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TOPOLOGICAL SEMIGROUPS AND UNIVERSAL SPACES RELATED TO EXTENSION DIMENSION

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It is proved that there is no structure of left (right) cancellative semigroup on $[L]$-dimensional universal space for the class of compact metrizable spaces of extensional dimension $\leq [L]$. Besides, we show that the homeomorphism group of every locally compact separable metric space whose every nonempty open subset is universal for the class of compact metric $[L]$-dimensional spaces is almost 0-dimensional and, therefore, at most one-dimensional.

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Доказано, что не существует структуры левой (правой) полугруппы с сокращениями на $[L]$-мерном универсальном пространстве для класса компактных метризуемых пространств экстенсиональной размерности $\leq [L]$. Кроме того, показано, что группа гомеоморфизмов каждого локально компактного сепарабельного метрического пространства, каждое непустое открытое подмножество которого является универсальным для класса компактных метризуемых пространств экстенсиональной размерности $\leq [L]$, является почти 0-мерной и, следовательно, не более чем одномерной.

1. Preliminaries

Let L be a CW-complex and X a Tychonov space. The *Kuratowski notation* $X\tau L$ means that, for any continuous map $f\colon A \to L$ defined on a closed subset A of X, there exists an extension $\bar{f}\colon X \to L$ onto X. This notation allows us to define the preorder relation $\preceq$ onto the class of CW-complexes: $L \preceq L'$ iff, for every Tychonov space X, $X\tau L$ implies $X\tau L'$ (see [5]).

The preorder relation $\preceq$ naturally generates the equivalence relation $\sim$: $L \sim L'$ iff $L \preceq L'$ and $L' \preceq L$. We denote by $[L]$ the equivalence class of L.

The following notion is introduced by A. Dranishnikov (see [5] and [3]). The *extension dimension* of a Tychonov space X is less than or equal to $[L]$ (briefly, ext-dim$(X) \leq [L]$) if $X\tau L$.

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We say that a Tychonov space Y is a *universal space for the class of compact metric spaces* X *with* ext-dim$(X) \leq [L]$ if Y contains a topological copy of compact metric space X with ext-dim$(X) \leq [L]$. See [1] and [2] for existence of universal spaces.

If $L = S^n$ then we obtain the usual definition of covering dimension.

As usual, by S^n we denote an n-dimensional sphere. In what follows we will need the following

Proposition 1.1. *Let* $i_0 = \min\{i : \pi_i(L) \neq 0\}$. *Then* ext-dim$(S^{i_0}) \leq [L]$.

For the proof see [4].

It easily follows from Proposition 1.1 that ext-dim$(X) > [S^0]$ implies ext-dim$(X) \geq [S^1]$.

2. Main theorem

Recall that a semigroup S is called a *left cancellation semigroup* if $xy = xz$ implies $y = z$ for every $x, y, z \in S$.

Theorem 2.1. *Let L be a CW-complex for which $[L] > [$two-point space$]$. Let Y be a universal space for the class of separable metric spaces X with* ext-dim$(X) \leq [L]$. *If* ext-dim$(Y) = [L]$, *then there is no structure of left (right) cancellation semigroup on Y compatible with its topology.*

Proof. Suppose the contrary and let Y be a left cancellation semigroup. Since Y is universal, Y contains a copy of the Alexandrov compactification $\alpha(\coprod_{j=1}^{\infty} S_j^{i_0})$, where $i_0 = \min\{i \colon \pi_i(L) \neq 0\}$. We will assume that $\alpha(\coprod_{j=1}^{\infty} S_j^{i_0}) \subset Y$. Besides, since ext-dim$(Y) > [S^0]$, we see that $i_0 > 0$ and hence Y contains an arc J. Let a, b be endpoints of J. There exists j_0 such that $aS_{j_0}^{i_0} \cap bS_{j_0}^{i_0} = \varnothing$. By Proposition 1.1, there exists a map $f \colon aS_{j_0}^{i_0} \cup bS_{j_0}^{i_0} \to L$ such that $f|aS_{j_0}^{i_0}$ is a constant map and $f|bS_{j_0}^{i_0}$ is not null-homotopic. Extend map f to a map $\bar{f} \colon Y \to L$. Let $g \colon [0,1] \to J$ be a homeomorphism, then the map $F \colon S_{j_0}^{i_0} \times [0,1] \to L$, $\bar{f}(x,t) = g(t)x$, is a homotopy that contradicts to the fact that $f|bS_{j_0}^{i_0}$ is not null-homotopic. □

3. Homeomorphism group

The homeomorphism group Homeo(X) of a compact space X is endowed with the compact-open topology.

A separable metrizable space X is called *almost* 0-*dimensional* [7] provided there exists a basis $\mathcal{B}$ for X such that for each $B \in \mathcal{B}$ the set $X \setminus \bar{B}$ is a union of clopen sets.

In a metric space (X, d) we denote by $B_\varepsilon(x)$ (respectively $\hat{B}_\varepsilon(x)$) the open (respectively closed) ball of radius ε and centered at $x \in X$.

Theorem 3.1. *Suppose X is a compact metric space,* ext-dim$(X) = [L]$ *and every nonempty open subset of X is universal for the class of compact metric spaces Y with* ext-dim$(Y) \leq [L]$. *Then the homeomorphism group* Homeo(X) *is almost 0-dimensional.*

Proof. We apply arguments from [7]. Endow Homeo(X) with the sup-metric generated by a metric d in X. Let $f \in$ Homeo(X). It is shown in [7, Lemma 3] that $\hat{B}_\varepsilon(f) = \overline{B_\varepsilon(f)}$ for all but a countable number of ε. Hence, without loss of generality, we may suppose that $\hat{B}_\varepsilon(f) = \overline{B_\varepsilon(f)}$ and let $g \notin \overline{B_\varepsilon(f)}$. Then there exists a nonempty open set U in X such that

$d(f(y), g(y)) > \varepsilon$ for every $y \in U$. By the properties of X, there exists an embedding of S^{i_0} into U, where i_0 is as in Proposition 1.1. We may suppose that $S^{i_0} \subset U$. There exists a map

$$h\colon \bigcup\{f'(S^{i_0}) : f' \in \overline{B_\varepsilon(f)}\} \cup g(S^{i_0}) \to L$$

such that the restriction $h|\bigcup\{f'(S^{i_0}) : f' \in \overline{B_\varepsilon(f)}\}$ is a singleton while the restriction $h|g(S^{i_0})$ is not null-homotopic. Extend h to a map $\bar{h}\colon X \to L$. The set

$$W = \{g' \in \mathrm{Homeo}(X) : \bar{h}|g'(S^{i_0}) \text{ is null-homotopic}\}$$

is an open and closed subset of Homeo(X) that contains g and does not intersect $\overline{B_\varepsilon(f)}$. This follows from the fact that close maps into a CW-complex are homotopic. □

Corollary 3.2. *In the assumptions of Theorem 3.1 the space* Homeo(X) *is at most one-dimensional.*

Proof. Follows from Theorem 3.1 and the fact that every almost 0-dimensional space is at most one-dimensional [7, Theorem 2]. □

Note that Theorem 3.1 and Corollary 3.2 can be extended onto the case of locally compact locally connected X.

4. Remarks and open problems

Note that the case $L = S^n$ corresponds to the covering dimension. In this case, the topology of homeomorphism groups of some universal spaces is investigated by many authors (see the survey [6]). Our Theorem 3.1 corresponds to the result by L. G. Oversteegen and E. D. Tymchatyn [7].

It is known (see [8] and [7]) that the homeomorphism group of the n-dimensional Menger compactum M^n (note that M^n satisfies the conditions of Theorem 3.1 with $L = S^n$) is one-dimensional. This naturally leads to the following question. Let $[L] \geq [S^1]$ and X be as in Theorem 3.1. Is $\dim(\mathrm{Homeo}(X)) = 1$?

Since there is no canonical model space that satisfies the conditions of Theorem 3.1 for arbitrary L, we can modify this question as follows: is there X that satisfies the conditions of Theorem 3.1 and such that $\dim(\mathrm{Homeo}(X)) = 1$?

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