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ON A FOURIER PROBLEM FOR COUPLED EVOLUTION SYSTEM OF EQUATIONS WITH TIME DELAYS

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A problem without initial conditions for coupled systems of parabolic-ordinary equations with time delays is investigated. Theorems on a priori estimate, uniqueness, and existence of a solution of such problem are proved. Besides, continuous dependence of the solution on the data-in of this problem is established.

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Исследована задача без начальных условий для систем дифференциальных уравнений, состоящих как из квазилинейных параболических, так и обыкновенных уравнений с опаздыванием. Доказаны теоремы об априорной оценке, единственности и существовании решения такой задачи. Кроме того, установлена неперерывная зависимость решения рассматриваемой задачи от ее исходных данных.

Introduction. Development of science and technique provokes to appearance of complicated mathematical models of real nature. To describe them it is not sufficient to use only the classical theory of differential equations. A lot of physical, biological and ecological processes are described by coupled systems of equations. Such systems consist of subsystems of different types, for example, subsystems of equations of parabolic type and ordinary differential equations [1–5]. Some of these processes are described by equations or systems of equations with time delays. Initial-boundary value problems for coupled systems of equations with time delays have been treated in articles [1, 2]. In this paper we investigate the correctness of Fourier Problem (problem without initial conditions) for such systems. Note that such problem for equations and systems of parabolic type has been considered in [6,7] and others.

Let us introduce notations and notions we need later. Let D be a domain in the space $\mathbb{R}^{n+1}_{x,t}$. Denote by $C^{\alpha,\alpha/2}(\overline{D})$, $C^{\alpha,1+\alpha/2}(\overline{D})$, $C^{2+\alpha,1+\alpha/2}(\overline{D})$, where α is a number from the interval $[0;1]$, the Banach spaces of real-valued functions which together with corresponding derivatives are continuous in $\overline{D}$, if $\alpha = 0$, and Hölder continuous functions in $\overline{D}$ with the exponent α, if $\alpha > 0$ (see definitions in [8], p.16). The norms in these spaces are denoted by $\|\cdot\|^{D}_{\alpha,\alpha/2}$, $\|\cdot\|^{D}_{\alpha,1+\alpha/2}$, and $\|\cdot\|^{D}_{2+\alpha,1+\alpha/2}$, respectively. If D is an unbounded domain then denote by $C^{\alpha,\alpha/2}_{\text{loc}}(\overline{D})$, $C^{\alpha,1+\alpha/2}_{\text{loc}}(\overline{D})$ and $C^{2+\alpha,1+\alpha/2}_{\text{loc}}(\overline{D})$ the spaces of functions

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defined in $\overline{D}$ whose restrictions onto the closure of any bounded subdomain D' of the domain D belong to $C^{\alpha,\alpha/2}(\overline{D'})$, $C^{\alpha,1+\alpha/2}(\overline{D'})$, and $C^{2+\alpha,1+\alpha/2}(\overline{D'})$, respectively ($\alpha \in [0;1]$). Set $C(\overline{D}) \overset{\text{def}}{=} C^{0,0}(\overline{D})$, $C_{\text{loc}}(\overline{D}) \overset{\text{def}}{=} C^{0,0}_{\text{loc}}(\overline{D})$. In the case when Q is the conjugation of the domain D and a part of its boundary we denote by $C^{\alpha,\alpha/2}_{\text{loc}}(Q)$, $C^{\alpha,1+\alpha/2}_{\text{loc}}(Q)$ and $C^{2+\alpha,1+\alpha/2}_{\text{loc}}(Q)$ the spaces of functions whose restrictions onto the closure of arbitrary bounded subdomain D' of the domain D such that $\overline{D'} \subset Q$, belong to the spaces $C^{\alpha,\alpha/2}(\overline{D'})$, $C^{\alpha,1+\alpha/2}(\overline{D'})$ and $C^{2+\alpha,1+\alpha/2}(\overline{D'})$, respectively ($\alpha \in [0;1]$).

The boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n_x$ belongs to the class $C^{2+\alpha}$ if it can be covered by a locally finite family of surfaces such that every of them is given by the equation $x_i = h(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ for some $i \in \{1, ..., n\}$, where $h \in C^{2+\alpha}(\overline{K}), K$ is a domain in a space of the corresponding variables.

For a set W we denote by $[W]^m$, where $m \in \mathbb{N}$, the m-th Cartesian power of W. The notation $w \in [W]^m$ means that $w = \text{col}\,(w_1, ..., w_m)$ is the vector-column with components $w_i \in W,\ i \in \{1, ..., m\}$ (as an exception, we write $\mathbb{R}^m$ instead of $[\mathbb{R}]^m$). Note that if W is also a linear space then $[W]^m$ is a linear space with corresponding linear operations.

Set $|w| = \max\limits_{1\leqslant i\leqslant m} |w_i|$, where $w = \text{col}\,(w_1, ..., w_m) \in \mathbb{R}^m$. Let us write $u < v$ for $u, v \in \mathbb{R}^m$, if $u_i < v_i,\ i \in \{1, ..., m\}$, and an inequality $u \leqslant v$ means that $u_i \leqslant v_i,\ i \in \{1, ..., m\}$.

1. Statement of problem and formulation of main results. Let $Q = \Omega \times (-\infty, T]$, where $0 < T < +\infty$ and Ω be a domain in the space $\mathbb{R}^n_x$ with smooth boundary $\partial\Omega$, $\Sigma = \partial\Omega \times (-\infty, T]$.

Consider Fourier Problem for coupled system of equations with time delays:

$$P_i w(x,t) \equiv \frac{\partial u_i(x,t)}{\partial t} - \sum_{k,l=1}^{n} a_{i,kl}(x,t)\frac{\partial^2 u_i(x,t)}{\partial x_k \partial x_l} + \sum_{k=1}^{n} a_{i,k}(x,t)\frac{\partial u_i(x,t)}{\partial x_k} + a_i(x,t)u_i(x,t) -$$
$$-f_i(x,t,w(x,t),w_\tau(x,t)) = \hat{f}_i(x,t), \qquad (x,t) \in Q, \qquad i \in \{1, ..., M\}, \tag{1}$$
$$G_j w(x,t) \equiv \frac{\partial v_j(x,t)}{\partial t} + c_j(x,t)v_j(x,t) - g_j(x,t,w(x,t),w_\tau(x,t)) =$$
$$= \hat{g}_j(x,t), \qquad (x,t) \in \overline{Q}, \qquad j \in \{1, ..., L\}, \tag{2}$$
$$u_i(x,t) = h_i(x,t), \qquad (x,t) \in \Sigma, \qquad i \in \{1, ..., M\}. \tag{3}$$

Here M, L are arbitrary natural numbers; $w(x,t) = \text{col}(u(x,t), v(x,t)), u(x,t) = \text{col}(u_1(x,t), ..., u_M(x,t)), v(x,t) = \text{col}(v_1(x,t), ..., v_L(x,t))$, $(x,t) \in \overline{Q}$; $\tau \overset{\text{def}}{=} (\tau_1, ..., \tau_M, \tau_{M+1}, ..., \tau_{M+L})$, $\tau_i \geq 0, i \in \{1, ..., M+L\}$; $w_\tau(x,t) = \text{col}(u_\tau(x,t), v_\tau(x,t))$, $u_\tau(x,t) = \text{col}(u_1(x,t-\tau_1), ..., u_M(x,t-\tau_M)), v_\tau(x,t) = \text{col}(v_1(x,t-\tau_{M+1}), ..., v_L(x,t-\tau_{M+L}))$, $(x,t) \in \overline{Q}$; for all $i \in \{1, ..., M\}$ and $j \in \{1, ..., L\}$ $f_i(x,t,\xi,\eta)$ and $g_j(x,t,\xi,\eta)$ are the functions which are defined for $(x,t) \in Q$ and $(x,t) \in \overline{Q}$, respectively, and $(\xi,\eta) \in \mathbb{R}^{2(M+L)}$.

Definition 1. A vector-function $w = \text{col}(u,v)$, where $u = \text{col}(u_1, ..., u_M) \in \left[C^{2,1}_{\text{loc}}(Q) \cap C_{\text{loc}}(\overline{Q})\right]^M, v = \text{col}(v_1, ..., v_L) \in \left[C^{0,1}_{\text{loc}}(\overline{Q})\right]^L$, is called a solution of Problem (1)–(3) if it satisfies the equations of system (1),(2) and boundary condition (3).

We impose the following main conditions on the data-in:

(**A1**) the functions $a_{i,kl}, a_{i,k}, a_i$ are continuous in Q, and the function c_j is continuous in $\overline{Q}$, $i \in \{1, ..., M\},\ j \in \{1, ..., L\},\ \{k,l\} \subset \{1, ..., n\}$;

(**A2**) for any $i \in \{1, ..., M\}$ $\quad a_{i,kl} \equiv a_{i,lk}$, $\quad \{k, l\} \subset \{1, ..., n\}$, moreover, for arbitrary points $(x, t) \in Q$ and for all $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ the following inequality holds

$$\sum_{k,l=1}^{n} a_{i,kl}(x, t)\xi_k\xi_l \geqslant \mu_i(t) \sum_{s=1}^{n} \xi_s^2,$$

where μ_i is a non-negative on $(-\infty, T]$ function;

(**A3**) for any $i \in \{1, ..., M\}$ and $j \in \{1, ..., L\}$ the functions $f_i(x, t, \xi, \eta)$, $(x, t, \xi, \eta) \in Q \times \mathbb{R}^{2(M+L)}$, and $g_j(x, t, \xi, \eta)$, $(x, t, \xi, \eta) \in \overline{Q} \times \mathbb{R}^{2(M+L)}$, are continuous by all their variables, non-decreasing by ξ, η and fulfill Lipschitz condition by these variables, more precisely, there exist non-negative and bounded on Q functions $K^f_{ik}(x, t)$, $L^f_{ik}(x, t)$, $(x, t) \in Q$, $K^g_{jk}(x, t)$, $L^g_{jk}(x, t)$, $(x, t) \in \overline{Q}$, such that for any $i \in \{1, ..., M\}$, $j \in \{1, ..., L\}$

$$\begin{aligned}
&|f_i(x, t, \xi + \beta e_{(k)}, \eta) - f_i(x, t, \xi, \eta)| \leqslant K^f_{ik}(x, t)|\beta|,\\
&|f_i(x, t, \xi, \eta + \beta e_{(k)}) - f_i((x, t, \xi, \eta)| \leqslant L^f_{ik}(x, t)|\beta|,\\
&|g_j(x, t, \xi + \beta e_{(k)}, \eta) - g_j(x, t, \xi, \eta)| \leqslant K^g_{jk}(x, t)|\beta|,\\
&|g_j(x, t, \xi, \eta + \beta e_{(k)}) - g_j(x, t, \xi, \eta)| \leqslant L^g_{jk}(x, t)|\beta|,
\end{aligned}$$

for arbitrary $\{\xi, \eta\} \subset \mathbb{R}^{2(M+L)}$, $\beta \in \mathbb{R}$, $k \in \{1, ..., M + L\}$ (here $e_{(k)} = \mathrm{col}\,(0; \ldots ; 0; 1; 0; \ldots ; 0)$ is a column vector with k-th component 1 and other components 0, $k \in \{1, \ldots, M + L\}$);

(**A4**)

$$\inf_{(x,t)\in Q}(a_i(x, t) - f^*_i(x, t)) \geqslant a_0 > 0, \quad i \in \{1, ..., M\},$$

$$\inf_{(x,t)\in \overline{Q}}(c_j(x, t) - g^*_j(x, t)) \geqslant b_0 > 0, \quad j \in \{1, ..., L\},$$

where a_0, b_0 are constants,

$$f^*_i(x, t) \stackrel{\text{def}}{=} \sum_{k=1}^{M+L} \left[K^f_{ik}(x, t) + L^f_{ik}(x, t)\right], \quad (x, t) \in Q, \quad i \in \{1, ..., M\},$$
$$g^*_j(x, t) \stackrel{\text{def}}{=} \sum_{k=1}^{M+L} \left[K^g_{jk}(x, t) + L^g_{jk}(x, t)\right], \quad (x, t) \in \overline{Q}, \quad j \in \{1, ..., L\};$$

(**A5**) $\hat{f} = \mathrm{col}(\hat{f}_1, ..., \hat{f}_M) \in [C_{\mathrm{loc}}(Q)]^M$, $\hat{g} = \mathrm{col}(\hat{g}_1, ..., \hat{g}_L) \in \left[C_{\mathrm{loc}}(\overline{Q})\right]^L$, $h = \mathrm{col}(h_1, ..., h_M) \in [C_{\mathrm{loc}}(\Sigma)]^M$.

For the convenience of formulating and proving the results, without loss of generality let us make the additional assumption

(**A0**) $f_i(x, t, 0, 0) = 0$, $(x, t) \in Q$, $i \in \{1, ..., M\}$; $g_j(x, t, 0, 0) = 0$, $(x, t) \in \overline{Q}$, $j \in \{1, ..., L\}$.

If Ω is an unbounded domain then we additionally assume

(**A6**) there exists a constant $m^* > 0$ such that $a_{i,kk}(x, t) \leqslant m^*(|x|^2 + 1)$, $|a_{i,k}(x, t)| \leqslant m^* \times$ $\times(|x| + 1)$, $(x, t) \in Q$, $i \in \{1, ..., M\}$, $j \in \{1, ..., L\}$, $\{k, l\} \subset \{1, ..., n\}$.

In the sequel we assume that conditions (**A0**)–(**A5**) (and (**A6**) in the case of unbounded domain Ω) hold.

The main results of the present paper concern the well-posedness of Problem (1)–(3). Before formulating them we introduce some notations and notions.

Let $Pw(x,t) \stackrel{\text{def}}{=} \text{col}\,(P_1w(x,t),...,P_Mw(x,t))$, $(x,t)\in Q$, $Gw(x,t) \stackrel{\text{def}}{=} \text{col}\,(G_1w(x,t),..., G_Lw(x,t))$, $(x,t)\in\overline{Q}$. Then Problem (1)–(3) can be briefly written

$$Pw(x,t)=\hat f(x,t),\ (x,t)\in Q,\ Gw(x,t)=\hat g(x,t),\ (x,t)\in\overline{Q};\ u(x,t)=h(x,t),\ (x,t)\in\Sigma,$$

where $w=\text{col}\,(u,v)\in W_{\text{loc}}(Q)\stackrel{\text{def}}{=}[C^{2,1}_{\text{loc}}(Q)\cap C_{\text{loc}}(\overline{Q})]^M\times[C^{0,1}_{\text{loc}}(\overline{Q})]^L$.

Set $f_0 \stackrel{\text{def}}{=} \max\limits_{1\leqslant i\leqslant M}\sup\limits_{(x,t)\in Q}\sum\limits_{k=1}^{M+L}L^f_{ik}(x,t)$, $g_0 \stackrel{\text{def}}{=} \max\limits_{1\leqslant j\leqslant L}\sup\limits_{(x,t)\in \overline{Q}}\sum\limits_{k=1}^{M+L}L^g_{jk}(x,t)$, $\tau_* \stackrel{\text{def}}{=} \max\limits_{1\leqslant k\leqslant M+L}\tau_k$, and let $\nu_0=\min\{\nu_1,\nu_2\}$, where ν_1 and ν_2 are respectively the solutions of equations

$$a_0-\nu-(e^{\nu\tau_*}-1)f_0=0 \quad\text{and}\quad b_0-\nu-(e^{\nu\tau_*}-1)g_0=0.$$

Because the functions

$$\varphi(\nu)=a_0-\nu-(e^{\nu\tau_*}-1)f_0 \quad\text{and}\quad \psi(\nu)=b_0-\nu-(e^{\nu\tau_*}-1)g_0$$

are continuous and decreasing by variable ν, $\varphi(0)=a_0>0$, $\varphi(a_0)=-(e^{a_0\tau_*}-1)\leqslant 0$, $\psi(0)=b_0$, $\psi(b_0)=-(e^{b_0\tau_*}-1)\leqslant 0$, every of these equations has one positive solution.

Let H be one of the sets Q, $\overline{Q}$ or Σ; m be an arbitrary natural number; ν be an arbitrary real number. Denote $E_\nu(D;m)=\{q\in\left[C_{\text{loc}}(H)\right]^m$: there exists a constant $C=C(q)\geqslant 0$ such that $|q(x,t)|\leqslant Ce^{-\nu t}$, $(x,t)\in H\}$.

Theorem 1. (A priori estimate of the solution) *Let for some* $\nu<\nu_0$ $\hat f\in E_\nu(Q;M)$, $\hat g\in E_\nu(\overline{Q};L)$ *and* $h\in E_\nu(\Sigma;M)$. *Then the solution* w *of Problem (1)-(3) from the class* $E_\nu(\overline{Q};M+L)$ *satisfies the following estimate*

$$|w(x,t)|\leqslant\max\Big\{\sup_{(y,s)\in\Sigma}|h(y,s)e^{\nu s}|,\ \sup_{(y,s)\in Q}\frac{|\hat f(y,s)e^{\nu s}|}{\varphi(\nu)},\ \sup_{(y,s)\in Q}\frac{|\hat g(y,s)e^{\nu s}|}{\psi(\nu)}\Big\}\cdot e^{-\nu t}\equiv M_0e^{-\nu t} \tag{4}$$

for all $(x,t)\in\overline{Q}$.

Theorem 2. (Uniqueness of the solution) *The solution of Problem (1)-(3) from the class* $E_\nu(\overline{Q};M)$, *where* $\nu<\nu_0$, *is unique.*

Let $\alpha\in(0;1]$. We denote by $S^{(\alpha)}$ the space of functions $f(x,t,\xi,\eta)$, $(x,t,\xi,\eta)\in\overline{Q}\times\mathbb{R}^{2(M+L)}$ which fulfill the condition: for an arbitrary compact $B\subset\mathbb{R}^{2(M+L)}$ there exists a constant $K=K(B)\geqslant 0$ such that the inequality

$$|f(x_1,t_1,\xi,\eta)-f(x_2,t_2,\xi,\eta)|\leqslant K\left[|x_1-x_2|^\alpha+|t_1-t_2|^{\alpha/2}\right]$$

holds for any $(x_1,t_1),(x_2,t_2)\in\overline{Q}$ and $(\xi,\eta)\in B$.

Theorem 3. (Existence of the solution) *Assume that for some* $\alpha\in(0;1]$ *and* $\nu<\nu_0$ *the following conditions hold:*

(**B1**) $\{a_{i,kl}, a_{i,k}, a_i, c_j\} \subset C^{\alpha,\alpha/2}(\overline{Q})$, $\partial a_{i,kl}/\partial x_s \in C(\overline{Q})$, $\mu_i(t) \geqslant \mu_0 \equiv const > 0$, $t \in (-\infty, T]$, $i \in \{1,...,M\}$, $j \in \{1,...,L\}$, $\{k,l,s\} \subset \{1,...,n\}$;

(**B2**) $\{f_i^{(\nu,\tau)}, g_j^{(\nu,\tau)}\} \subset S^{(\alpha)}$, $i \in \{1,...,M\}$, $j \in \{1,...,L\}$, *where*

$$f_i^{(\nu,\tau)}(x,t,\xi,\eta) \stackrel{\text{def}}{=} f_i(x,t,\xi^{(\nu,t)},\eta^{(\nu,t,\tau)})e^{\nu t}, \quad g_j^{(\nu,\tau)}(x,t,\xi,\eta) \stackrel{\text{def}}{=} g_j(x,t,\xi^{(\nu,t)},\eta^{(\nu,t,\tau)})e^{\nu t},$$

$(x,t,\xi,\eta) \in \overline{Q} \times \mathbb{R}^{2(M+L)}$, $\xi^{(\nu,t)} = \xi e^{-\nu t}$, $\eta^{(\nu,t,\tau)} = col(\eta_1 e^{-\nu(t-\tau_1)}, ..., \eta_{M+L} e^{-\nu(t-\tau_{M+L})})$;

(**B3**) $\partial\Omega \in C^{2+\alpha}$;

(**B4**) $col(e^{\nu t}\hat{f}, e^{\nu t}\hat{g}) \in \left[C^{\alpha,\alpha/2}(\overline{Q})\right]^{M+L}$, $e^{\nu t}h \in \left[C^{2+\alpha,1+\alpha/2}(\overline{Q})\right]^{M}$.
Then there exists a solution $w = col(u,v)$ *of Problem (1)–(3) and it belongs to the space* $E_\nu(\overline{Q}; M+L) \cap \left([C_{loc}^{2+\alpha,1+\alpha/2}(\overline{Q})]^M \times [C_{loc}^{\alpha,1+\alpha/2}(\overline{Q})]^L\right)$.

Let Π_ν be a space of vector-functions $\operatorname{col}(\hat{f},\hat{g},h)$ such that for an arbitrary ν we have $\operatorname{col}(e^{\nu t}\hat{f}, e^{\nu t}\hat{g}, e^{\nu t}h) \in [C^{\alpha,\alpha/2}(\overline{Q})]^{M+L} \times [C^{2+\alpha,1+\alpha/2}(\overline{Q})]^M$. Assume that conditions (**B1**)–(**B3**) hold. Then for any vector-functions $\operatorname{col}(\hat{f},\hat{g},h) \in \Pi_\nu$, where $\nu < \nu_0$, there exists a unique solution w of Problem (1)–(3) from the class $E_\nu(\overline{Q}; M+L)$. In short, we write this as $w = RS_\nu(\hat{f},\hat{g},h)$.

Theorem 4. (Continuous dependence of solution on data-in) *Let conditions* (**B1**)–(**B3**) *of Theorem 3 be fulfilled. Then for an arbitrary value* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for arbitrary* $\{col(\hat{f}^1,\hat{g}^1,h^1), col(\hat{f}^2,\hat{g}^2,h^2)\} \subset \Pi_\nu$ *satisfying the conditions*

$$\sup_{(x,t)\in Q} |\hat{f}^1(x,t) - \hat{f}^2(x,t)|e^{\nu t} < \delta, \quad \sup_{(x,t)\in\overline{Q}} |\hat{g}^1(x,t) - \hat{g}^2(x,t)|e^{\nu t} < \delta,$$

$$\sup_{(x,t)\in\Sigma} |h^1(x,t) - h^2(x,t)|e^{\nu t} < \delta,$$

the following inequality holds

$$\sup_{(x,t)\in Q} |w^1(x,t) - w^2(x,t)|e^{\nu t} < \varepsilon,$$

where $w^i = RS_\nu(\hat{f}^i,\hat{g}^i,h^i)$, $i \in \{1,2\}$.

2. Auxiliary statements.

Remark 1. Let $\xi^l = \operatorname{col}(\xi_1^l,...,\xi_{M+L}^l) \in \mathbb{R}^{M+L}$, $\eta^l = \operatorname{col}(\eta_1^l,...,\eta_{M+L}^l) \in \mathbb{R}^{M+L}$, $l \in \{1,2\}$. Set $\xi_{(k)}^{1,2} = \operatorname{col}(\xi_1^2,...,\xi_k^2,\xi_{k+1}^1,...,\xi_{M+L}^1)$, $\eta_{(k)}^{1,2} = \operatorname{col}(\eta_1^2,...,\eta_k^2,\eta_{k+1}^1,...,\eta_{M+L}^1)$, $k \in \{1,...,M+L-1\}$, $\xi_{(0)}^{1,2} = \xi^1$, $\xi_{(M+L)}^{1,2} = \xi^2$, $\eta_{(0)}^{1,2} = \eta^1$, $\eta_{(M+L)}^{1,2} = \eta^2$. It is clear that for an arbitrary $i \in \{1,...,M\}$

$$f_i(x,t,\xi^1,\eta^1) - f_i(x,t,\xi^2,\eta^2) = f_i(x,t,\xi^1,\eta^1) - f_i(x,t,\xi^2,\eta^1) + f_i(x,t,\xi^2,\eta^1) - f_i(x,t,\xi^2,\eta^2) =$$

$$= \sum_{k=1}^{M+L} [(f_i(x,t,\xi_{(k-1)}^{1,2},\eta^1) - f_i(x,t,\xi_{(k)}^{1,2},\eta^1)) + (f_i(x,t,\xi^2,\eta_{(k-1)}^{1,2}) - f_i(x,t,\xi^2,\eta_{(k)}^{1,2})] =$$

$$= \sum_{k=1}^{M+L} [\Phi_{ik}(x,t,\xi^1,\xi^2,\eta^1)(\xi_k^1 - \xi_k^2) + \Psi_{ik}(x,t,\xi^2,\eta^1,\eta^2)(\eta_k^1 - \eta_k^2)],$$

where

$$\Phi_{ik}(x,t,\xi^1,\xi^2,\eta^1) \stackrel{\text{def}}{=} \frac{f_i(x,t,\xi^{1,2}_{(k-1)},\eta^1) - f_i(x,t,\xi^{1,2}_{(k)},\eta^1)}{\xi^1_k - \xi^2_k}, \quad \text{if} \quad \xi^1_k \neq \xi^2_k,$$

$\Phi_{ik}(x,t,\xi^1,\xi^2,\eta^1) \stackrel{\text{def}}{=} 0,$ if $\xi^1_k = \xi^2_k,$ and

$$\Psi_{ik}(x,t,\xi^2,\eta^1,\eta^2) = \frac{f_i(x,t,\xi^2,\eta^{1,2}_{(k-1)}) - f_i(x,t,\xi^2,\eta^{1,2}_{(k)})}{\eta^1_k - \eta^2_k}, \quad \text{if} \quad \eta^1_k \neq \eta^2_k,$$

$\Psi_{ik}(x,t,\xi^1,\eta^1,\eta^2) \stackrel{\text{def}}{=} 0,$ if $\eta^1_k = \eta^2_k,$ $k \in \{1,...,M+L\}.$

Similarly, for arbitrary $j \in \{1,...,L\}$ we have

$$g_j(x,t,\xi^1,\eta^1) - g_j(x,t,\xi^2,\eta^2) = \sum_{k=1}^{M+L} [S_{jk}(x,t,\xi^1,\xi^2,\eta^1)(\xi^1_k - \xi^2_k) + \Lambda_{jk}(x,t,\xi^2,\eta^1,\eta^2)(\eta^1_k - \eta^2_k)],$$

where

$$S_{jk}(x,t,\xi^1,\xi^2,\eta^1) \stackrel{\text{def}}{=} \frac{g_j(x,t,\xi^{1,2}_{(k-1)},\eta^1) - g_j(x,t,\xi^{1,2}_{(k)},\eta^1)}{\xi^1_k - \xi^2_k}, \quad \text{if} \quad \xi^1_k \neq \xi^2_k,$$

$S_{jk}(x,t,\xi^1,\xi^2,\eta^1) \stackrel{\text{def}}{=} 0,$ if $\xi^1_k = \xi^2_k,$ and

$$\Lambda_{jk}(x,t,\xi^2,\eta^1,\eta^2) = \frac{g_j(x,t,\xi^2,\eta^{1,2}_{(k-1)}) - g_j(x,t,\xi^1,\eta^{1,2}_{(k)})}{\eta^1_k - \eta^2_k}, \quad \text{if} \quad \eta^1_k \neq \eta^2_k,$$

$\Lambda_{jk}(x,t,\xi^2,\eta^1,\eta^2) \stackrel{\text{def}}{=} 0,$ if $\eta^1_k = \eta^2_k,$ $k \in \{1,...,M+L\}.$

Remark 2. In view of definitions of the functions f_i^* and g_j^* and condition (**A3**), it follows that for any $\{\xi^1,\xi^2,\eta^1,\eta^2\} \subset \mathbb{R}^{M+L}$

$$\sum_{k=1}^{M+L} [\Phi_{ik}(x,t,\xi^1,\xi^2,\eta^1) + \Psi_{ik}(x,t,\xi^2,\eta^1,\eta^2)] \leqslant f_i^*(x,t), \quad (x,t) \in Q, \quad i \in \{1,...,M\},$$

$$\sum_{k=1}^{M+L} [S_{jk}(x,t,\xi^1,\xi^2,\eta^1) + \Lambda_{jk}(x,t,\xi^2,\eta^1,\eta^2)] \leqslant g_j^*(x,t), \quad (x,t) \in \overline{Q}, \quad j \in \{1,...,L\}.$$

Let t_0 be an arbitrary number from the interval $(-\infty,T)$, $Q^0 = \Omega \times (t_0,T]$, $\Omega_0 = \Omega \times \{t_0\} \equiv Q \cap \{t = t_0\}$, $\Sigma^0 = \partial\Omega \times (t_0,T]$. Set for any $k \in \{1,...,M+L\}$ $D^0_k = \overline{\Omega} \times (t_0 - \tau_k, T]$, $G^0_k = \overline{\Omega} \times (t_0 - \tau_k, t_0]$, if $\tau_k > 0$, and $G^0_k = \overline{\Omega_0}$, if $\tau_k = 0$. We denote by W^0 the space of functions $w = \mathrm{col}(u,v) \in \Big(\big[C^{2,1}_{\mathrm{loc}}(Q^0) \cap C(\overline{Q^0})\big]^M \times \big[C^{0,1}(\overline{Q^0}]^L \Big) \cap \big[C_{\mathrm{loc}}(D^0_1) \times C_{\mathrm{loc}}(D^0_2) \times \cdots \times C_{\mathrm{loc}}(D^0_{M+L})\big]$.

We next establish some properties of functions from the space W^0.

Lemma 1. *Let Ω be a bounded domain and vector-functions $\{\tilde{w} = col(\tilde{u},\tilde{v}),\ \hat{w} = col(\hat{u},\hat{v})\} \subset W^0$ satisfy the following inequalities*

$$P\tilde{w}(x,t) < P\hat{w}(x,t), \quad (x,t) \in Q^0, \quad G\tilde{w}(x,t) < G\hat{w}(x,t), \quad (x,t) \in \overline{Q^0}, \tag{5}$$

$$\tilde{u}(x,t) < \hat{u}(x,t), \qquad (x,t) \in \Sigma^0, \tag{6}$$

$$\tilde{u}_i(x,t) < \hat{u}_i(x,t), \quad (x,t) \in G^0_i, \quad i \in \{1,...,M\},$$

$$\tilde{v}_j(x,t) < \hat{v}_j(x,t), \quad (x,t) \in G^0_{M+j}, \quad j \in \{1,...,L\}. \tag{7}$$

Then $\tilde{w}(x,t) < \hat{w}(x,t)$ for all $(x,t) \in \overline{Q^0}$.

Proof. Let us assume the converse. Then in view of (7) there exists $t^* \in (t_0, T]$, the maximum value of the variable t, such that $\tilde{w}(x,t) < \hat{w}(x,t)$ for all $(x,t) \in \overline{Q^0} \cap \{(x,t) : t_0 \leqslant t < t^*\}$, and $x^* \in \overline{\Omega}$ such that $\tilde{u}_\mu(x^*, t^*) = \hat{u}_\mu(x^*, t^*)$ for some $\mu \in \{1, ..., M\}$ or $\tilde{v}_s(x^*, t^*) = \hat{v}_s(x^*, t^*)$ for some $s \in \{1, ..., L\}$.

If $\tilde{u}_\mu(x^*, t^*) = \hat{u}_\mu(x^*, t^*)$, then $(x^*, t^*) \notin \Sigma^0$, because of inequality (6). The difference $\tilde{u}_\mu - \hat{u}_\mu$ in $\overline{Q^0} \cap \{(x,t) : t_0 \leqslant t \leqslant t^*\}$ takes its maximum value at the point $(x^*, t^*) \in Q^0$ and this value equals zero. Thus, using condition (**A2**) we have

$$\partial(\tilde{u}_\mu - \hat{u}_\mu)/\partial t\big|_{(x^*,t^*)} \geqslant 0, \quad \partial(\tilde{u}_\mu - \hat{u}_\mu)/\partial x_m\big|_{(x^*,t^*)} = 0, \quad m \in \{1, ..., n\},$$
$$\sum_{k,l=1}^{n} a_{\mu,kl}(x^*, t^*)\partial^2(\tilde{u}_\mu - \hat{u}_\mu)/\partial x_k \partial x_l\big|_{(x^*,t^*)} \leqslant 0.$$

From this and condition (**A3**) we obtain

$$P_\mu \tilde{w}(x^*, t^*) - P_\mu \hat{w}(x^*, t^*) = \frac{\partial}{\partial t}(\tilde{u}_\mu - \hat{u}_\mu)\Big|_{(x^*,t^*)} - \sum_{k,l=1}^{n} a_{\mu,kl}(x^*, t^*)\frac{\partial^2(\tilde{u}_\mu - \hat{u}_\mu)}{\partial x_k \partial x_l}\Big|_{(x^*,t^*)} +$$
$$+ \sum_{k=1}^{n} a_{\mu,k}(x^*, t^*)\frac{\partial(\tilde{u}_\mu - \hat{u}_\mu)}{\partial x_k}\Big|_{(x^*,t^*)} + a_\mu(x^*, t^*)(\tilde{u}_\mu - \hat{u}_\mu)\Big|_{(x^*,t^*)} -$$
$$-(f_\mu(x^*, t^*, \tilde{w}(x^*, t^*), \tilde{w}_\tau(x^*, t^*)) - f_\mu(x^*, t^*, \hat{w}(x^*, t^*), \hat{w}_\tau(x^*, t^*))) \geqslant$$
$$\geqslant f_\mu(x^*, t^*, \hat{w}(x^*, t^*), \hat{w}_\tau(x^*, t^*)) - f_\mu(x^*, t^*, \tilde{w}(x^*, t^*), \tilde{w}_\tau(x^*, t^*))) \geqslant 0,$$

but this contradicts (5). If $\tilde{v}_s(x^*, t^*) = \hat{v}_s(x^*, t^*)$ then it can be similarly proved that $G_s \tilde{w}(x^*, t^*) \geqslant G_s \hat{w}(x^*, t^*)$, which also contradicts (5). □

Lemma 2. *Assume that all conditions of Lemma 1 hold, but inequalities (5)–(7) are non-strict. Then* $\tilde{w}(x,t) \leqslant \hat{w}(x,t)$ *for all* $(x,t) \in \overline{Q^0}$.

Proof. Let $\theta(\mathrm{w}) = \mathrm{col}(e^{-\mathrm{w}_1}, ..., e^{-\mathrm{w}_{M+L}})$ for an arbitrary $\mathrm{w} = \mathrm{col}(\mathrm{w}_1, ..., \mathrm{w}_{M+L}) \in \mathbb{R}^{2(M+L)}$. Consider an auxiliary vector-function $\hat{w}^\lambda(x,t) = \hat{w}(x,t) + \lambda e^t \theta(0)$, $\lambda > 0$. Using Remarks 1 and 2 and conditions (**A3**), (**A4**) we obtain

$$P_i \hat{w}^\lambda(x,t) = P_i \hat{w}(x,t) + \lambda e^t + \lambda e^t a_i(x,t) -$$
$$-[f_i(x,t,\hat{w}(x,t) + \lambda e^t \theta(0), \hat{w}_\tau(x,t) + \lambda e^t \theta(\tau)) - f_i(x,t,\hat{w}(x,t), \hat{w}_\tau(x,t))] =$$
$$= P_i \hat{w}(x,t) + \lambda e^t + \lambda e^t a_i(x,t) - \sum_{k=1}^{M+L} \big[\Phi_{ik}(x,t,\hat{w}(x,t) + \lambda e^t \theta(0), \hat{w}(x,t), \hat{w}_\tau(x,t) + \lambda e^t \theta(\tau)) +$$
$$+\Psi_{ik}(x,t,\hat{w}(x,t), \hat{w}_\tau(x,t) + \lambda e^t \theta(\tau), \hat{w}_\tau(x,t))e^{-\tau_k}\big]\lambda e^t \geqslant$$
$$\geqslant P_i \hat{w}(x,t) + \lambda e^t + \lambda e^t (a_i(x,t) - f_i^*(x,t)) > P_i \hat{w}(x,t),$$

$(x,t) \in Q^0$, $i \in \{1, ..., M\}$. Similar arguments yield $G\hat{w}^\lambda(x,t) > G\hat{w}(x,t)$, $(x,t) \in \overline{Q^0}$. Since $P\tilde{w}(x,t) \leqslant P\hat{w}(x,t)$, and $P\hat{w}(x,t) < P\hat{w}^\lambda(x,t)$, we have $P\tilde{w}(x,t) < P\hat{w}^\lambda(x,t)$ for $(x,t) \in Q^0$. Similarly $G\tilde{w}(x,t) < G\hat{w}^\lambda(x,t)$, $(x,t) \in \overline{Q^0}$. Using this and the fact that $\tilde{u}_i(x,t) < \hat{u}_i^\lambda(x,t)$, $(x,t) \in \Sigma^0$, $\tilde{u}_i(x,t) < \hat{u}_i^\lambda(x,t)$, $(x,t) \in G_i^0$, $i \in \{1, ..., M\}$, $\tilde{v}_j(x,t) < \hat{v}_j^\lambda(x,t)$, $(x,t) \in G_{M+j}^0$, $j \in \{1, ..., L\}$, in view of Lemma 1 we obtain $\tilde{w}(x,t) < \hat{w}^\lambda(x,t)$, if $(x,t) \in \overline{Q^0}, \lambda > 0$. Since $\lim_{\lambda \to 0} \hat{w}^\lambda(x,t) = \hat{w}(x,t)$, we have $\tilde{w}(x,t) \leqslant \hat{w}(x,t), (x,t) \in \overline{Q^0}$. □

Lemma 3. *Let Ω be an unbounded domain and vector-functions $\{\tilde{w} = col(\tilde{u}, \tilde{v}),\ \hat{w} = col(\hat{u}, \hat{v})\} \subset W^0$ satisfy inequalities*

$$P\tilde{w}(x,t) \leqslant P\hat{w}(x,t), \quad (x,t) \in Q^0, \quad G\tilde{w}(x,t) \leqslant G\hat{w}(x,t), \quad (x,t) \in \overline{Q^0}, \tag{8}$$

$$\tilde{u}(x,t) \leqslant \hat{u}(x,t), \qquad (x,t) \in \Sigma^0, \tag{9}$$

$$\tilde{u}_i(x,t) \leqslant \hat{u}_i(x,t), \quad (x,t) \in G_i^0, \quad i \in \{1,...,M\},$$
$$\tilde{v}_j(x,t) \leqslant \hat{v}_j(x,t), \quad (x,t) \in G_{M+j}^0, \quad j \in \{1,...,L\}. \tag{10}$$

Then $\tilde{w}(x,t) \leqslant \hat{w}(x,t)$ for all $(x,t) \in \overline{Q^0}$.

Proof. Let K be a constant such that $|\tilde{w}(x,t)| \leqslant K$, $|\hat{w}(x,t)| \leqslant K$, $(x,t) \in \overline{Q^0}$. Denote $\Omega_R = \Omega \cap \{x \in \mathbb{R}^n : |x| < R\}$, $Q_R^0 = \Omega_R \times (t_0, T]$, $\Sigma_R^0 = \partial\Omega_R \times (t_0, T]$, $G_{k,R}^0 = \overline{\Omega_R} \times (t_0 - \tau_k, t_0]$, if $\tau_k > 0$, and $G_{k,R}^0 = \overline{\Omega_R} \times \{t_0\}$, if $\tau_k = 0$, $k \in \{1,...,M+L\}$. Let us consider an auxiliary vector-function $\hat{w}^{R,\lambda}(x,t) = \mathrm{col}\,(\hat{u}^{R,\lambda}(x,t), \hat{v}^{R,\lambda}(x,t)) = \hat{w}(x,t) + q^{R,\lambda}(x,t)$ $(\hat{u}^{R,\lambda}(x,t) = \mathrm{col}\,(\hat{u}_1^{R,\lambda}(x,t),...,\hat{u}_M^{R,\lambda}(x,t)),\ \hat{v}^{R,\lambda}(x,t) = \mathrm{col}\,(\hat{v}_1^{R,\lambda}(x,t),...,\hat{v}_L^{R,\lambda}(x,t)))$, where $q^{R,\lambda}(x,t) = \dfrac{2K}{R^2}(|x|^2+1)e^{\lambda(t-t_0)}\theta(0)$, $R > 1$, $\lambda > 8m^*n$ $(\theta(0) = \mathrm{col}\,(1,...,1) \in \mathbb{R}^{M+L})$. Using Remarks 1 and 2 and conditions (**A3**), (**A4**), (**A6**) we obtain

$$\begin{aligned}
P_i\hat{w}^{R,\lambda}(x,t) &= P_i\hat{w}(x,t) + \frac{2K}{R^2}e^{\lambda(t-t_0)}\big[\lambda(|x|^2+1) - 2\sum_{k=1}^{n} a_{i,kk}(x,t) + 2\sum_{k=1}^{n} a_{i,k}(x,t)x_k\big] + \\
&+a_i(x,t)\frac{2K}{R^2}(|x|^2+1)e^{\lambda(t-t_0)} - \big[f_i(x,t,\hat{w}(x,t) + q^{R,\lambda}(x,t), \hat{w}_\tau(x,t) + q_\tau^{R,\lambda}(x,t)) - \\
&-f_i(x,t,\hat{w}(x,t),\hat{w}_\tau(x,t))\big] \geqslant P_i\hat{w}(x,t) + \frac{2K}{R^2}(\lambda - 8m^*n)e^{\lambda(t-t_0)}(|x|^2+1) + (a_i(x,t) - \\
&-\sum_{k=1}^{M+L}\big[\Phi_{ik}(x,t,\hat{w}(x,t) + q^{R,\lambda}(x,t), \hat{w}(x,t), \hat{w}_\tau(x,t) + q_\tau^{R,\lambda}(x,t)) + \\
&+\Psi_{ik}(x,t,\hat{w}(x,t),\hat{w}_\tau(x,t) + q_\tau^{R,\lambda}(x,t),\hat{w}_\tau(x,t))e^{-\lambda\tau_k}\big]\big)\frac{2K}{R^2}(|x|^2+1)e^{\lambda(t-t_0)} \geqslant \\
&\geqslant P_i\hat{w}(x,t) + \frac{2K}{R^2}(\lambda - 8m^*n)e^{\lambda(t-t_0)}(|x|^2+1) + \\
&+(a_i(x,t) - f_i^*(x,t))\frac{2K}{R^2}(|x|^2+1)e^{\lambda(t-t_0)} > P_i\hat{w}(x,t),
\end{aligned}$$

$(x,t) \in Q_R^0$, $i \in \{1,...,M\}$. Here $q_\tau^R(x,t) = \dfrac{2K}{R^2}(|x|^2+1)e^{\lambda(t-t_0)}\theta(\lambda\tau)$. Recall that $\theta(\lambda\tau) = \mathrm{col}(e^{-\lambda\tau_1},...,\ e^{-\lambda\tau_M},...,e^{-\lambda\tau_{M+L}})$. It can be proved similarly that $G\hat{w}^{R,\lambda}(x,t) > G\hat{w}(x,t)$, $(x,t) \in \overline{Q_R^0}$. Thus $P\tilde{w}(x,t) < P\hat{w}^{R,\lambda}(x,t)$, $(x,t) \in Q_R^0$, and $G\tilde{w}(x,t) < G\hat{w}^{R,\lambda}(x,t)$, $(x,t) \in \overline{Q_R^0}$. Since $|\tilde{w}(x,t)| \leqslant K$, $(x,t) \in \overline{Q_R^0}$, we have $\tilde{u}(x,t) \leqslant \hat{u}^{R,\lambda}(x,t)$, $(x,t) \in \Sigma_R^0$. It is obvious that $\tilde{u}_i(x,t) \leqslant \hat{u}_i^{R,\lambda}(x,t)$, $(x,t) \in G_{i,R}^0$, $i \in \{1,...,M\}$, and $\tilde{v}_j(x,t) \leqslant \hat{v}_j^{R,\lambda}(x,t)$, $(x,t) \in G_{M+j,R}^0$, $j \in \{1,...,L\}$. Using Lemma 2 we obtain $\tilde{w}(x,t) \leqslant \hat{w}^{R,\lambda}(x,t)$, $(x,t) \in Q_R^0$, for arbitrary $R > 1$. By letting $R \to +\infty$ in the last inequality we complete the proof. □

Lemma 4. *An arbitrary function* $w = col(u, v) \in W^0$ *satisfies the estimate*

$$|w(x,t)| \leqslant \max\Big\{ \sup_{(y,s)\in\Sigma^0} |u(y,s)|, \max_{i\in\{1,...,M\}} \sup_{(y,s)\in G_i^0} |u_i(y,s)|,$$
$$\max_{j\in\{1,...,L\}} \sup_{(y,s)\in G_{M+j}^0} |v_j(y,s)|, \sup_{(y,s)\in Q^0} \frac{|Pw(y,s)|}{a_0}, \sup_{(y,s)\in\overline{Q^0}} \frac{|Gw(y,s)|}{b_0} \Big\}, \quad (x,t)\in\overline{Q^0}. \tag{11}$$

Proof. Let

$$C = \max\Big\{ \sup_{(y,s)\in\Sigma^0} |u(y,s)|, \max_{i\in\{1,...,M\}} \sup_{(y,s)\in G_i^0} |u_i(y,s)|,$$
$$\max_{j\in\{1,...,L\}} \sup_{(y,s)\in G_{M+j}^0} |v_j(y,s)|, \sup_{(y,s)\in Q^0} \frac{|Pw(y,s)|}{a_0}, \sup_{(y,s)\in\overline{Q^0}} \frac{|Gw(y,s)|}{b_0} \Big\}.$$

Consider the vector-function $\hat{w} = \mathrm{col}\,(\hat{u}, \hat{v})$, where $\hat{u} = \mathrm{col}\,(C, ..., C) \in \mathbb{R}^M, \hat{v} = \mathrm{col}\,(C, ..., C) \in \mathbb{R}^L$, $(x,t)\in\overline{Q}$. Using Remarks 1 and 2 and condition (**A4**) it can be easily proved that

$$P_i\hat{w}(x,t) = a_i(x,t)\cdot C - f_i(x,t,\hat{w}(x,t),\hat{w}_\tau(x,t)) =$$
$$= C\cdot\Big(a_i(x,t) - \frac{f_i(x,t,\hat{w}(x,t),\hat{w}_\tau(x,t)) - f_i(x,t,0,0)}{C}\Big) \geqslant$$
$$\geqslant C(a_i(x,t) - f_i^*(x,t)), \quad (x,t)\in Q^0,\ i\in\{1,...,M\}. \tag{12}$$

By condition (**A4**), inequality (12), and the choice of C it follows that

$$P_i\hat{w}(x,t) \geqslant C\cdot a_0 \geqslant P_i w(x,t), \qquad (x,t)\in Q^0, \quad i\in\{1,...,M\}. \tag{13}$$

It can be similarly proved that

$$G_j\hat{w}(x,t) \geqslant G_j w(x,t), \qquad (x,t)\in\overline{Q^0}, \quad j\in\{1,...,L\}. \tag{14}$$

In view of definition of vector-function $\hat{w} = \mathrm{col}\,(\hat{u},\hat{v})$ we have

$$\hat{u}(x,t) \geqslant u(x,t), \qquad (x,t)\in\Sigma^0, \tag{15}$$

$$\hat{u}_i(x,t) \geqslant u_i(x,t), \quad (x,t)\in G_i^0, \quad i\in\{1,...,M\},$$

$$\hat{v}_j(x,t) \geqslant v_j(x,t), \quad (x,t)\in G_{M+j}^0, \quad j\in\{1,...,L\}. \tag{16}$$

By (13)–(16) in view of Lemmas 2 and 3 we obtain $\hat{w}(x,t) \geqslant w(x,t)$, $(x,t)\in\overline{Q^0}$. It can be similarly proved that $w(x,t) \geqslant -\hat{w}(x,t)$, $(x,t)\in\overline{Q^0}$. Two last inequalities imply estimate (11). □

Remark. If $Pw(x,t) = 0$ and $Gw(x,t) = 0$ then $\sup_{(x,t)\in\overline{Q^0}} |w(x,t)|$ is estimated by $\sup_{(x,t)\in\Sigma^0} |u(x,t)|$, $\max_{i\in\{1,...,M\}} \sup_{(x,t)\in G_i^0} |u_i(x,t)|$ and $\max_{j\in\{1,...,L\}} \sup_{(x,t)\in G_{M+j}^0} |v_j(x,t)|$ and independent of a_0 and b_0.

Lemma 5. *Let for some* $\nu < \nu_0$ *vector-functions* $w^1 = col(u^1, v^1)$, $w^2 = col(u^2, v^2)\} \subset E_\nu(\overline{Q}; M+L) \cap W_{loc}(Q)$ *be such that* $Pw^1 - Pw^2 \in E_\nu(Q; M)$, $Gw^1 - Gw^2 \in E_\nu(\overline{Q}; L)$, $(u^1 - u^2)\big|_\Sigma \in E_\nu(\Sigma; M)$. *Then*

$$|w^1(x,t) - w^2(x,t)| \leqslant \max\Big\{ \sup_{(y,s)\in\Sigma} |u^1(y,s) - u^2(y,s)| e^{\nu s},$$
$$\sup_{(y,s)\in Q} \frac{|Pw^1(y,s) - Pw^2(y,s)| e^{\nu s}}{\varphi(\nu)}, \sup_{(y,s)\in\overline{Q}} \frac{|Gw^1(y,s) - Gw^2(y,s)| e^{\nu s}}{\psi(\nu)} \Big\} e^{-\nu t}, \ (x,t) \in Q. \quad (17)$$

Proof. Denote $\hat{f}^k(x,t) \stackrel{\text{def}}{=} Pw^k(x,t)$, $(x,t) \in Q$, $\hat{g}^k(x,t) \stackrel{\text{def}}{=} Gw^k(x,t)$, $(x,t) \in \overline{Q}$, $h^k(x,t) \stackrel{\text{def}}{=} u^k(x,t)$, $(x,t) \in \Sigma$, $k \in \{1,2\}$. Having set $u^{1,2} \stackrel{\text{def}}{=} u^1 - u^2$, $v^{1,2} \stackrel{\text{def}}{=} v^1 - v^2$, $w^{1,2} \stackrel{\text{def}}{=} w^1 - w^2$, by equations (1), (2) and condition (3) for w^1 and w^2, using Remark 1 we obtain

$$L_i w^{1,2}(x,t) - \tilde{f}_i(x,t,w^{1,2}(x,t), w^{1,2}_\tau(x,t)) = \hat{f}^{1,2}_i(x,t), \ (x,t) \in Q, \ i \in \{1,...,M\}, \quad (18)$$

$$F_j w^{1,2}(x,t) - \tilde{g}_j(x,t,w^{1,2}(x,t), w^{1,2}_\tau(x,t)) = \hat{g}^{1,2}_j(x,t), \ (x,t) \in \overline{Q}, \ j \in \{1,...,L\}, \quad (19)$$

$$u^{1,2}_i(x,t) = h^{1,2}_i(x,t), \quad (x,t) \in \Sigma, \quad i \in \{1,...,M\}, \quad (20)$$

where $L_i w^{1,2}(x,t) \stackrel{\text{def}}{=} P_i w^{1,2}(x,t) + f_i(x,t,w^{1,2}(x,t), w^{1,2}_\tau(x,t))$, $(x,t) \in Q$, $i \in \{1,...,M\}$, $F_j w^{1,2}(x,t) \stackrel{\text{def}}{=} G_j w^{1,2}(x,t) + g_j(x,t,w^{1,2}(x,t), w^{1,2}_\tau(x,t))$, $(x,t) \in \overline{Q}$, $j \in \{1,...,L\}$;

$$\tilde{f}_i(x,t,\xi,\eta) \stackrel{\text{def}}{=} \sum_{k=1}^{M+L} \big[\Phi_{ik}(x,t,w^1(x,t),w^2(x,t),w^1_\tau(x,t))\xi_k +$$
$$+\Psi_{ik}(x,t,w^2(x,t),w^1_\tau(x,t),w^2_\tau(x,t))\eta_k\big], \ (x,t,\xi,\eta) \in Q \times \mathbb{R}^{2(M+L)}, \ i \in \{1,...,M\},$$

$$\tilde{g}_j(x,t,\xi,\eta) \stackrel{\text{def}}{=} \sum_{k=1}^{M+L} \big[S_{jk}(x,t,w^1(x,t),w^2(x,t),w^1_\tau(x,t))\xi_k +$$
$$+\Lambda_{jk}(x,t,w^2(x,t),w^1_\tau(x,t),w^2_\tau(x,t))\eta_k\big], \ (x,t,\xi,\eta) \in \overline{Q} \times \mathbb{R}^{2(M+L)}, \ j \in \{1,...,L\};$$

$$\hat{f}^{1,2}(x,t) \stackrel{\text{def}}{=} \hat{f}^1(x,t) - \hat{f}^2(x,t), \ (x,t) \in Q; \ \hat{g}^{1,2}(x,t) \stackrel{\text{def}}{=} \hat{g}^1(x,t) - \hat{g}^2(x,t), \ (x,t) \in \overline{Q};$$

$$h^{1,2}(x,t) \stackrel{\text{def}}{=} h^1(x,t) - h^2(x,t), \quad (x,t) \in \Sigma.$$

First we consider the case when $\nu = 0$ and use an idea from article [6]. Let λ be an arbitrary number for now. Multiply (18), (19) and (20) by $e^{\lambda t}$. After simple transformations we obtain

$$P^\lambda_i \tilde{w}^{1,2}(x,t) \equiv L_i \tilde{w}^{1,2}(x,t) - \lambda \tilde{u}^{1,2}_i(x,t) -$$
$$-\tilde{f}^\lambda_i(x,t,\tilde{w}^{1,2}(x,t), \tilde{w}^{1,2}_\tau(x,t)) = \hat{f}^{1,2}_i(x,t) e^{\lambda t}, \quad (x,t) \in Q, \quad i \in \{1,...,M\}, \quad (21)$$

$$G^\lambda_j \tilde{w}^{1,2}(x,t) \equiv F_j \tilde{w}^{1,2}(x,t) - \lambda \tilde{v}^{1,2}_j(x,t) -$$
$$-\tilde{g}^\lambda_j(x,t,\tilde{w}^{1,2}(x,t), \tilde{w}^{1,2}_\tau(x,t)) = \hat{g}^{1,2}_j(x,t) e^{\lambda t}, \quad (x,t) \in \overline{Q}, \quad j \in \{1,...,L\}, \quad (22)$$

$$\tilde{u}^{1,2}_i(x,t) = h^{1,2}_i(x,t) e^{\lambda t}, \quad (x,t) \in \Sigma, \quad i \in \{1,...,M\},$$

where $\tilde{w}^{1,2}(x,t) = \mathrm{col}\,(\tilde{u}^{1,2}(x,t), \tilde{v}^{1,2}(x,t)) \stackrel{\text{def}}{=} \mathrm{col}\,(u^{1,2}(x,t)e^{\lambda t}, v^{1,2}(x,t)e^{\lambda t})$, $(x,t) \in Q$, and

$$\tilde{f}^\lambda_i(x,t,\xi,\eta) \stackrel{\text{def}}{=} \sum_{k=1}^{M+L} \big[\tilde{K}^f_{ik}(x,t)\xi_k + \tilde{L}^f_{ik}(x,t)\eta_k\big], \quad (x,t,\xi,\eta) \in Q \times \mathbb{R}^{2(M+L)};$$

$$\tilde{g}^\lambda_j(x,t,\xi,\eta) \stackrel{\text{def}}{=} \sum_{k=1}^{M+L} \big[\tilde{K}^g_{jk}(x,t)\xi_k + \tilde{L}^g_{jk}(x,t)\eta_k\big], \quad (x,t,\xi,\eta) \in \overline{Q} \times \mathbb{R}^{2(M+L)},$$

where

$$\tilde{K}^f_{ik}(x,t) \stackrel{\text{def}}{=} \Phi_{ik}(x,t,w^1(x,t),w^2(x,t),w^1_\tau(x,t)), \quad (x,t) \in Q;$$

$$\tilde{L}^f_{ik}(x,t) \stackrel{\text{def}}{=} \Psi_{ik}(x,t,w^2(x,t),w^1_\tau(x,t),w^2_\tau(x,t))e^{\lambda\tau_k}, \quad (x,t) \in Q;$$

$$\tilde{K}^g_{jk}(x,t) \stackrel{\text{def}}{=} S_{jk}(x,t,w^1(x,t),w^2(x,t),w^1_\tau(x,t)), \quad (x,t) \in Q;$$

$$\tilde{L}^g_{jk}(x,t) \stackrel{\text{def}}{=} \Lambda_{jk}(x,t,w^2(x,t),w^1_\tau(x,t),w^2_\tau(x,t))e^{\lambda\tau_k}, \quad (x,t) \in Q.$$

It is easily seen that the coefficients of the differential operators P^λ_i, $i \in \{1,...,M\}$, G^λ_j, $j \in \{1,...,L\}$, satisfy the conditions analogous to conditions (**A1**)–(**A3**) for the coefficients of the operators P_i, $i \in \{1,...,M\}$, G_j, $j \in \{1,...,L\}$. Let us ensure that if $\lambda \in (0,\nu_0)$ then a condition analogous to condition (**A4**) holds.

In view of condition (**A3**) and definition ν_0 we have

$$a_i(x,t) - \lambda - \sum_{k=1}^{M+L}[\tilde{K}^f_{ik}(x,t)+\tilde{L}^f_{ik}(x,t)] \geqslant a_i(x,t) - \lambda - \sum_{k=1}^{M+L}\left[K^f_{ik}(x,t)+L^f_{ik}(x,t)e^{\lambda\tau_k}\right] \geqslant$$

$$\geqslant \inf_{(x,t)\in Q}\big(a_i(x,t)-f^*(x,t)\big) - \lambda - \sup_{(x,t)\in Q}\sum_{k=1}^{M+L}(e^{\lambda\tau_k}-1)L^f_{ik}(x,t) \geqslant$$

$$\geqslant a_0 - \lambda - (e^{\lambda\tau_*}-1)f_0 > 0, \quad i \in \{1,...,M\},$$

$$c_j(x,t) - \lambda - \sum_{k=1}^{M+L}\left[\tilde{K}^g_{jk}(x,t)+\tilde{L}^g_{jk}(x,t)\right] \geqslant c_j(x,t) - \lambda - \sum_{k=1}^{M+L}\left[K^g_{jk}(x,t)+L^g_{jk}(x,t)e^{\lambda\tau_k}\right] \geqslant$$

$$\geqslant \inf_{(x,t)\in \overline{Q}}\big(c_j(x,t)-g^*(x,t)\big) - \lambda - \sup_{(x,t)\in Q}\sum_{k=1}^{M+L}(e^{\lambda\tau_k}-1)\sup_{(x,t)\in Q}L^g_{jk}(x,t) \geqslant$$

$$\geqslant b_0 - \lambda - (e^{\lambda\tau_*}-1)g_0 > 0, \quad j \in \{1,...,L\}.$$

Hence it is clear that the condition analogous to condition (**A4**) is satisfied (with $\varphi(\lambda)$ and $\psi(\lambda)$ instead of $a_0 > 0$ and $b_0 > 0$, respectively).

Let t_* be an arbitrary negative number, $Q^* = \Omega \times (t_*,T]$, $\Sigma^* = \partial\Omega \times (t_*,T], G^*_k = \overline{\Omega} \times (t_* - \tau_k, t_*]$, if $\tau_k > 0$, and $G^*_k = \overline{\Omega} \times \{t_*\}$, if $\tau_k = 0$, $k \in \{1,...,M+L\}$. Using above mentioned arguments in the same way as in the proof of Lemma 4 we obtain for $(x,t) \in \overline{Q^*}$:

$$|\tilde{w}^{1,2}(x,t)| \leqslant \max\Big\{e^{\lambda T}\cdot \sup_{(y,s)\in\Sigma^*}|u^{1,2}(y,s)|,\ e^{\lambda t_*}\cdot \max_{i\in\{1,...,M\}}\sup_{(y,s)\in G^*_i}|u^{1,2}_i(y,s)|,$$

$$e^{\lambda t_*}\cdot \max_{j\in\{1,...,L\}}\sup_{(y,s)\in G^*_{M+j}}|v^{1,2}_j(y,s)|,\ \frac{e^{\lambda T}}{\varphi(\lambda)}\cdot \sup_{(y,\tau)\in Q^*}|\hat{f}^{1,2}(y,s)|, \frac{e^{\lambda T}}{\psi(\lambda)}\cdot \sup_{(y,s)\in Q^*}|\hat{g}^{1,2}(y,s)|\Big\}. \tag{24}$$

Since w^1, $w^2 \in E_0(\overline{Q}; M+L)$, we have $|w^{1,2}(x,t)| \leqslant C_1$ for all $(x,t) \in \overline{Q}$, where $C_1 \geqslant 0$ is a constant. It leads to the fact that $e^{\lambda t_*}\cdot \max_{i\in\{1,...,M\}}\sup_{(y,s)\in G^*_i}|u^{1,2}_i(y,s)| \to 0$ and $e^{\lambda t_*}\cdot$

$\max\limits_{j\in\{1,...,L\}}\sup\limits_{(y,s)\in G^*_{M+j}}|v^{1,2}_j(y,s)|\to 0$ as $t_*\to-\infty$. Hence letting $t_*\to-\infty$ in (24) we obtain

$$|w^{1,2}(x,t)|\leqslant\max\{e^{\lambda(T-t)}\cdot\sup_{(y,s)\in\Sigma}|u^{1,2}(y,s)|,\quad \frac{e^{\lambda(T-t)}}{\varphi(\lambda)}\cdot\sup_{(y,s)\in Q}|\hat f^{1,2}(y,s)|,$$
$$\frac{e^{\lambda(T-t)}}{\psi(\lambda)}\cdot\sup_{(y,s)\in Q}|\hat g^{1,2}(y,s)|\},\quad (x,t)\in Q. \tag{25}$$

For every fixed point $(x,t)\in Q$ by letting $\lambda\to+0$ in (25) we obtain (17) for $\nu=0$.

Let now ν be an arbitrary number, $\nu<\nu_0$, $\nu\neq 0$. Multiply (18), (19) and (20) by $e^{\nu t}$. After simple transformations we obtain (cf. (21)–(23))

$$P^\nu_i\hat w^{1,2}(x,t)=\hat f^{1,2}_i(x,t)e^{\nu t},\ \ (x,t)\in Q,\quad i\in\{1,...,M\},$$

$$G^\nu_j\hat w^{1,2}(x,t)=\hat g^{1,2}_j(x,t)e^{\nu t},\quad (x,t)\in\overline{Q},\quad j\in\{1,...,L\},$$

$$\hat u^{1,2}_i(x,t)=h^{1,2}_i(x,t),\quad (x,t)\in\Sigma,\quad i\in\{1,...,M\},$$

where $\hat w^{1,2}(x,t)=\operatorname{col}(\hat u^{1,2}(x,t),\hat v^{1,2}(x,t))\overset{\text{def}}{=}\operatorname{col}(u^{1,2}(x,t)e^{\nu t},v^{1,2}(x,t)e^{\nu t})$, $(x,t)\in Q$. It is seen that the coefficients of differential operators P^ν_i, $i\in\{1,...,M\}$, G^ν_j, $j\in\{1,...,L\}$, satisfy the conditions analogous to conditions (**A1**)-(**A4**) for the coefficients of the operators P_i, $i\in\{1,...,M\}$, G_j, $j\in\{1,...,L\}$, (with $\varphi(\nu)>0$ and $\psi(\nu)>0$ instead of $a_0>0$ and $b_0>0$). It is obvious that $\hat w^{1,2}\in E_0(\overline{Q};M+L)$. The proof for $\nu=0$ yields

$$|\hat w^{1,2}(x,t)|\leqslant\max\Big\{\sup_{(y,s)\in\Sigma}|u^{1,2}(y,s)e^{\nu s}|,\ \sup_{(y,s)\in Q}\frac{|\hat f^{1,2}(y,s)e^{\nu s}|}{\varphi(\nu)},$$
$$\sup_{(y,s)\in Q}\frac{|\hat g^{1,2}(y,s)e^{\nu s}|}{\psi(\nu)}\Big\},\quad (x,t)\in Q.$$

This leads to (17). □

3. Proofs of main results.

Proof of Theorem 1. Let $\hat w=\operatorname{col}(0,...,0)\in\mathbb{R}^{M+L}$, and w be the solution of Problem (1)–(3). It is clear that $P\hat w=0$ and $G\hat w=0$. Using Lemma 5 for vector-functions w and $\hat w$ we complete the proof. □

Proof of Theorem 2. Assume that there exist two solutions $w^1=\operatorname{col}(u^1,v^1)$ and $w^2=\operatorname{col}(u^2,v^2)$ of Problem (1)–(3). This gives $u^1(x,t)=u^2(x,t)$, if $(x,t)\in\Sigma$, $Pw^1(x,t)=Pw^2(x,t)$, if $(x,t)\in Q$, and $Gw^1(x,t)=Gw^2(x,t)$, if $(x,t)\in\overline{Q}$. Hence in view of Lemma 5 it follows that $w^1(x,t)=w^2(x,t)$, $(x,t)\in\overline{Q}$. □

Proof of Theorem 3. First consider the case $\nu=0$. For every $k\in\mathbb{N}$ we denote $Q^k=Q\cap\{(x,t):t>-k\}$, $\Sigma^k=\Sigma\cap\{(x,t):t>-k\}$, $G^k_l=\overline{\Omega}\times(-k-\tau_l,-k]$, if $\tau_l>0$,

$G_l^k = \overline{\Omega} \times \{-k\}$, if $\tau_l = 0,\ l \in \{1, ..., M + L\}$. Define the vector-function $w^k = \mathrm{col}\,(u^k, v^k)$ as a solution of problem

$$P_i w^k(x,t) = \hat{f}_i^k(x,t), \qquad (x,t) \in Q^k, \qquad i \in \{1, ..., M\}, \tag{26}$$

$$G_j w^k(x,t) = \hat{g}_j^k(x,t), \qquad (x,t) \in \overline{Q^k}, \qquad j \in \{1, ..., L\}, \tag{27}$$

$$u_i^k(x,t) = h_i^k(x,t), \qquad (x,t) \in \Sigma^k, \qquad i \in \{1, ..., M\}, \tag{28}$$

$$\begin{gathered} u_i^k(x,t) = 0, \quad (x,t) \in G_i^k, \quad i \in \{1, ..., M\}, \\ v_j^k(x,t) = 0, \quad (x,t) \in G_{M+j}^k, \quad j \in \{1, ..., L\}. \end{gathered} \tag{29}$$

Here $h^k(x,t) = \zeta(t+k)h(x,t)$, if $(x,t) \in \Sigma$, $\hat{f}^k(x,t) = \hat{f}(x,t)\zeta(t+k)$, $\hat{g}^k(x,t) = \hat{g}(x,t)\zeta(t+k)$, if $(x,t) \in \overline{Q}$, where ζ is a smooth and monotonic on $\mathbb{R}$ function such that $\zeta(t) = 0$ for $t \leqslant 1/2$, $\zeta(t) = 1$ for $t \geqslant 1$.

It follows from the results of papers [1, 2] that for every $k \in \mathbb{N}$ problem (26)–(29) has a unique solution $w^k \in [C^{2+\alpha,1+\alpha/2}(\overline{Q^k})]^M \times [C^{\alpha,1+\alpha/2}(\overline{Q^k})]^L$, moreover, (based on Lemma 4) $w^k(x,t) = 0$ for all $(x,t) \in \overline{\Omega} \times (-k, -k+1/2)$. Let us extend u^k and v^k by zero on $\overline{Q} \setminus \overline{Q^k}$ and denote these extensions again by u^k and v^k ($k \in \mathbb{N}$). It is obvious that $w^k = \mathrm{col}\,(u^k, v^k) \in [C^{2+\alpha,1+\alpha/2}(\overline{Q})]^M \times [C^{\alpha,1+\alpha/2}(\overline{Q})]^L$, $w^k = RS_0(\hat{f}^k, \hat{g}^k, h^k)$, $k \in \mathbb{N}$. Moreover, according to Theorem 1 we have

$$|w^k(x,t)| \leqslant M_0, \quad (x,t) \in \overline{Q}, \quad k \in \mathbb{N}. \tag{30}$$

We next show that

$$\sum_{i=1}^{M} ||u_i^k||^Q_{2+\alpha,1+\alpha/2} + \sum_{j=1}^{L} ||v_j^k||^Q_{\alpha,1+\alpha/2} \leqslant C_2, \quad k \in \mathbb{N}, \tag{31}$$

where $C_2 > 0$ is a constant independent of k.

We rewrite (26) in the following way

$$\begin{gathered} \frac{\partial u_i^k(x,t)}{\partial t} - \sum_{m,l=1}^{n} a_{i,ml}(x,t) \frac{\partial^2 u_i^k(x,t)}{\partial x_m \partial x_l} + \sum_{m=1}^{n} a_{i,m}(x,t) \frac{\partial u_i^k(x,t)}{\partial x_m} + a_i(x,t) u_i^k(x,t) = \\ = f_i(x,t,w^k(x,t), w_\tau^k(x,t)) + \hat{f}_i^k(x,t), \qquad (x,t) \in Q^k, \qquad i \in \{1, ..., M\}, \end{gathered} \tag{32}$$

and consider the right-hand member to be an absolute term. By the conditions of theorem it follows from (30) that the right-hand side of (32) is a continuous and bounded on $\overline{Q}$ function. In view of Theorem 3.1 of the monograph [8; p. 665] for system (32) we obtain

$$\sum_{i=1}^{M} ||u_i^k||^Q_{\alpha,\alpha/2} \leqslant C_3, \tag{33}$$

where $C_3 > 0$ is a constant which depends on n, M_0, $||a_{i,ml}||^Q_{0,0}$, $||a_{i,m}||^Q_{0,0}$, $||a_i||^Q_{0,0}$, $||\partial a_{i,ml}/\partial x_s||^Q_{0,0}$, $\inf\limits_{t \in (-\infty,T]} \mu_i(t)$, $\sup\{f_i(x,t,\xi,\eta) : (x,t) \in Q, |\xi| \leqslant M_0, |\eta| \leqslant M_0\}$, $||\hat{f}_i||^Q_{0,0}$, $||h_i||^\Sigma_{0,0}$, $i \in \{1, ..., M\}$, $\{m, l, s\} \subset \{1, ..., n\}$, and does not depend on k.

Consider a sequence $\{v^k\}$. For arbitrary (x_1, t_1), $(x_2, t_2) \in \overline{Q}$ we have

$$|v_j^k(t_1, x_1) - v_j^k(t_2, x_2)| \leqslant |v_j^k(t_1, x_2) - v_j^k(t_2, x_2)| + |v_j^k(t_1, x_1) - v_j^k(t_1, x_2)|. \tag{34}$$

We subtract equations (27) for x_2 from the same equations for x_1. Using Remark 1 this yields

$$\frac{\partial(v_j^k(x_1,t)-v_j^k(x_2,t))}{\partial t}+c_j(x_2,t)(v_j^k(x_1,t)-v_j^k(x_2,t))-$$
$$-\sum_{m=M+1}^{M+L}S_{jm}(x_1,t,w^k(x_1,t),w^k(x_2,t),w_\tau^k(x_1,t))(v_m^k(x_1,t)-v_m^k(x_2,t))-$$
$$-\sum_{m=M+1}^{M+L}\Lambda_{jm}(x_1,t,w^k(x_2,t),w_\tau^k(x_1,t),w_\tau^k(x_2,t))(v_m^k(x_1,t-\tau_m)-v_m^k(x_2,t-\tau_m))=$$
$$=[g_j(x_1,t,w^k(x_2,t),w_\tau^k(x_2,t))-g_j(x_2,t,w^k(x_2,t),w_\tau^k(x_2,t))]+$$
$$+\sum_{l=1}^{M}S_{jl}(x_1,t,w^k(x_1,t),w^k(x_2,t),w_\tau^k(x_1,t))(u_l^k(x_1,t)-u_l^k(x_2,t))+$$
$$+\sum_{l=1}^{M}\Lambda_{jl}(x_1,t,w^k(x_2,t),w_\tau^k(x_1,t)),w_\tau^k(x_2,t))(u_l^k(x_1,t-\tau_l)-u_l^k(x_2,t-\tau_l))+$$
$$+(c_j(x_1,t)-c_j(x_2,t))v_j^k(x_1,t)+(\hat{g}_j(x_1,t)-\hat{g}_j(x_2,t)),\quad j\in\{1,...,L\}. \tag{35}$$

Analogously to the proof of Lemma 4, using Remark 2, conditions (**B1**)–(**B3**), (**B5**) and estimate (30), from (35) we obtain

$$|v_j^k(x_1,t)-v_j^k(x_2,t)|\leqslant C_4|x_1-x_2|^\alpha,\quad t\in(-\infty,T], \tag{36}$$

where $C_4>0$ is a constant independent of k.

Now rewrite equation (27) in the next way

$$\frac{\partial v_j^k(x,t)}{\partial t}=g_j(x,t,w^k(x,t),w_\tau^k(x,t))+\hat{g}_j^k(x,t)-$$
$$-c_j(x,t)v_j^k(x,t),\quad (x,t)\in\overline{Q},\ j\in\{1,...,L\}. \tag{37}$$

From condition (**B2**) and estimate (30) it follows that the right-hand side of (37) is bounded uniformly in $k\in\mathbb{N}$. This leads to the fact that derivatives $\partial v_j^k/\partial t$, $k\in\mathbb{N}$, $j\in\{1,...,L\}$, on Q are uniformly bounded by k. Thus, using Lagrange theorem on finite decrement from (34), (36), (37) and (**B2**) we obtain

$$\sum_{j=1}^{L}||v_j^k||_{\alpha,1+\alpha/2}^{Q}\leqslant C_5, \tag{38}$$

where $C_5\geqslant 0$ is a constant independent of k.

Let $\tilde{f}_i^k(x,t)\overset{\text{def}}{=}f_i(x,t,w^k(x,t),w_\tau^k(x,t))$, $(x,t)\in\overline{Q}$, $i\in\{1,...,M\}$, $k\in\mathbb{N}$. By (33), (38) and (**B2**), (**B3**) we have $||\tilde{f}_i^k||_{\alpha,\alpha/2}^{Q}\leqslant C_6$, $i\in\{1,...,M\}$, $k\in\mathbb{N}$, where $C_6\geqslant 0$ is a constant independent of k. From this and (32) based on Theorem 10.1 of monograph [8; p. 400] we obtain

$$\sum_{i=1}^{M}||u_i^k||_{2+\alpha,1+\alpha/2}^{Q}\leqslant C_7, \tag{39}$$

where $C_7 \geqslant 0$ is a constant independent of k.

By (38) and (39) it follows estimate (31). Thus, using a diagonal process, we conclude that there exists a subsequence of the sequence w^k (we denote it by w^k again) such that for arbitrary $m \in \mathbb{N}$ we have $u_i^k \underset{k\to\infty}{\to} u_i$ in $C^{2+\gamma,1+\gamma/2}(\overline{Q^m})$ and $v_j^k \underset{k\to\infty}{\to} v_j$ in $C^{\gamma,1+\gamma/2}(\overline{Q^m})$, $i \in \{1,...,M\}$, $j \in \{1,...,L\}$, where $0 < \gamma < \alpha$. Set $w(x,t) = \mathrm{col}\,(u_1(x,t),\dots,u_M(x,t), v_1(x,t),...,v_L(x,t))$, $(x,t) \in \overline{Q}$. It is easy to see that this vector-function w is a solution of Problem (1)–(3). By (31) we have

$$\sum_{i=1}^{M} ||u_i||^{Q}_{2+\alpha,1+\alpha/2} + \sum_{j=1}^{L} ||v_j||^{Q}_{\alpha,1+\alpha/2} \leqslant C_2. \tag{40}$$

This completes the proof of theorem in the case $\nu = 0$.

Consider Problem (1)–(3) in the case when ν is arbitrary, $\nu < \nu_0,\ \nu \neq 0$. Multiply every equation of Problem by $e^{\nu t}$. This yields

$$L_i\tilde{w}(x,t) - \nu\tilde{u}_i(x,t) - f_i^{(\nu,\tau)}(x,t,\tilde{w}(x,t),\tilde{w}_\tau(x,t)) = \hat{f}_i(x,t)e^{\nu t}, \quad (x,t) \in Q,\ i \in \{1,...,M\}, \tag{41}$$

$$F_j\tilde{w}(x,t) - \nu\tilde{v}_j(x,t) - g_j^{(\nu,\tau)}(x,t,\tilde{w}(x,t),\tilde{w}_\tau(x,t)) = \hat{g}_j(x,t)e^{\nu t}, \quad (x,t) \in \overline{Q},\ j \in \{1,...,L\}, \tag{42}$$

$$\tilde{u}_i(x,t) = h_i(x,t)e^{\nu t}, \quad (x,t) \in \Sigma, \quad i \in \{1,...,M\}, \tag{43}$$

where L_i, $i \in \{1,...,M\}$, F_j, $j \in \{1,...,L\}$, are the same as in (18) and (19), $\tilde{w}(x,t) = w(x,t)e^{\nu t}$, the functions $f_i^{(\nu,\tau)}(x,t,\xi,\eta)$, $i \in \{1,...,M\}$, and $g_j^{(\nu,\tau)}(x,t,\xi,\eta)$, $j \in \{1,...,L\}$, are from condition (**B2**). To ensure that the functions $f_i^{(\nu,\tau)}$ $i \in \{1,...,M\}$, $g_j^{(\nu,\tau)}$, $j \in \{1,...,L\}$, satisfy conditions (**A3**) and (**A4**) with $\varphi(\nu) > 0$ and $\psi(\nu) > 0$ instead of $a_0 > 0$ and $b_0 > 0$, respectively, we use similar arguments as in the case of functions from equations (21) and (22). It is obvious that the functions $\hat{f}(x,t)e^{\nu t}$, $h(x,t)e^{\nu t}$, $\hat{g}(x,t)e^{\nu t}$ belong to the spaces $E_0(Q;M)$, $E_0(\Sigma;M)$, $E_0(\overline{Q};L)$, respectively. It follows from the proof of theorem in the case $\nu = 0$ that there exists solution $\tilde{w}$ of problem (41)–(43) from the class $E_0(\overline{Q};M+L)$. Thus the vector-function $w(x,t) = \tilde{w}(x,t)e^{-\nu t}$, $(x,t) \in Q$, belongs to the class $E_\nu(\overline{Q};M+L)$ and is a solution of Problem (1)-(3) which completes the proof of theorem. □

Proof of Theorem 4. Let $\varepsilon > 0$ be an arbitrary number, and vector-functions $\{\mathrm{col}\,(\hat{f}^1,\hat{g}^1,h^1), \mathrm{col}\,(\hat{f}^2,\hat{g}^2,h^2)\} \subset \Pi_\nu$ are such that

$$\sup_{(y,s)\in Q} \frac{|\hat{f}^1(y,s) - \hat{f}^2(y,s)|e^{\nu s}}{\varphi(\nu)} < \varepsilon, \quad \sup_{(y,s)\in Q} \frac{|\hat{g}^1(y,s) - \hat{g}^2(y,s)|e^{\nu s}}{\psi(\nu)} < \varepsilon,$$
$$\sup_{(y,s)\in\Sigma} |h^1(y,s) - h^2(y,s)|e^{\nu s} < \varepsilon,$$

and $w^i = RS_\nu(\hat{f}^i,\hat{g}^i,h^i)$, $i \in \{1,2\}$. Using Lemma 5 we complete the proof of theorem. □

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