УДК 517.98

T. S. Kudryk*, W. E. Lyantse**

STANDARD FILLING OF A PRODUCT SPACE

T. S. Kudryk, W. E. Lyantse. *Standard filling of a product space*, Matematychni Studii, **16** (2001) 169–184.

Sometimes it is useful to substitute an infinite mathematical structure by a hyperfinite one. But for a hyperfinite set the conception of standardness (in the usual sense) is not defined. This disadvantage can be removed by a suitable standard filling construction. The last conception was introduced in [7] (see also [9] or [10]). It is based on some easy Q-П-formalism. Here we propose a standard filling for a hyperfinite product of a family of finite probability spaces. This gives a way to replace an infinite stochastic process by a hyperfinite one.

Т. С. Кудрик, В. Э. Лянце. *Стандартное наполнение произведения пространств* // Математичні Студії. – 2001. – Т.16, №2. – С.169–184.

Иногда полезно заменять бесконечную математическую структуру гиперконечной. Но для гиперконечного множества понятие стандартности (в обычном смысле) не определено. Этот недостаток можно устранить подходящей конструкцией стандартного наполнения. Последнее понятие было введено в [7] (см. также [9] или [10]). Оно базируется на некотором простом Q-П-формализме. Здесь мы предлагаем стандартное наполнение для гиперконечного произведения семьи конечных вероятностных пространств. Это дает возможность заменять бесконечный стохастический процесс гиперконечным.

1. Measure spaces.[1] Let $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ be a *standard* family of finite-dimensional probability spaces $\mathfrak{X}_n = (X_n, 2^{X_n}, p_n)$. Thus for any natural n $\mathrm{card} X_n \in \mathbb{N}$ and $p_n X_n = 1$. Consider some infinite $m \in \mathbb{N}$ and denote by $\mathfrak{Y}$ the product space $\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n$. Then $\mathfrak{Y} = (Y, 2^Y, p)$, where $Y = X_1 \times \cdots \times X_m$ and for each $y = (y_1, \ldots, y_m) \in Y$

$$p_y := p\{y\} = p_1\{y_1\} \cdots p_m\{y_m\}; \tag{1.1}$$

for $A \in 2^Y$ we have $pA = \sum_{y\in A} p_y$. Note that $\mathfrak{Y}$ is an *internal* (hyper)finite probability space: $pY = 1$. We want to consider $\mathfrak{Y}$ as a finite substitution of the *standard* product $\mathfrak{Z} = (Z, \Lambda, \lambda)$ of the infinite family $(\mathfrak{X}_n)_{n\in\mathbb{N}}$: $\mathfrak{Z} = \prod_{n\in\mathbb{N}} \mathfrak{X}_n$. By definition, the elements $z \in Z$ are sequences $z = (z_n)_{n\in\mathbb{N}}$ where $z_n \in X_n$ for $n \in \mathbb{N}$. Λ is the minimal σ-algebra (subalgebra of 2^Z) which contains each cylindrical set $B \in 2^Z$. Recall that a set $B \in 2^Z$ is said to be *cylindrical* iff it is of the form $B = \prod_{n\in\mathbb{N}} A_n$, where $A_n \in 2^{X_n}$ and there is a set $N_B := \{n_1, \ldots, n_r\} \subset \mathbb{N}$,

2000 *Mathematics Subject Classification:* 46G99, 60G99.
* The first author was partially supported by a fellowship of the University of Aveiro, Portugal.
** Робота частково фінансувалася Державним фондом фундаментальних досліджень.
[1]We work in the framework of Nelson's Internal Set Theory [11].

$n_1 < \cdots n_r$ such that $\forall\, n \in \mathbb{N}\backslash N_B\ A_n = X_n$. If it is the case, then the product $A_{n_1} \times \cdots \times A_{n_r}$ is said to be the *base* of B and we write

$$B = \mathrm{csb}(A_{n_1} \times \cdots \times A_{n_r}). \tag{1.2}$$

By definition, λ is the (unique) σ-additive measure defined on the σ-algebra Λ such that for a set B of form (1.2)

$$\lambda B = p_{n_1} A_{n_1} \cdots p_{n_r} A_{n_r}. \tag{1.3}$$

It is well known that such a measure λ does exist, is unique, and $\lambda Z = 1$.

1.1. Remark. A function $F \in \mathbb{C}^Z$ of the form

$$\forall\, z = (z_n)_{n\in\mathbb{N}} \in Z \quad F(z) = \Phi(z_{n_1}, \ldots , z_{n_r})$$

where $n_1 < \cdots n_r$, $\Phi \in \mathbb{C}^{X_{n_1}\times\cdots\times X_{n_r}}$, is said to be *cylindrical*, with the *base* $X_{n_1} \times \cdots \times X_{n_r}$. For such a function

$$\int_Z F\, d\lambda = \sum_{x\in \mathrm{base}\, F} \Phi(x_1, \ldots , x_r) p_{n_1}\{x_1\} \ldots p_{n_r}\{x_r\}. \tag{1.4}$$

Indeed, Z is a disjoint union $Z = \bigcup_{x\in\mathrm{base}\, F} B_x$, where $B_x = \mathrm{csb}(\{x_1\}, \ldots , \{x_r\})$. Now for $z \in Z$ we have $z \in B_x \iff z_{n_1} = x_1, \ldots , z_{n_r} = x_r$. Since F is constant on each B_x, we get $\int_Z F\, d\lambda = \sum_{x\in\mathrm{base}\, F} \Phi(x_1, \ldots , x_r) \int_{B_x} d\lambda$. Hence our assertion follows from (1.3).

1.2. Remark. Note that the above definition determines the base not uniquely. If $X_{n_1} \times \cdots \times X_{n_r}$ is a base of F and $\{n_1, \ldots , n_r\} \subset \{n'_1, \ldots , n'_{r'}\}$, then $X_{n'_1} \times \cdots \times X_{n'_{r'}}$ is also a base of F.

2. Maps Π and Q. At first we define the map Π by

$$\forall\, z \in Z \quad \Pi z = (z_1, \ldots , z_m).$$

Therefore $\Pi \in Y^Z$, i. e. Π transforms $Z = X_1 \times X_2 \times \cdots$ into $Y = X_1 \times \cdots \times X_m$. The pre-image of $y \in Y$ under Π is denoted by Qy:

$$\begin{aligned} \forall\, y \in Y \quad Qy = \{z \in Z :\ \Pi z = y\} = \mathrm{csb}(\{y_1\}, \ldots , \{y_m\}) = \\ \{y_1\} \times \cdots \times \{y_m\} \times X_{m+1} \times X_{m+2} \times \cdots \end{aligned} \tag{2.1}$$

Observe that

$$\forall\, y', y'' \in Y \quad y' \neq y'' \Rightarrow Qy' \cap Qy'' = \varnothing.$$

Therefore, defining

$$\forall\, A \in 2^Y \quad QA = \bigcup_{y\in A} Qy,$$

we find that $QY = Z$, Q preserves the operations $\cup, \cap, \backslash$. Moreover, Q is a boolean isomorphism $2^Y \to \Lambda$ which preserves measure. Namely, from (2.1), (1.3), and (1.1) it follows that

$$\forall\, y \in Y\ \ \lambda Qy = p_y, \quad \forall\, A \in 2^Y\ \ \lambda QA = pA.$$

In particular, $\lambda Z = \lambda QY = pY = 1$. The quadruple (Z, Λ, λ, Q) is said to be a *standard filling* of the set $Y = X_1 \times \cdots \times X_m$. (In our first publications about standard filling we started from Q and then introduced Π (see [7]). It was Y. Feneyrol-Perrin who suggested us to regard Π as the initial step.)

The second step is the extension of Q defined at first on the algebra 2^Y of internal sets $A \subset Y$ to the algebra $\mathbb{C}^Y$ of internal complex-valued function f, defined on Y, by

$$\forall f \in \mathbb{C}^Y \quad \forall z \in Z \quad Qf(z) = f(\Pi z). \tag{2.2}$$

Since $\Pi z = y$ is equivalent to $z \in Qy$, we have

$$\forall f \in \mathbb{C}^Y \quad \forall y \in Y \ \ \forall z \in Qy \quad Qf(z) = f(y). \tag{2.3}$$

Consider a simple *example*. Supposing that $\forall n \in \mathbb{N}\ X_n \subset \mathbb{C}$, define for $y \in Y$, $z \in Z$, $n, s \in \mathbb{N}$, $s \leq m$, $\eta_s(y) = y_s$, $\zeta_n(z) = z_n$. By (2.2) $Q\eta_s(z) = \eta_s(\Pi z) = z_s$, i. e.

$$\forall s \leq m \quad Q\eta_s = \zeta_s. \tag{2.4}$$

A natural extension of the map $\Pi \in Z^Y$ to a map defined on some subset of $\mathbb{C}^Z$ is as follows. Put $H(Y) := L_2(Y, 2^Y, p)$, $H(Z) := L_2(Z, \Lambda, \lambda)$. Consider $H(Y)$ and $H(Z)$ as Hilbert spaces in the usual way. Scalar products and norms in $H(Y)$ and $H(Z)$ are denoted by $(\cdot|\cdot)$ and $\|\cdot\|$:

$$\forall f, g \in H(Y) \quad (f|g) = \int_Y f\overline{g}\, dp = \sum_{y\in Y} f(y)\overline{g(y)}\, p_y,$$
$$\forall F, G \in H(Z) \quad (F|G) = \int_Z F\overline{G}\, d\lambda.$$

A direct verification shows that the operator Q is *isometric* $H(Y) \to H(Z)$:

$$\forall f \in H(Y) \quad \|Qf\| = \|f\|.$$

We define the map $\Pi\colon H(Z) \to H(Y)$ as the Hilbert adjoint to $Q\colon H(Y) \to H(Z)$, i. e.

$$\forall f \in H(Y)\ \forall F \in H(Z) \quad (Qf|F) = (f|\Pi F). \tag{2.5}$$

The explicit formula for Π is

$$\forall F \in H(Z)\ \forall y \in Y\ \ \Pi F(y) = Q^*F(y) = \frac{1}{p_y}\int_{Qy} F\, d\lambda; \tag{2.6}$$

$\Pi F(y)$ is the mean of F on the cylindrical set Qy. The proof is a simple computation based on definition (2.5). An immediate corollary of (2.6) is

$$\Pi Q = \mathbb{I}_{H(Y)}, \tag{2.7}$$

where Q is considered as defined on $H(Y)$ and Π as defined on $H(Z)$.

2.1. Example. Let $F \in \mathbb{C}^Z$ be a cylindrical function of the form

$$\forall z \in Z \quad F(z) := Qf(z) := f(\Pi z),$$

where $f \in H(Y)$. Then

$$\forall y \in Y \quad \forall z \in Qy \quad \Pi F(y) = f(y).$$

It is obvious in view of (2.7). For instance, (see (2.4)) $\forall\, s \leq m\ \Pi\zeta_s = \eta_s$. The case $s > m$ is different. For simplicity let $F(z) = \Phi(z_n)$ for some natural $n > m$ and some $\Phi \in \mathbb{C}^{X_n}$. We claim that

$$\forall\, y \in Y\ \ \Pi F(y) = \int\limits_X \Phi\, dp_n := \sum_{x \in X_n} \Phi(x) p_n\{x\}, \tag{2.8}$$

i. e. $\Pi F(y)$ is (independent of y) the mathematical expectation of the random variable Φ of the probability space $\mathfrak{X}_n = (X_n, 2^{X_n}, p_n)$.

Indeed, let χ_k be the indicator of the one-point set $\{y_k\}$ as a subset of X_k. Then $\Pi F(y) = \frac{1}{p_y} \int_Z \chi_1(z_1) \cdots \chi_m(z_m) \Phi(z_n) \lambda(dz)$. According to (1.4)

$$\Pi F(y) = \frac{1}{p_y} p_1\{y_1\} \cdots p_m\{y_m\} \sum_{x \in X_n} \Phi(x) p_n\{x\}.$$

A particular case of (2.8) is

$$\forall\, n > m \quad \forall\, y \in Y \quad \Pi\zeta_n(y) = \sum_{x \in X_n} x p_n\{x\}.$$

Explicit formula (2.5) implies

$$\forall\, F \in H(Z) \quad \|\Pi F\| \leq \|F\|,$$

i. e. Π is only *unstretching* (whereas Q is isometric).

3. Orthoprojector P. For $Q\colon H(Y) \to H(Z)$ and $\Pi = Q^*$ define

$$P = Q\Pi. \tag{3.1}$$

Since $\Pi Q = \mathbb{I}_{H(Y)}$, we have $P^2 = P$. Besides $P^* = P$, therefore P is an *orthoprojector* of the Hilbert space $H(Z)$. In view of (2.3) and (2.5),

$$\forall\, F \in H(Z)\ \forall\, y \in Y\ \forall\, z \in Qy \quad PF(z) = \frac{1}{p_y} \int\limits_{Qy} F\, d\lambda. \tag{3.2}$$

3.1. Remark. For $F \in H(Z)$ the projection PF is the mathematical expectation of the random variable F with respect to the subalgebra $(Qy)_{y \in Y}$ of the σ-algebra Λ.

3.2. Proposition. *The image $PH(Z)$ of P is the whole $QH(Y)$. Thus for $F \in H(Z)$*

$$PF = F \tag{3.3}$$

if and only if F is a cylindrical function with the base $Y = X_1 \times \cdots \times X_m$, i. e. if and only if there exists a function $f \in H(Y)$ such that

$$\forall\, z \in Z \quad F(z) = Qf(z) = f(\Pi z).$$

Proof. Let $F \in H(Z)$ and (3.3) holds. Define $f = \Pi F$. In view of (3.1), $F = Q\Pi F = Qf$. Conversely, let $F = Qf$ for some $f \in H(Y)$. Then $PF = Q\Pi Q f = Qf = F$. □

3.3. Remark. If $F \in \mathbb{C}^Z$ is a cylindrical function with base X_n for some $n > m$, then PF is a constant function. Indeed, let $F(z) = \Phi(z_n)$, $n > m$, $\Phi \in \mathbb{C}^{X_n}$. By (3.2) we have $\forall\, y \in Y$ $\forall\, z \in Qy$ $PF(z) = \frac{1}{p_y}\int_Z \chi_{Qy}(z')\Phi(z'_n)\lambda(dz')$, where χ_{Qy} is the indicator of Qy considered as a subset of Z. But χ_{Qy} is a cylindrical function: $\forall\, z \in Z$ $\chi_{Qy}(z) = \chi_1(z_1)\cdots\chi_m(z_m)$, where for $k \le m$ χ_k is the indicator of the one-point set $\{y_k\}$ considered as a subset of X_k. Hence, by (1.4), we have

$$PF(z) = \frac{1}{p_y}\sum_{(x,x_n)\in Y\times X_n}\chi_1(x_1)\cdots\chi_m(x_m)\Phi(x_n)p_1\{x_1\}\cdots p_m\{x_m\}p_n\{x_n\}.$$

Therefore,

$$\forall\, z \in Z \quad PF(z) = \int_{X_n}\Phi\, dp_n := \sum_{x_n\in X_n}\Phi(x_n)p_n\{x_n\}.$$

3.4. Corollary. *If a cylindrical function F is standard, then $PF = F$.*

Indeed, the base $X_1\times\cdots\times X_n$ of a cylindrical standard function F is standard, therefore $n < m$ and Proposition 3.2 is applicable (see Remark 1.2).

3.5. Theorem. *The orthoprojector P is an S-unity in the following sense:*

$$\forall\, F \in {}^{\mathrm{st}}H(Z) \quad \|PF - F\| \approx 0.$$

Proof. The set of cylindrical functions is dense in $H(Z)$. (For instance, in the next section we shall describe an orthonormal basis of $H(Z)$ which consists of cylindrical functions.) Let $F \in {}^{\mathrm{st}}H(Z)$, then by transfer there exists a standard sequence $(F_n)_{n\in\mathbb{N}}$ of cylindrical functions F_n such that $\|F_n - F\| \to 0$ as $n \to \infty$. For standard n, F_n is standard, therefore by Corollary 3.4 $PF_n = F_n$. By overspill this holds also for some $n \in \mathbb{N}\setminus{}^{\mathrm{st}}\mathbb{N}$. But for such n $\|F_n - F\| \approx 0$, hence $PF \approx PF_n = F_n \approx F$. □

4. Bases. In each Hilbert space $H_n := L_2(X_n, 2^{X_n}p_n)$ we choose an orthonormal basis

$$(\varepsilon_{n0},\dots,\varepsilon_{nc_n}), \quad 1 + c_n = \mathrm{card}X_n \tag{4.1}$$

such that

$$\forall\, n \in \mathbb{N} \quad \forall\, x \in X_n \quad \varepsilon_{n0}(x) = 1. \tag{4.2}$$

It is essential for us that in view of (4.2)

$$\forall\, j \ne 0 \quad \int_{X_n}\varepsilon_{nj}\, dp_n := \sum_{x\in X_n}\varepsilon_{nj}(x)p_n\{x\} = 0. \tag{4.3}$$

We use (4.1) to construct a basis of the Hilbert space $H(Z)$ as follows.

4.1. Definition. A sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ such that $\forall\, n \in \mathbb{N}$ $\alpha_n \in \{0,\dots,c_n\}$ is called a *multiindex* (for $(X_n)_{n\in\mathbb{N}}$). A multiindex α is said to be (hyper)*finite* iff its "support"

$$\mathbb{N}_\alpha := \{n \in \mathbb{N} :\ \alpha_n \ne 0\}$$

is finite: $\operatorname{card} \mathbb{N}_\alpha \in \mathbb{N}$. The set of finite multiindices is denoted by $\mathcal{A}$. For $\alpha \in \mathcal{A}$ we put[2]

$$\forall\, z \in Z \quad \varepsilon_\alpha(z) := \prod_{n \in \mathbb{N}_\alpha} \varepsilon_{n\alpha_n}(z_n). \tag{4.4}$$

In what follows we assume (without loss of generality) that the sequence $((\varepsilon_{n0}, ..., \varepsilon_{nc_n}))_{n\in\mathbb{N}}$ is standard. Therefore the family $(\varepsilon_\alpha)_{\alpha\in\mathcal{A}}$ is *standard* too.

4.2. Proposition. *$(\varepsilon_\alpha)_{\alpha\in\mathcal{A}}$ is a standard orthonormal basis of the space $H(Z)$.*

Proof. For cylindrical functions F, G of the form $F(z) = F_1(z_1)\cdots F_n(z_n)$, $G(z) = G_1(z_1)\cdots \cdot G_n(z_n)$ we have $(F|G) = (F_1|G_1)\cdots(F_n|G_n)$, where

$$(F_j|G_j) := \int\limits_{X_j} F_j\overline{G_j}\, dp_j = \sum_{x\in X_j} F_j(x)\overline{G_j(x)} p_j\{x\}.$$

Therefore the orthogonality of ε_α's follows from orthogonality of basis (4.1). Thus we ought to check the completeness only. It is sufficient to show that for each cylindrical set $B \in \Lambda$ its indicator χ_B belongs to the linear hull of ε_α's. Let $B = \operatorname{csb}(A_1, \dots, A_r)$, where $A_j \in 2^{X_j}$. Then $\chi_B(z) = \chi_1(z_1)\cdots\chi_r(z_r)$, where χ_j is the indicator of A_j as a subset of X_j. Since $\chi_j \in \operatorname{span}(\varepsilon_{j0}, \dots, \varepsilon_{jc_j})$, we have $\chi_B \in \operatorname{span}\{\varepsilon_\alpha\}_{\alpha\in\mathcal{A}}$. □

4.3. Definition. Denote by $\mathcal{B}$ the set of multiindices β of length m:

$$\beta = (\beta_1, \dots, \beta_m) \quad \beta_j \in \{0, \dots, c_j\}.$$

For $\beta \in \mathcal{B}$ define

$$\forall\, y \in Y \quad \phi_\beta(y) = \varepsilon_{1\beta_1}(y_1)\dots\varepsilon_{m\beta_m}(y_m). \tag{4.5}$$

Evidently, $(\phi_\beta)_{\beta\in\mathcal{B}}$ is an internal orthonormal basis of the space $H(Y)$.

We shall use the letters Π and Q for one more purpose. For $\alpha \in \mathcal{A}$ let $\Pi\alpha$ be the restriction of the sequence α from $\mathbb{N}$ to $\{1, \dots, m\}$ and for $\beta \in \mathcal{B}$ let $Q\beta$ be the extension of β from $\{1, \dots, m\}$ to $\mathbb{N}$ by zeros:

$$\forall\, \alpha \in \mathcal{A} \quad \Pi\alpha := (\alpha_1, \dots, \alpha_m), \quad \forall\, \beta \in \mathcal{B} \quad Q\beta := (\beta_1, \dots, \beta_m, 0, 0, \dots).$$

Thus,

$$\forall\, \alpha \in \mathcal{A} \quad \Pi\alpha \in \mathcal{B}, \quad \forall\, \beta \in \mathcal{B} \quad Q\beta \in \mathcal{A}.$$

4.11. Proposition. *Let $\alpha \in \mathcal{A}$, then*

$$\Pi\varepsilon_\alpha = \begin{cases} \phi_{\Pi\alpha} & \text{for } \alpha \in Q\mathcal{B}, \\ 0 & \text{for } \alpha \in \mathcal{A} \setminus Q\mathcal{B}. \end{cases} \tag{4.6}$$

Let $\beta \in \mathcal{B}$, then

$$Q\phi_\beta = \varepsilon_{Q\beta}. \tag{4.7}$$

[2]This construction is as in [1].

Proof. Let $\alpha \in Q\mathcal{B}$, then $\forall y \in Y\ \forall z \in Qy\ \varepsilon_\alpha(z) = \phi_{\Pi\alpha}$, hence $\Pi\varepsilon_\alpha(y) = \frac{1}{p_y}\int_{Qy} \varepsilon_\alpha\, d\lambda = \phi_{\Pi\alpha}(y)$. And if $\alpha \in \mathcal{A} \setminus Q\mathcal{B}$, then

$$\frac{1}{p_y}\int\limits_{Qy} \varepsilon_\alpha\, d\lambda = \varepsilon_{1,\alpha_1}(y_1)\cdots\varepsilon_{m,\alpha_m}(y_m)\int\limits_{X_{m+1}} \varepsilon_{m+1,\alpha_{m+1}}\, dp_{m+1}\cdots\int\limits_{X_{m+r}} \varepsilon_{m+r,\alpha_{m+r}}\, dp_{m+r}.$$

In view of (4.3) this product is equal to 0, because $\alpha \notin Q\mathcal{B}$ implies that some of the numbers $\alpha_{m+1}, \dots, \alpha_{m+r}$ is different from 0.

Now let $\beta \in \mathcal{B}$. Consider the expansion $Q\phi_\beta = \sum_{\alpha\in\mathcal{A}}(Q\phi_\beta|\varepsilon_\alpha)\varepsilon_\alpha$. Since $(Q\phi_\beta|\varepsilon_\alpha) = (\phi_\beta|Q\varepsilon_\alpha)$, in view of (4.6) $(Q\phi_\beta|\varepsilon_\alpha) = (\phi_\beta|\phi_{\Pi\alpha}) = 1$ if $\Pi\alpha = \beta$ and $= 0$ otherwise. Therefore (4.7) holds. □

4.5. Corollary. *Since* $P = Q\Pi$,

$$P\varepsilon_\alpha = \begin{cases} \varepsilon_\alpha & \text{for } \alpha \in Q\mathcal{B}, \\ 0 & \text{for } \alpha \in \mathcal{A} \setminus Q\mathcal{B}. \end{cases}$$

5. Shadow on $H(Y)$. For a normed space $\mathcal{N}$ we denote by ${}^<\mathcal{N}$ the external set of S-finite vectors $x \in \mathcal{N}$:

$${}^<\mathcal{N} := \{x \in \mathcal{N} :\ \|x\| \ll \infty\}. \tag{5.1}$$

At first observe that

$$\forall f \in {}^<H(Y)\ \ Qf \in {}^<H(Z),\ \ \forall F \in {}^<H(Z)\ \Pi F \in {}^<H(Y), \tag{5.2}$$

because Q is isometric and Π unstretching.

Consider an $f \in {}^<H(Y)$. In view of (5.2), by the Riesz representation theorem, there exists a unique standard $F \in H(Z)$ such that for any standard $G \in H(Z)$ $(Qf|G) \approx (F|G)$. This F is denoted by ${}^\bullet f$ and called the *black shadow* of f. The *(white) shadow* ${}^\circ f$ is defined as ${}^\circ f = \Pi F$. Thus we accept the following definition:

$$\begin{gathered} \forall f \in {}^<H(Y)\ \ {}^\bullet f \in {}^{\mathrm{st}}H(Z) \text{ and} \\ \forall G \in {}^{\mathrm{st}}H(Z)\ \ ({}^\bullet f|G) = {}^\circ(Qf|G),\ \ {}^\circ f = \Pi{}^\bullet f. \end{gathered} \tag{5.3}$$

Note that ${}^\circ f$ is not standard in the usual sense: for elements of $H(Y)$ the standardness notion at present is not defined.

5.1. Remark. Let $f \in {}^<H(Y)$, then $\|{}^\bullet f\| ={}^\circ \|{}^\circ f\|$.

Proof. Since Q is isometric and P is an S-unity, we get $\|{}^\circ f\| = \|\Pi{}^\bullet f\| = \|Q\Pi{}^\bullet f\| = \|P{}^\bullet f\| \approx \|{}^\bullet f\|$. □

Now we shall describe the Fourier expansion of the shadow. For $f \in H(Y)$ and $\beta \in \mathcal{B}$ set $f_\beta := (f|\phi_\beta)$ (see (4.5)). Then

$$f = \sum_{\beta\in\mathcal{B}} f_\beta\phi_\beta, \quad \|f\|^2 = \sum_{\beta\in\mathcal{B}} |f_\beta\|^2.$$

If $f \in {}^<H(Y)$, then $\forall \beta \in \mathcal{B}$ $|f_\beta| \ll \infty$, hence there exists the standard part ${}^\circ f_\beta$ of the complex number f_β. Denote by $(\overset{\circ}{f}_\alpha)_{\alpha \in \mathcal{A}}$ the standard family of complex numbers $\overset{\circ}{f}_\alpha$ such that

$$\forall \alpha \in {}^{\mathrm{st}}\mathcal{A} \quad \overset{\circ}{f}_\alpha = {}^\circ(f_{\Pi\alpha}). \tag{5.4}$$

By standardization principle of IST (see [11]) such a family exists and is unique.

5.2. Proposition. *Let $f \in {}^<H(Y)$ and $(\overset{\circ}{f}_\alpha)_{\alpha \in \mathcal{A}}$ be the standard family, for which (5.4) holds. Then*

$$^\bullet f = \sum_{\alpha \in \mathcal{A}} \overset{\circ}{f}_\alpha\, \varepsilon_\alpha, \quad {}^\circ f = \sum_{\beta \in \mathcal{B}} \overset{\circ}{f}_{Q\beta}\, \phi_\beta. \tag{5.5}$$

In particular,

$$^\bullet\phi_\beta = \begin{cases} \varepsilon_{Q\beta} & \text{for } Q\beta \in {}^{st}\mathcal{A}, \\ 0 & \text{otherwise,} \end{cases}$$

$$^\circ\phi_\beta = \begin{cases} \phi_\beta & \text{for } Q\beta \in {}^{st}\mathcal{A}, \\ 0 & \text{otherwise,} \end{cases} \tag{5.6}$$

Proof. Let $\forall \alpha \in \mathcal{A}$ $c_\alpha = ({}^\bullet f|\varepsilon_\alpha)$, i. e. the Fourier expansion of ${}^\bullet f$ is ${}^\bullet f = \sum_{\alpha \in \mathcal{A}} c_\alpha \varepsilon_\alpha$. Since for $\alpha \in {}^{\mathrm{st}}\mathcal{A}$ $\varepsilon_\alpha \in {}^{\mathrm{st}}H(Z)$, by (5.3) we have $c_\alpha = {}^\circ(Qf|\varepsilon_\alpha) = {}^\circ(f|\Pi\varepsilon_\alpha) = {}^\circ(f|\phi_{\Pi\alpha})$. Therefore $\forall \alpha \in {}^{\mathrm{st}}\mathcal{A}$ $c_\alpha = {}^\circ(f_{\Pi\alpha}) = \overset{\circ}{f}_\alpha$, i. e. the first equality in (5.5) holds. Since ${}^\circ f = \Pi {}^\bullet f$, in view of (4.6) the second one holds too. □

5.3. Remark. The shadow ${}^\bullet f$ of a function $f \in {}^<H(Y)$ can be characterized as the unique standard solution $F \in H(Z)$ of equations

$$\forall \beta \in {}^{\mathrm{st}}\mathcal{B} \quad (F|\varepsilon_{Q\beta}) = {}^\circ(f|\phi_\beta),$$

where $\beta \in {}^{\mathrm{st}}\mathcal{B}$ means that $Q\beta \in {}^{\mathrm{st}}\mathcal{A}$.

5.4. Definition. A function $f \in {}^<H(Y)$ is said to be conventionally standard (*c-standard*) and we write $f \in {}^{\mathrm{st}}H(Y)$) iff $f = {}^\circ f$.

For instance, from (5.6) we see that ϕ_β is c-standard for each $\beta \in {}^{\mathrm{st}}\mathcal{B}$.

5.5. Remark. Remark 5.3 can be reformulated as follows: for $f \in {}^<H(Y)$ the shadow ${}^\circ f$ is the unique c-standard solution $g \in {}^<H(Y)$ of equations

$$\forall \beta \in {}^{\mathrm{st}}\mathcal{B} \quad (g|\phi_\beta) = {}^\circ(f|\phi_\beta). \tag{5.7}$$

5.6. Proposition. *A function $f \in {}^<H(Y)$ is c-standard iff for some $F \in {}^{\mathrm{st}}H(Z)$ $f = \Pi F$. If it is the case, then $F = {}^\bullet f$.*

Proof. If $f \in {}^{\mathrm{st}}H(Z)$, then $f = {}^\circ f = \Pi {}^\bullet f$, therefore $f = \Pi F$ for $F = {}^\bullet f \in {}^{\mathrm{st}}H(Z)$. Conversely, let $F \in {}^{\mathrm{st}}H(Z)$ and $f := \Pi F$. Then $\|f\| \le \|F\| \ll \infty$, therefore $f \in {}^<H(Y)$ and ${}^\bullet f$ is determined. We have $\forall \alpha \in {}^{\mathrm{st}}\mathcal{A}$ $({}^\bullet f|\varepsilon_\alpha) = ({}^\bullet f|Q\phi_{\Pi\alpha}) = (\Pi {}^\bullet f|\phi_{\Pi\alpha}) = ({}^\circ f|\phi_{\Pi\alpha}) \approx (f|\phi_{\Pi\alpha})$ in view of (5.7). Thus $({}^\bullet f|\varepsilon_\alpha) \approx (\Pi F|\phi_{\Pi\alpha}) = (F|Q\phi_{\Pi\alpha}) = (F|\varepsilon_\alpha)$. Hence by transfer $\forall \alpha \in \mathcal{A}$ $({}^\bullet f|\varepsilon_\alpha) = (F|\varepsilon_\alpha)$, whence ${}^\bullet f = F$ and ${}^\circ f = \Pi {}^\bullet f = \Pi F = f$. □

5.7. Definition. A function $f \in {}^{<}H(Y)$ is said to be conventionally nearstandard (*c-nearstandard*, and we write $f \in {}^{\mathrm{ns}}H(Y)$) iff $\|f - {}^{\circ}f\| \approx 0$.

5.8. Lemma. *Let* $f \in {}^{<}H(Y)$, *then*

$$(f|{}^{\circ}f) \approx \|{}^{\circ}f\|^2, \quad \|f - {}^{\circ}f\|^2 \approx \|f\|^2 - \|{}^{\circ}f\|^2. \tag{5.8}$$

Proof. We have $(f|{}^{\circ}f) = (f|\Pi^{\bullet}f) = (Qf|{}^{\bullet}f) \approx \|{}^{\bullet}f\|^2$. By Remark 5.1 this implies (5.8). □

5.9. Corollary.*Let* $f \in {}^{<}H(Y)$, *then*

$$f \in {}^{\mathrm{ns}}H(Y) \quad \Longleftrightarrow \quad \|{}^{\circ}f\| = {}^{\circ}\|f\|. \tag{5.9}$$

Notice that for an arbitrary *standard* Hilbert space H and vector $h \in H$ the equality $\|{}^{\circ}h\| = {}^{\circ}\|h\|$ is necessary and sufficient for a (genuine) nearstandardness of h.

Now we shall expose the criterion of c-nearstandardness in the language of Fourier coefficients. To this end let us introduce a *lexicographical order* in the set $\mathcal{A}$ of finite multiindices. By definition for $\alpha, \alpha' \in \mathcal{A}$ the inequality $\alpha < \alpha'$ means that there is an $n \in \mathbb{N}$ such that $\alpha_k = \alpha'_k$ for $k < n$ but $\alpha_n < \alpha'_n$. For $\beta, \beta' \in \mathcal{B}$ we write $\beta < \beta'$ iff $Q\beta < Q\beta'$.

5.10. Proposition. Let $f \in {}^{<}H(Y)$, then $f \in {}^{\mathrm{ns}}H(Y)$ iff

$$\forall\, \beta \in \mathcal{B} \setminus {}^{\mathrm{st}}\mathcal{B} \quad \sum_{\beta' > \beta} |f_{\beta'}|^2 \approx 0.$$

(Notice that this criterion is a copy of the criterion of nearstandardness of a vector in a standard (in the usual sense) Hilbert space and the proof is almost the same; see, e. g., [4]).

Proof. From Proposition 5.2 it follows that

$$\forall\, f \in {}^{<}H(Y) \;\; \forall\, \alpha \in \mathcal{A} \setminus {}^{\mathrm{st}}\mathcal{A} \quad \sum_{\alpha' > \alpha} |\, \mathring{f}_{\alpha'} \,|^2 \approx 0.$$

Indeed, the remainder of a standard convergent series, which corresponds to nonstandard α, is infinitesimal. By the Robinson lemma there is a $\beta_0 \in \mathcal{B} \setminus {}^{\mathrm{st}}\mathcal{B}$ such that

$$\forall\, \beta < \beta_0 \quad \sum_{\beta' < \beta} |f_{\beta'}|^2 \approx \sum_{\beta' < \beta} |\, \mathring{f}_{\beta'} \,|^2.$$

Therefore if $\beta < \beta_0$ and $\beta \in \mathcal{B} \setminus {}^{\mathrm{st}}\mathcal{B}$, then

$$\|f\|^2 - \|{}^{\circ}f\|^2 \approx \sum_{\beta' > \beta} |f_{\beta'}|^2.$$

Hence our assertion follows from (5.9). □

6. Random variables. A function $f \in H(Y)$ can be considered as a random variable on the probability space $\mathfrak{Y} = (Y, 2^Y, p)$. If $f \in {}^{<}H(Y)$ and therefore has the shadow, it can be regarded also as an (internal) finite-dimensional substitution of a standard infinite-dimensional variable. Let us pass to details.

Denote by Ef the mathematical expectation of $f \in H(Y)$: $Ef := \int_Y f\,dp = \sum_{y\in Y} f(y)p_y$. By the Cauchy-Buniakowsky inequality an S-finite variable has an S-finite mathematical expectation, i. e.,

$$\forall\, f \in {}^{<}H(Y) \quad |Ef| \ll \infty. \tag{6.1}$$

6.1. Remark. Let $f \in {}^{<}H(Y)$, then

$$E({}^\circ f) = {}^\circ(Ef) = \mathcal{E}^\bullet f, \tag{6.2}$$

where $\mathcal{E}F$ denotes $\int_Z F\,d\lambda$ for $F \in L(Z, \Lambda, \lambda)$.

Proof. Denote by $\phi_0 \in \mathbb{C}^Y$ the function $\forall\, y \in Y \;\; \phi_0(y) = 1$. Since ϕ_0 is c-standard ($\phi_0 = \Pi\varepsilon_0$, where $\forall\, z \in Z \;\; \varepsilon_0(z) = 1$), by Remark 5.5 we have $E({}^\circ f) = ({}^\circ f|\phi_0) = {}^\circ(f|\phi_0) = {}^\circ(Ef)$. Besides $E({}^\circ f) = E(\Pi^\bullet f) = \sum_{y\in Y} \frac{1}{p_y} \int_{Qy} {}^\bullet f\,d\lambda\, p_y = \int_Z {}^\bullet f\,d\lambda = \mathcal{E}^\bullet f$. □

The dispersions of $f \in H(Y)$ and $F \in H(Z)$ are denoted by Df and $\mathcal{D}F$ respectively:

$$\begin{aligned} Df &= E|f - Ef|^2 = \|f\|^2 - |Ef|^2, \\ \mathcal{D}F &= \mathcal{E}|F - \mathcal{E}F|^2 = \|F\|^2 - |\mathcal{E}F|^2. \end{aligned} \tag{6.3}$$

From (6.1) one can see that

$$\forall\, f \in {}^{<}H(Y) \quad Df \ll \infty, \quad \forall\, F \in {}^{<}H(Z) \quad \mathcal{D}F \ll \infty.$$

In view of Remarks 5.1 and 6.1 we obtain

$$\forall\, f \in {}^{<}H(Y) \quad \mathcal{D}^\bullet f = {}^\circ(D^\circ f).$$

According to (6.2) and (6.3) $D({}^\circ f) - Df \approx \|f\|^2 - \|{}^\circ f\|^2$. Consequently, from 5.9 we obtain the following

6.2. Proposition. *A random variable $\forall\, f \in {}^{<}H(Y)$ is c-nearstandard iff*

$$D({}^\circ f) = {}^\circ Df.$$

Consider the relation between values of variable $f \in {}^{<}H(Y)$ and values of its shadow. To this end we use the concept of rare set (see e. g. [3] or [10]). Let $(\Omega, \mathfrak{A}, \mu)$ be an internal measure space. An internal or external set $A \subset \Omega$ is said to be *rare* iff for any $\varepsilon \gg 0$ there is a set $A_\varepsilon \subset \mathfrak{A}$ such that $A \subseteq A_\varepsilon$ and $\mu A_\varepsilon < \varepsilon$. Obviously, an internal A is rare iff $\mu A \approx 0$. Using the Čebyshev inequality and a suitable permanence principle, we can prove that

(I) Let $F \in L(\Omega, \mathfrak{A}, \mu)$ and $\int_\Omega |F|\,d\mu \ll \infty$, then the set $\{\omega \in \Omega : \; |F(\omega)| \approx \infty\}$ is rare.

(II) Let $F \in L(\Omega, \mathfrak{A}, \mu)$ and $\int_\Omega |F|\,d\mu \approx 0$, then the set $\{\omega \in \Omega : \; |F(\omega)| \gg 0\}$ is rare.

This implies the following assertion.

6.3. Proposition. *1) Let $f \in {}^{<}H(Y)$, then the set $\{y \in Y : \; |f(y)| \approx \infty\}$ is p-rare.*

2) Let $f \in {}^{ns}H(Y)$, then the set $\{y \in Y : \; |f(y)| \neq ({}^\circ f)(y)\}$ is p-rare.

6.4. Remark. For $f \in {}^{<}H(Y)$ there exists a set $\mathcal{S}_f \subset Z$ such that $Z \setminus \mathcal{S}_f$ is λ-rare and

$$\forall\, y \in Y \;\; \forall\, z \in \mathcal{S}_f \cap Qy \quad ({}^\bullet f)(z) \approx ({}^\circ f)(y).$$

Proof. Since $^\circ f = \Pi^\bullet f$, we have $\|Q^\circ f - {}^\bullet f\| = \|P^\bullet f - {}^\bullet f\| \approx 0$. Therefore $Q^\circ f(z) = {}^\bullet f(z)$ except a λ-rare set of values of $z \in Z$. □

Let V be a random variable on a probability space $(\Omega, \mathfrak{A}, \mu)$. The distribution function of V is denoted by Φ_V:

$$\forall\, \gamma \in \mathbb{R} \quad \Phi_V(\omega) = \mu\{\omega \in \Omega :\ V(\omega) < \gamma\}.$$

Notice that by this definition Φ_V is increasing and left continuous.

6.5. Remark. Let $F \in H(Y)$, then

$$\Phi_{Qf} = \Phi_f. \tag{6.4}$$

Proof. Because Z is a disjunctive (hyper)finite union of Qy's and $Qf(z) = f(y)$ for $z \in Qy$, we have

$$\Phi_{Qf}(\gamma) = \lambda\{z \in Z :\ Qf(z) < \gamma\} = \sum_{f(y)<\gamma} \lambda Qy =$$
$$= \sum_{f(y)<\gamma} p_y = p\{y \in Y :\ f(y) < \gamma\} = \Phi_f(\gamma).$$

□

6.6. Theorem. *Let $f \in H(Y)$ be c-nearstandard. Then for each standard $\gamma \in \mathbb{R}$*

$$\Phi_{\bullet f}(\gamma) \le {}^\circ[\Phi_f(\gamma)] \le \Phi_{\bullet f}(\gamma + 0). \tag{6.5}$$

Proof. For a positive ε denote

$$Z_{f,\varepsilon} := \{z \in Z :\ |Qf(z) - {}^\bullet f(z)| < \varepsilon\}. \tag{6.6}$$

We have $\lambda(Z \setminus Z_{f,\varepsilon}) = \int\limits_{|Qf - {}^\bullet f| \ge \varepsilon} d\lambda \le \frac{1}{\varepsilon} \int\limits_Z |Qf - {}^\bullet f|\, d\lambda$. Therefore, $\lambda(Z \setminus Z_{f,\varepsilon}) < \frac{1}{\varepsilon}\|Qf - {}^\bullet f\|$. Hence

$$\forall\, f \in {}^{\mathrm{ns}}H(Y)\ \ \forall\, \varepsilon \gg 0 \quad \lambda(Z \setminus Z_{f,\varepsilon}) \approx 0.$$

From (6.6) we see that $\forall\, \varepsilon \gg 0$

$${}^\bullet f(z) - \varepsilon < Qf(z) < {}^\bullet f(z) + \varepsilon,$$

except the set $Z \setminus Z_{f,\varepsilon}$ of infinitesimal λ-measure. Thus $\forall^{\mathrm{st}} \varepsilon > 0$

$$\Phi_{\bullet f}(\gamma - \varepsilon) \le {}^\circ[\Phi_{Qf}(\gamma)] \le \Phi_{\bullet f}(\gamma + \varepsilon).$$

Since by transfer this holds for each $\varepsilon > 0$, in view of (6.4) we get (6.5). □

6.7. Corollary. *At each point of continuity of the distribution $\Phi_{\bullet f}$ we have ${}^\circ\phi_f(\gamma) = \Phi_{\bullet f}(\gamma)$.*

7. Example. We shall get a very simple example whenever the measure spaces $\mathfrak{X}_n$ are independent of n, $\mathfrak{X}_n = (X, 2^X, p)$ and, for instance, $X = \{-1, +1\}$, $p\{-1\} = p_-$, $p\{+1\} = p_+$, where p_-, p_+ are standard positive numbers such that $p_- + p_+ = 1$. Keep our previous notation $\mathfrak{Y} = \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_m$, $\mathfrak{Z} = \prod_{n\in\mathbb{N}} \mathfrak{X}_n$, $\eta_n(y) = y_n$, $\zeta_n(z) = z_n$, $y \in Y = X^m$, $z \in Z = X \times X \times \cdots$ and consider the random variable

$$\sigma = s_1\eta_1 + \cdots + s_m\eta_m \quad (\text{i. e. } \sigma(y) = s_1y_1 + \cdots + s_my_m), \tag{7.1}$$

where $s_1, \dots, s_m$ are given positive real numbers; σ is a random walk on the real line $\mathbb{R}$. The walking particle at a moment $k \in \{1, \dots, m\}$ carries out the step $-s_k$ with probability p_- or the step $+s_k$ with probability p_+, and σ is the resulting position after m steps. Let us look whether σ can be considered as a substitution of a standard random walk unbounded in the time.

We shall say that a sequence $(s_n)_{n\le m}$ is *S-convergent* iff

$$s := s_1 + \cdots + s_m \ll \infty \tag{7.2}$$

and

$$\forall\, n \le m \quad s_{n+1} + \cdots + s_m \approx 0 \;\text{ for }\; n \approx \infty. \tag{7.3}$$

We claim that the variable σ is c-nearstandard whenever $(s_n)_{n\le m}$ is S-convergent. Moreover, for S-convergent $(s_n)_{n\le m}$ denote by $(t_n)_{n\in\mathbb{N}}$ the (unique) *standard* sequence such that

$$\forall\, n \in {}^{\mathrm{st}}\mathbb{N} \quad t_n = {}^\circ s_n \tag{7.4}$$

and define

$$\tau = \sum_{n\in\mathbb{N}} t_n\zeta_n, \tag{7.5}$$

where $\forall\, z \in Z$ $\zeta_n(z) = z_n$. Series (7.5) is convergent and ${}^\bullet\sigma = \tau$.

Proof. The two-dimensional standard Hilbert space $H = L_2(X, 2^X, p)$ has an orthonormal basis $(\varepsilon_-, \varepsilon_+)$ such that

$$\forall\, x \in X \;\; \varepsilon_-(x) = 1 \quad \text{and} \quad \varepsilon_+(x) = \begin{cases} -q & \text{for } x = -1, \\ q^{-1} & \text{for } x = +1, \end{cases}$$

where $q := \sqrt{p_+/p_-}$. Proceeding from this, construct a basis $(\phi_\beta)_{\beta\in\mathcal{B}}$ of $H(Y)$ and $(\varepsilon_\alpha)_{\alpha\in\mathcal{A}}$ of $H(Z)$ as it was described above. Then it is easy to check that η_n and ζ_n have Fourier expansions

$$\begin{aligned} \eta_n &= p_+\phi^- + \sqrt{p_+p_-}\,\phi_n^+, \\ \zeta_n &= p_+\varepsilon^- + \sqrt{p_+p_-}\,\varepsilon_n^+, \end{aligned} \tag{7.6}$$

where $\phi^-(y) = \varepsilon_-(y_n)$, $\phi_n^+(y) = \varepsilon_-(z_n)$, $\varepsilon^-(z) = \varepsilon_-(z_n)$, $\varepsilon_n^+(z) = \varepsilon_+(z_n)$; in particular, $\phi^-(y) \equiv 1$, $\varepsilon^-(z) \equiv 1$. Note that ϕ^-, ϕ_n^+ are elements of the basis $(\phi_\beta)_{\beta\in\mathcal{B}}$, and $\varepsilon^-, \varepsilon_n^+$ are elements of the basis $(\varepsilon_\alpha)_{\alpha\in\mathcal{A}}$. (Observe also that (7.6) implies $p_+ = E\eta_n = \|\eta_n\|^2$, $p_+p_- = D\eta_n$, $p_+ = \mathcal{E}\zeta_n = \|\zeta_n\|^2$, $p_+p_- = \mathcal{D}\zeta_n$.) From (7.6) we conclude that the Fourier expansion of σ is

$$\sigma = sp_+\phi^- + \sqrt{p_+p_-}(s_1\phi_1^+ + \cdots + s_m\phi_m^+),$$

where $s = s_1 + \cdots + s_m$. Since $s_1^2 + \cdots + s_m^2 \le (\max_{k\le m} s_k) \cdot s$, from (7.2) it follows that $\|\sigma\|^2 = s^2 + s_2^2 + \cdots + s_m^2$ is S-finite: $\|\sigma\| \ll \infty$. Therefore the shadow ${}^\bullet\sigma$ is determined. By Proposition 5.2 we have

$$^\bullet\sigma = {}^\circ s p_+\varepsilon^- + \sqrt{p_+p_-}\sum_{n\in\mathbb{N}} t_n\varepsilon_n^+, \tag{7.7}$$

where $(t_n)_{n\in\mathbb{N}}$ is the standard sequence determined by condition (7.4). According to Proposition 5.10 σ is c-nearstandard iff for any infinite $n \le m$ $s_{n+1}^2 + \dots + s_m^2 \approx 0$. Evidently, the last follows from (7.3). Thus the c-nearstandardness of σ is proved. It rests to prove that ${}^{\bullet}\sigma = \tau$ where τ is defined by (7.5) . To this end calculate ${}^{\circ}s$. Let S be a standard real such that $s = s_1 + \dots + s_m < S$. Then $\forall n \in {}^{\mathrm{st}}\mathbb{N}$ $t_1 + \dots + t_n = {}^{\circ}s_1 + \dots + {}^{\circ}s_n < S$. By transfer the standard series $\sum\limits_{n\in\mathbb{N}} t_n$ is convergent. By Robinson lemma there is $n \in \mathbb{N} \setminus {}^{\mathrm{st}}\mathbb{N}$ such that $t_1 + \dots + t_n \approx s_1 + \dots + s_n$. But for such n $\sum_{k\ge n} t_k \approx 0$. In view of (7.3) we have $\sum\limits_{n\in\mathbb{N}} t_n \approx s_1 + \dots + s_m = s$. i. e., ${}^{\circ}s = \sum\limits_{n\in\mathbb{N}} t_n$. Therefore by (7.7) ${}^{\bullet}\sigma = \big(\sum\limits_{n\in\mathbb{N}} t_n\big)p_+\varepsilon^- + \sqrt{p_+p_p}\sum\limits_{n\in\mathbb{N}} t_n\varepsilon_n^+ = \sum\limits_{n\in\mathbb{N}} t_n\zeta_n = \tau$ in view of the second equality in (7.6) and definition (7.5) of τ. $\square$

8. Spaces H_w. It may well occurs that for the variable σ defined by (7.1) we have $\|\sigma\| \approx \infty$. Then the shadow of σ is not defined. This can be remedied by use of a weighted scalar product (compare [1] or [2]).

Choose a *standard* function $w\colon \alpha \mapsto w_\alpha$ defined on the set $\mathcal{A}$ of finite multiindices α with positive values w_α. Denote by $H_w(Z)$ the Hilbert space whose elements are (formal) series of the form $F = \sum\limits_{\alpha\in\mathcal{A}} F_\alpha\varepsilon_\alpha$, where $F_\alpha \in \mathbb{C}$ and $(\varepsilon_\alpha)_{\alpha\in\mathcal{A}}$ is the basis of $H(Z)$ constructed above (see (4.4)). By *definition* the scalar product $(\cdot|\cdot)_w$ of $H_w(Z)$ corresponds to the norm $\|\cdot\|_w$:

$$\|F\|_w^2 = \sum_{\alpha\in\mathcal{A}} |F_\alpha|^2 w_\alpha.$$

Note that the Hilbert space $H_w(Z)$ is standard and $(w_\alpha^{-1/2}\varepsilon_\alpha)_{\alpha\in\mathcal{A}}$ is its standard orthonormal basis.

Analogously let $H_w(Y)$ be the Hilbert space with elements that are the sums of the form $f = \sum\limits_{\beta\in\mathcal{B}} f_\beta\phi_\beta$, where $f_\beta \in \mathbb{C}$ and $(\phi_\beta)_{\beta\in\mathcal{B}}$ is defined by (4.5). By *definition* the scalar product $(\cdot|\cdot)_w$ of $H_w(Y)$ corresponds to the norm $\|\cdot\|_w$:

$$\|f\|_w^2 = \sum_{\beta\in\mathcal{B}} |f_\beta|^2 w_{Q\beta}.$$

Note that $H_w(Y)$ is internal and hyperfinite-dimensional and $(w_{Q\beta}^{-1/2}\phi_\beta)_{\beta\in\mathcal{B}}$ is its orthonormal basis. In account of Proposition 4.4 we extend the maps Q and Π as follows.

8.1. Definition. For $f = \sum\limits_{\beta\in\mathcal{B}} f_\beta\phi_\beta \in H_w(Y)$ set $Qf = \sum\limits_{\alpha\in Q\mathcal{B}} f_{\Pi\alpha}\varepsilon_\alpha$, and for $F = \sum\limits_{\alpha\in\mathcal{A}} F_\alpha\varepsilon_\alpha$ set

$$\Pi F = \sum_{\beta\in\mathcal{B}} F_{Q\beta}\phi_\beta. \tag{8.1}$$

8.2. Remark. It is easy to check that $\forall f \in H_w(Y)\ \forall F \in H_w(Z)$

$$\|Qf\|_w = \|f\|_w, \quad \|\Pi F\|_w \le \|F\|_w,$$
$$(Qf|F)_w = (f|\Pi F)_w,$$

i. e. $\Pi\colon H_w(Z) \to H_w(Y)$ is adjoint to $Q\colon H_w(Y) \to H_w(Z)$.

8.3. Remark. For the above defined Q and Π we have $\Pi Q = \mathbb{I}_{H_w(Y)}$. Therefore, setting $P := Q\Pi$, we find that P is the *orthoprojector* $H_w(Z) \to QH_w(Y)$ such that

$$\forall\,\alpha \in \mathcal{A} \quad P\varepsilon_\alpha = \begin{cases} \varepsilon_\alpha & \text{for } \alpha \in Q\mathcal{B}, \\ 0 & \text{for } \alpha \in \mathcal{A} \setminus Q\mathcal{B}. \end{cases}$$

8.4. Remark. Consider an arbitrary $f \in {}^{<}H_w(Y)$. By the same reasoning as in Section 5 there exists a unique standard $F \in H_w(Z)$ such that $\forall\, G \in {}^{\mathrm{st}}H_w(Z)$ $(Qf|G)_w \approx (F|G)$. Therefore we can accept a definition of ${}^{\bullet}f$ and ${}^{\circ}f$ as follows (see (5.3)):

$$\forall\, f \in {}^{<}H_w(Y) \;\; {}^{\bullet}f \in {}^{\mathrm{st}}H_w(Z) \;\text{and}\, \forall\, G \in {}^{\mathrm{st}}H_w(Z) \;\; ({}^{\bullet}f|G) = {}^{\circ}(Qf|G), \;\; {}^{\circ}f = \Pi^{\bullet}f.$$

Now for $f \in {}^{<}H_w(Y)$ we define the *c-standardness* (write $f \in {}^{\mathrm{st}}H_w(Y)$) by $f = {}^{\circ}f$ and the *c-nearstandardness* (write $f \in {}^{\mathrm{ns}}H_w(Y)$) by $\|f - {}^{\circ}f\|_w \approx 0$.

8.5. Proposition. *The operator $\Pi\colon H_w(Z) \to H_w(Y)$ is S-injective and the orthoprojector $P\colon H_w(Z) \to QH_w(Y)$ is an S-unity in the following sense:*

$$\forall\, F \in {}^{st}H_w(Z) \quad \|\Pi F\|_w \approx 0 \;\Rightarrow\; F = 0, \tag{8.2}$$
$$\forall\, F \in {}^{\mathrm{st}}H_w(Z) \quad \|PF - F\| \approx 0. \tag{8.3}$$

Proof. Let $F \in {}^{\mathrm{st}}H_w(Z)$ and $\|\Pi F\|_w \approx 0$. Then (see (8.1)) $\sum_{\beta\in\mathcal{B}} |F_{Q\beta}|^2 w_{Q\beta} \approx 0$. Since the series $\sum_{\alpha\in\mathcal{A}} |F_\alpha|^2 w_\alpha \approx 0$ is standard and convergent, we have $\sum_{\alpha>\tilde{\alpha}} |F_\alpha|^2 w_\alpha \approx 0$, where $\tilde{\alpha}$ is the maximum of $Q\mathcal{B}$ in the sense of the lexicographical order. This implies $\|F\|_w^2 = \sum_{\alpha\in\mathcal{A}} |F_\alpha|^2 w_\alpha \approx 0$. Since F is standard, (8.2) holds.

Now let $F = \sum_{\alpha\in\mathcal{A}} F_\alpha \varepsilon_\alpha \in {}^{\mathrm{st}}H_w(Z)$. Denote $G = PF$ and the expansion of G is $G = \sum_{\alpha\in\mathcal{A}} G_\alpha\varepsilon_\alpha$. We have $G_\alpha = (G|w_\alpha^{-1/2}\varepsilon_\alpha)_w = (PF|w_\alpha^{-1/2}\varepsilon_\alpha) = (F|w_\alpha^{-1/2}P\varepsilon_\alpha)$. For $\alpha \in {}^{\mathrm{st}}\mathcal{A}$ we have $P\varepsilon_\alpha = \varepsilon_\alpha$. By overspill this holds up to $\alpha_0 \in \mathcal{A} \setminus {}^{\mathrm{st}}\mathcal{A}$. Thus $\forall\,\alpha \le \alpha_0$ $G_\alpha = F_\alpha$. Putting $\tilde{F} := \sum_{\alpha\le\alpha_0} F_\alpha\varepsilon_\alpha$, $\tilde{G} := \sum_{\alpha\le\alpha_0} G_\alpha\varepsilon_\alpha$, we get $\tilde{F} = \tilde{G}$. But $\|F - \tilde{F}\|_w \approx 0$ and, since $\|G - \tilde{G}\|_w = \|P(F - \tilde{F})\|_w \le \|F - \tilde{F}\|_w$, we have $\|G - \tilde{G}\|_w \approx 0$. Therefore $G \approx \tilde{G} = \tilde{F} \approx F$, i. e. (8.3) holds. □

8.6. Remark. Proposition 8.5 implies (as it is easy to check) that assertions 5.1, 5.3, 5.5, 5.6, 5.9, and 5.10 retain their validity whenever one replaces H by H_w, $(\cdot|\cdot)$ by $(\cdot|\cdot)_w$, and so on. For instance, the following is true. Let $f := \sum_{\beta\in\mathcal{B}} f_\beta\phi_\beta \in {}^{<}H_w(Y)$, then $f \in {}^{\mathrm{st}}H_w(Y)$ iff $\forall\,\beta \in {}^{\mathrm{st}}\mathcal{B}$ $\sum_{\beta'>\beta} |f_{\beta'}|^2 w_{Q\beta} \approx 0$.

Again consider the variable $\sigma = s_1\eta_1 + \cdots + s_m\eta_m$. Now we reject the restriction that the spaces $\mathfrak{X}_n$ are the same for different n, but assume (as above) that they are finite and the sequence $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ is standard. Define for $x \in X_n$ $\xi_n(x) = x$. Let the expansion of ξ_n in $H_n = L_2(X_n, 2^{X_n}, p_n)$ be $\xi_n = \xi_{n0}\varepsilon_{n0} + \cdots + \xi_{nc_n}\varepsilon_{nc_n}$. Then

$$\eta_n = \xi_{n0}\phi_n^0 + \cdots + \xi_{nc_n}\phi_n^{c_n}, \quad \zeta_n = \xi_{n0}\varepsilon_n^0 + \cdots + \xi_{nc_n}\varepsilon_n^{c_n}, \tag{8.4}$$

where

$$\forall\, y \in Y \;\; \phi_n^j(y) = \varepsilon_{nj}(y_n), \quad \forall\, z \in Z \;\; \varepsilon_n^j(z) = \varepsilon_{nj}(z_n).$$

Note that ϕ_n^j and ε_n^j are elements of bases $(\phi_\beta)_{\beta\in\mathcal{B}}$ and $(\varepsilon_\alpha)_{\alpha\in\mathcal{A}}$ respectively. Namely, $\phi_n^j = \phi_\beta$ for $\beta = (\beta_1, \dots, \beta_m)$ such that $\beta_n = j$ and $\beta_k = 0$ for $k \neq n$. Also $\varepsilon_n^j = \varepsilon_\alpha$ where $\alpha = (\alpha_1, \alpha_2, \dots)$ is such that $\alpha_n = j$ and $\alpha_k = 0$ for $k \neq n$. From (8.4) it follows that

$$\|\eta_n\|_w^2 = \|\xi_n\|_w^2 = |\xi_{n0}|^2 w_n^0 + \cdots + |\xi_{nc_n}|^2 w_n^{c_n}, \tag{8.5}$$

where $w_n^j = w_\alpha$ with the above relation between (n, j) and α. To simplify assume that w_n^j is independent of j, i.e.,

$$\forall\, j \leq c_n \quad w_n^j = w_n,$$

where $(w_n)_{n\in\mathbb{N}}$ is a standard sequence of positive numbers. Denote

$$\forall\, n \in \mathbb{N} \quad \xi_n = +\sqrt{|\xi_{n0}|^2 + \cdots + |\xi_{nc_n}|^2}. \tag{8.18}$$

Then (8.5) becomes

$$\|\eta_n\|_w^2 = \|\zeta_n\|_w^2 = \xi_n^2 w_n.$$

Hence

$$\|\sigma\|_w^2 = \sum_{n\leq m} s_n^2 \xi_n^2 w_n.$$

Now we choose a weight $w = (w_n)_{n\in\mathbb{N}}$ so that $\sum_{n\leq m} s_n^2\xi_n^2 w_n \ll \infty$. Then $\|\sigma\|_w \ll \infty$ and the shadow ${}^\bullet\sigma \in {}^{\mathrm{st}}H_w(Z)$ is determined. The result is almost the same as in Example 7. Namely suppose that $\|\sigma\|_w \ll \infty$. Then $\forall\, n \in \mathbb{N}$ $s_n^2\xi_n^2 w_n \ll \infty$. For standard n numbers ξ_n and w_n are standard. Therefore $\forall\, n \in {}^{\mathrm{st}}\mathbb{N}$ $\xi_n^2 w_n^2 \gg 0$, hence $\forall\, n \in {}^{\mathrm{st}}\mathbb{N}$ $|s_n| \ll \infty$. This allows us to define $(t_n)_{n\in\mathbb{N}}$ as a standard sequence of numbers t_n such that (7.4) holds, i. e. $\forall\, n \in {}^{\mathrm{st}}\mathbb{N}$ $t_n = {}^\circ s_n$. By transfer the standard series $\sum_{n\in\mathbb{N}} t_n^2\xi_n^2 w_n$ is convergent. Accept definition (7.5), i. e. $\tau = \sum_{n\in\mathbb{N}} t_n\zeta_n$. We claim that the equality $\tau = {}^\bullet\sigma$ holds again. For proof define $\sigma' := \sum_{k\leq m} t_k\eta_k$. By Definition 8.1 (of Π) we have $\sigma' = \Pi\tau$. Since $\tau \in {}^{\mathrm{st}}H(Z)$, in view of Proposition 5.6 (it is now also true) we see that $\sigma' \in {}^{\mathrm{st}}H_w(Y)$ and ${}^\bullet\sigma' = \tau$. Let $n \in {}^{\mathrm{st}}\mathbb{N}$, then $(Q\sigma|\varepsilon_n^j)_w = (\sigma|\Pi\varepsilon_n^j)_w = (\sigma|\phi_n^j)_w = \xi_{nj}s_n$ in view of (8.4). Therefore $(Q\sigma|\varepsilon_n^j)_w \approx \xi_{nj}t_n = (Q\sigma'|\varepsilon_n^j) \approx ({}^\bullet\sigma'|\varepsilon_n^j) = (\tau|\varepsilon_n^j)$, hence ${}^\bullet\sigma = {}^\bullet\sigma' = \tau$.

As above, the necessary and sufficient condition for $\sigma \in {}^{\mathrm{ns}}H_w(Z)$ is

$$\forall\, n \leq m \quad n \approx \infty \;\Rightarrow \sum_{n<n'\leq m} s_n^2 w_n \approx 0.$$

This condition ensures that $\|\sigma - \sigma'\|_w \approx 0$.

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Faculty of Mechanics and Mathematics, Lviv National University,
1, Universytetska str., Lviv, 79000, Ukraine
wlanc@litech.lviv.ua, tkudryk@franko.lviv.ua

Received 25.04.2001