{14}$$

Now we estimate $M_f(x_k)$. From $R_f = 1$ it follows that $a_{n_k} \leq (1 + o(1))^{n_k}$, $k \to \infty$. Thus, in view of (11) and (10),

$$z^{n_0} + \sum_{p \leq k-1} \sum_{n=m_{p+1}}^{n_{p+1}} a_{n_p} c_p^{n_p - n} x_k^n \leq \mu_f(x_k) n_k \leq \mu_f(c_k) n_k \leq a_{n_k} n_k \leq e^{n_k} =$$

$$= e^{o(\ln n_{k+1})}, \qquad k \to \infty. \tag{15}$$

Further, (9) implies $a_{n_p} \leq a_{n_k} c_k^{n_k - n_p}$, $p \geq k$. Therefore, from (11), (13) and (10) we obtain

$$\sum_{p \geq k} \sum_{n=m_{p+1}}^{n_{p+1}} a_{n_p} c_p^{n_p - n} x_k^n \leq \sum_{p \geq k} \sum_{n=m_{p+1}}^{n_{p+1}} \frac{a_{n_k}}{c_k^{n_p - n_k} c_p^{n - n_p}} x_k^n \leq$$

$$\leq a_{n_k} c_k^{n_k} \sum_{p \geq k} \sum_{n=m_{p+1}}^{n_{p+1}} \left(\frac{x_k}{c_k}\right)^n \leq a_{n_k} \sum_{n \geq m_{k+1}} \left(\frac{x_k}{c_k}\right)^n = a_{n_k}\left(\frac{x_k}{c_k}\right)^{m_{k+1}} \frac{c_k}{c_k - x_k} \leq$$

$$\leq e^{n_k}\left(\frac{1}{n_{k+1}}\right)^{1+o(1)} \frac{n_{k+1}}{2n_{k+1} - 1} = e^{o(\ln n_{k+1})}, \qquad k \to \infty.$$

Together with (15) this yields $\ln M_f(x_k) = o(\ln n_{k+1})$ $(k \to \infty)$. On the other hand, from (14) we have $\ln M_f(y_k) \geq (1/2 + o(1)) \ln n_{k+1}$ $(k \to \infty)$. Therefore,

$$\ln M_f(y_k) / \ln M_f(x_k) \to +\infty, \qquad k \to \infty,$$

i. e. $\ln M_f(r)$ is not a slowly growing function.

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Received 15.10.2001