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ON THE BEST POSSIBLE DESCRIPTION OF EXCEPTIONAL SET IN WIMAN-VALIRON THEORY FOR ENTIRE FUNCTIONS

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We show that known estimates of exceptional sets in some relations in Wiman-Valiron theory for entire functions are unimprovable. Several assumptions on the interplay between a size of an exceptional set and the speed of the growth of an entire Dirichlet series are formulated.

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Доказана неулучшаемость известных оценок исключительных множеств в некоторых соотношениях теории Вимана-Валирона для целых функций. Сформулировано несколько предположений относительно связи размера исключительного множества и скоростью возрастания целого ряда Дирихле.

Let $H(\lambda)$ be the class of entire Dirichlet series absolutely convergent in $\mathbb{C}$,

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n},$$

where $\lambda = (\lambda_n)$ is a fixed sequence such that $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \leqslant n \uparrow +\infty)$, $H = \bigcup_\lambda H(\lambda)$ the class of all entire Dirichlet series.

For $F \in H(\lambda)$ and $x \in \mathbb{R}$ we denote

$$\mu(x, F) = \max\{|a_n|e^{x\lambda_n} : n \geqslant 0\}, \quad M_0(x, F) = \inf\{|F(x+iy)| : y \in \mathbb{R}\},$$

$$M(x, F) = \sup\{|F(x+iy)| : y \in \mathbb{R}\}.$$

Let $h(x)$ be a positive continuously differentiable on $[0, +\infty)$ function. For a Lebesgue measurable set $E \subset [0, +\infty)$ we call

$$m_h E \equiv \int_E dh(x)$$

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its h-measure (see h-measure by Hausdorff [1]), when $h(x) \equiv x$ the h-measure of a set E is its Lebesgue measure.

1. Borel relations. It is known [2] that for every entire function $F \in H(\lambda)$ the relation

$$\ln M(x, F) = (1 + o(1)) \ln \mu(x, F) \tag{1}$$

holds as $x \to +\infty$ outside some set E, $mE < +\infty$, if and only if

$$\sum_{n=1}^{+\infty} \frac{1}{n\lambda_n} < +\infty. \tag{2}$$

We remark that for every set $E = \bigcup_{n=1}^{+\infty} [a_n, b_n]$ such that $a_n < b_n < a_{n+1}$ $(n \geqslant 1)$ and $mE < +\infty$ there exists a positive continuously differentiable function $h(x)$ for which $h'(x) \uparrow +\infty$ $(x \to +\infty)$ and $m_h E < +\infty$. Indeed, if $\sum_{n=1}^{+\infty} (b_n - a_n) < +\infty$, then there exists a sequence $1 < c_n \uparrow +\infty$ $(n \to +\infty)$ such that $\sum_{n=1}^{+\infty} c_n(b_n - a_n) < +\infty$. Since

$$m_h E = \sum_{n=1}^{+\infty} \int_{a_n}^{b_n} h'(x)dx \leqslant \sum_{n=1}^{+\infty} h'(b_n)(b_n - a_n),$$

it is sufficient to satisfy $h'(b_n) \leqslant c_n$.

In this case the question on possibility of obtaining a sharp estimate of an exceptional set E in relation (1) arises.

It is found that the characterization of an exceptional set in relation (1) given by the quoted theorem from [2] is the best possible, in some sense, in the class H. More precisely, from Theorem 3 [3] the next proposition follows.

Proposition 1. *Let $h(x)$ be a positive continuously differentiable on $[0, +\infty)$ function such that $\varliminf_{x\to+\infty} \ln h'(x)/ \ln x > 0$. Then there exist a sequence λ satisfying condition (2), a function $F \in H(\lambda)$, a constant $d > 0$ and a set E such that $m_h E = +\infty$ and for all $x \in E$*

$$\ln M(x, F) \geqslant (1 + d) \ln \mu(x, F). \tag{3}$$

Proof. By Theorem 3 [3] for each $q \in (0, 1]$ there exist a sequence λ satisfying (2), a function $F \in H(\lambda)$, sequences $c_k > 0$ with $\sum_{k=1}^{+\infty} c_k = +\infty$ and $x_k \uparrow +\infty$ and a constant $d > 0$ such that for all $x \in \left[x_k, x_k + c_k x_k^{-q}\right]$, $k \geqslant 1$, inequality (3) is valid. Now, if a constant $\varepsilon > 0$ is such that $h'(x) \geqslant x^{\varepsilon}$ $(x \geqslant x_0)$, then choosing $q \leqslant \varepsilon$ for h-measure of the set

$$E = \bigcup_{k=1}^{+\infty} [x_k, x_k + c_k x_k^{-q}]$$

we obtain

$$m_h E \geqslant \sum_{k=1}^{+\infty} \Big(h\Big(x_k + \frac{c_k}{x_k^q}\Big) - h(x_k)\Big) = \sum_{k=1}^{+\infty} \frac{c_k}{x_k^q} h'\Big(x_k + \frac{\theta_k c_k}{x_k^q}\Big) \geqslant \sum_{x_k \geqslant x_0} c_k = +\infty,$$

where $0 < \theta_k < 1$. The proposition is proved. □

Now we remark that if $h(x) = x^{1+\varepsilon}$, $\varepsilon > 0$ is arbitrary, then we have $m_h E = (1+\varepsilon)\int_E x^\varepsilon dx$, i. e. the quoted theorem from [2] is sharp in the sense that the Lebesgue measure mE cannot be replaced by any other measure of the form $\int_E x^\varepsilon dx$.

Conjecture 1. *For every continuously differentiable positive on $[0, +\infty)$ function $h(x)$ such that $h'(x) \uparrow +\infty$ $(x \to +\infty)$, there exist a sequence $\lambda = (\lambda_n)$, satisfying condition (2), a function $F \in H(\lambda)$, a set E and a constant $d > 0$ such that $m_h E = +\infty$ and for all $x \in E$ inequality (3) holds.*

The authors know that the conjecture is true under the additional condition

$$\int^{+\infty} \frac{dx}{xh'(x)} < +\infty. \tag{4}$$

Proposition 2. *Under the supplementary condition (4) Conjecture 1 is true.*

Proof. Our arguments are similar to that in the proof of Theorem 3 from [3]. Without loss of generality, we assume that $xh'(x) \geqslant 3$ $(x \geqslant 1)$. Let us define $\lambda_n = \ln nh'(\ln n/\ln\ln n)$ for $n \geqslant e^e \equiv n_0$. It is not difficult to check that condition (4) implies that

$$\sum_{n>n_0} \frac{1}{n\lambda_n} < +\infty,$$

i. e. condition (2) is satisfied. Now, take a positive continuous increasing to $+\infty$ on $[0, +\infty)$ function ψ such that $\psi(\lambda_n) = \frac{3}{8}\ln n/\ln\ln n$. Then

$$\frac{\ln n}{\psi(\lambda_n)} \uparrow +\infty, \quad \frac{\ln n}{\lambda_n \psi(\lambda_n)} \searrow 0, \quad n \to +\infty. \tag{5}$$

Similarly to [3], we can check that the choice of the increasing sequences

$$\{n_k\} \subset \{2s : s \in \mathbb{N}\}, \quad m_k = 0{,}5 n_k \geqslant n_q, \quad k \geqslant 1$$

satisfying the following conditions with $d > 0$ is non-contradictive:

$$\frac{\ln n_k}{\lambda_{m_k}\psi(\lambda_{n_k})} \leqslant \frac{4}{3}d, \tag{6}$$

$$\psi(\lambda_{n_{k+1}}) \geqslant \left(2 + \frac{3}{8d}\right)\psi(\lambda_{n_k}), \tag{7}$$

$$\ln n_{k+1} \geqslant 2\ln n_k, \tag{8}$$

$$\frac{\ln n_{k+1}}{\lambda_{n_k}\psi(\lambda_{n_{k+1}})} \geqslant 4\left(1 + \frac{16d}{3}\right). \tag{9}$$

Without loss of generality, we assume that

$$\lambda_k\psi(\lambda_k) \geqslant \ln k \geqslant \psi(\lambda_k) \geqslant 2, \quad k \geqslant m_1. \tag{10}$$

Now, for $k \geqslant 1$ we put

$$\varkappa_k = \frac{1}{2h}\frac{\ln n_k}{\lambda_{n_k}} + \frac{8}{3}\psi(\lambda_{n_k}),$$

$$b_{n_k} = \exp\Big\{-\frac{8}{3}\lambda_{n_k}\psi(\lambda_{n_k})\Big\},\ b_n = b_{n_k}\exp\{x_k(\lambda_{n_k}-\lambda_n)\},\quad m_k \leqslant n \leqslant n_k,$$
$$b_n = 0,\quad n_k < n < m_{k+1},\quad n < m_1,$$

and a sequence $c_k > 0$ such that

$$\sum_{k=1}^{+\infty} c_k = +\infty,\quad c_k \leqslant \frac{\ln n_k}{\lambda_{n_k}} h'(\varkappa_k).$$

Since $\ln n_k h'(\varkappa_k)/\lambda_{n_k} \geqslant 1$, the choice of (c_k) is possible. In particular, we assume that $c_k \leqslant 1$ $(k \geqslant 1)$. Let, in addition, $c_k^* > 0$ be such that $\sum_{k=1}^{+\infty} c_k^* = +\infty$, $c_k^* \leqslant c_k$ $(k \geqslant 1)$ and $c_k^* = o(c_k)$ $(k \to +\infty)$.

Now, we remark that by inequality (10)

$$\frac{8}{3}\psi(\lambda_{n_k}) \leqslant \varkappa_k \leqslant \Big(\frac{1}{2d}+\frac{8}{3}\Big)\psi(\lambda_{n_k}). \tag{11}$$

Taking into account that for all $k \geqslant 1$ and $m_k \leqslant n \leqslant n_k$

$$b_n e^{\varkappa_k\lambda_n} = (n_k)^{1/(2d)}, \tag{12}$$

holds, by inequality (6), omitting the first summand in braces, we obtain for $m_k \leqslant n \leqslant n_k$

$$\ln b_n = \ln b_{n_k} + \varkappa_k\lambda_{n_k} - \varkappa_k\lambda_n = \frac{1}{2d}\ln n_k - \Big(\frac{1}{2d}\frac{\ln n_k}{\lambda_{n_k}} + \frac{8}{3}\psi(\lambda_{n_k})\Big)\lambda_n \leqslant$$
$$\leqslant \frac{2}{3}\lambda_{m_k}\psi(\lambda_{n_k}) - \frac{8}{3}\lambda_n\psi(\lambda_{n_k}) \leqslant -2\lambda_n\psi(\lambda_{n_k}) \leqslant -2\lambda_n\psi(\lambda_n).$$

Hence, using (5) we deduce that the function $F(\sigma) = \sum\limits_{n=m_1}^{+\infty} b_n e^{\sigma\lambda_n}$ belongs to the class $H(\lambda)$.

Now, let us check that $\mu(\sigma, F) = b_{n_k}e^{\sigma\lambda_{n_k}}$ for all $\sigma \in \big[\varkappa_k, \varkappa_k + c_k/h'(\varkappa_k)\big]$.

When $j \geqslant 1$ and $m_{k+j} \leqslant n \leqslant n_{k+j}$, using equality (12), inequalities (11) and conditions (7), (6) of the choice, we obtain

$$\ln(b_{n_k}e^{\varkappa_k\lambda_{n_k}}) - \ln(b_n e^{\varkappa_k\lambda_n}) = \frac{1}{2d}\ln n_k - (\ln b_{n_{k+j}} + \varkappa_{k+j}\lambda_{n_{k+j}}) + (\varkappa_{k+j}-\varkappa_k)\lambda_n =$$
$$= \frac{1}{2d}(\ln n_k - \ln n_{k+j}) + (\varkappa_{k+j}-\varkappa_k)\lambda_n \geqslant$$
$$\geqslant -\frac{1}{2d}\ln n_{k+j} + \Big(\frac{8}{3}\psi(\lambda_{n_{k+j}}) - \Big(\frac{8}{3}+\frac{1}{2d}\Big)\psi(\lambda_{n_k})\Big)\lambda_n \geqslant -\frac{1}{2d}\ln n_{k+j} + \frac{4}{3}\psi(\lambda_{n_{k+j}})\lambda_n \geqslant$$
$$\geqslant -\frac{1}{2d}\ln n_{k+j} + \frac{4}{3}\psi(\lambda_{n_{k+j}})\lambda_{m_{k+j}} \geqslant \frac{1}{2d}\ln n_{k+j} > 0,$$

i. e. for $j \geqslant 1$ and $m_{k+j} \leqslant n \leqslant n_{k+j}$ we have

$$b_n e^{\varkappa_k\lambda_n} < b_{n_k}e^{\varkappa_k\lambda_{n_k}}. \tag{13}$$

Similarly, using successively equality (12), the right inequality from (11) and inequalities (8), (9), for $1 \leqslant j < k$, $m_{k-j} \leqslant n \leqslant n_{k-j}$, we obtain

$$\ln(b_{n_k}e^{\varkappa_k\lambda_{n_k}}) - \ln(b_n e^{\varkappa_k\lambda_n}) = \frac{1}{2d}\ln n_k - (\ln b_{n_{k-j}} + \varkappa_{k-j}\lambda_{n_{k-j}}) + (\varkappa_{k-j}-\varkappa_k)\lambda_n =$$

$$= \frac{1}{2d}(\ln n_k - \ln n_{k-j}) - (\varkappa_k - \varkappa_{k-j})\lambda_{n_{k-j}} \geqslant \frac{1}{4d}\ln n_k - \varkappa_k \lambda_{n_{k-j}} \geqslant$$
$$\geqslant \frac{1}{4d}\ln n_k - \left(\frac{8}{3} + \frac{1}{2d}\right)\psi(\lambda_{n_k})\lambda_{n_{k-j}} \geqslant \frac{1}{8d}\ln n_k > 0,$$

i. e. for $1 \leqslant j < k$, $m_{k-j} \leqslant n \leqslant n_{k-j}$ inequality (13) is valid.

Thus, for $m_k \leqslant n \leqslant n_k$ from (12) and (13) we obtain

$$\ln \mu(\varkappa_k, F) = \ln(b_n e^{\varkappa_k \lambda_n}) = \frac{1}{2d}\ln n_k. \tag{14}$$

Using successively inequalities (11), (7), and $xh'(x) \geqslant 3$ $(x \geqslant 1)$, we have

$$\varkappa_{k+1} - \varkappa_k \geqslant \frac{8}{3}\psi(\lambda_{n_{k+1}}) - \left(\frac{1}{2h} + \frac{8}{3}\right)\psi(\lambda_{n_k}) \geqslant \frac{4}{3}\psi(\lambda_{n_{k+1}}) >$$
$$> \frac{8}{3}\psi(\lambda_{n_k}) \geqslant 3/h'\left(\frac{8}{3}\psi(\lambda_{n_k})\right) \geqslant \frac{3c_k}{h'(\varkappa_k)} \quad (k \geqslant 1) \tag{15}$$

and, similarly, using again the inequality $xh'(x) \geqslant 3$

$$\frac{c_k}{\psi(\lambda_{n_{k+1}})h'(\varkappa_k)} \leqslant \frac{c_k}{2\psi(\lambda_{n_k})h'(\varkappa_k)} \leqslant \frac{1}{2\psi(\lambda_{n_k})h'(\frac{8}{3}\psi(\lambda_{n_k}))} \leqslant \frac{4}{9}.$$

By the last inequality and inequality (15) we have

$$\ln b_{m_{k+1}} + \left(\varkappa_k + \frac{c_k}{h'(\varkappa_k)}\right)\lambda_{m_{k+1}} = \frac{1}{2d}\ln n_{k+1} - \left(\varkappa_{k+1} - \varkappa_k - \frac{C_k}{h'(\varkappa_k)}\right)\lambda_{m_{k+1}} \leqslant$$
$$\leqslant \frac{1}{2d}\ln n_{k+1} - \left(\frac{4}{3}\psi(\lambda_{n_{k+1}}) - \frac{c_k}{h'(\varkappa_k)}\right)\frac{3}{4d}\frac{\ln n_{k+1}}{\psi(\lambda_{n_{k+1}})} =$$
$$= \frac{1}{d}\left(-\frac{1}{2} + \frac{3c_k}{4h'(\varkappa_k)\psi(\lambda_{n_{k+1}})}\right)\ln n_{k+1} \leqslant -\frac{1}{6d}\ln n_{k+1} < 0, \quad k \geqslant 1.$$

Taking into account the last inequality, inequality (15), and the fact that $b_n e^{\sigma\lambda_n} \leqslant \mu(\sigma, F)$ $(\sigma \geqslant \varkappa_k, n < n_k)$ and $b_{m_{k+1}} e^{\sigma\lambda_{m_{k+1}}} \leqslant b_n e^{\sigma\lambda_n}$ $(\sigma < \varkappa_{k+1}, m_{k+1} \leqslant n \leqslant n_{k+1})$ we finally obtain (recalling (14)), that for $\sigma \in [\varkappa_k, \varkappa_k + c_k/h'(\varkappa_k))$ (also for $\sigma \in [\varkappa_k, \varkappa_k + c_k^*/h'(\varkappa_k)]$)

$$\mu(\sigma, F) = b_{n_k} e^{\sigma\lambda_{n_k}}. \tag{16}$$

Now we note that for $\sigma = \varkappa_k + c_k^*/h'(\varkappa_k)$, using the inequality $c_k \leqslant \dfrac{\ln n_k}{\lambda_{n_k}} h'(\varkappa_k)$, one can obtain

$$\ln \mu(\sigma, F) = \frac{1}{2d}\ln n_k + \frac{c_k^*}{h'(\varkappa_k)}\lambda_{n_k} \leqslant \left(\frac{1}{2d} + \frac{c_k^*}{c_k}\right)\ln n_k. \tag{17}$$

From equalities (14) and (16) we deduce for $\sigma \in [\varkappa_k, \varkappa_k + c_k^*/h'(\varkappa_k)]$

$$F(\sigma) \geqslant \sum_{n=m_k}^{n_k} b_n e^{\sigma\lambda_n} \geqslant \sum_{n=m_k}^{n_k} b_n e^{\varkappa_k\lambda_n} = (n_k - m_k + 1)\mu(\varkappa_k, F) \geqslant 0,5(n_k)^{1+1/2d},$$

whence, by (17), for all k satisfying $\ln 2/\ln n_k + (1+d)c_k^*/c_k \leqslant 1/2$, we obtain

$$\ln F(\sigma) \geqslant \left(1+\frac{1}{2d}\right)\ln n_k - \ln 2 \geqslant (1+d)\ln\mu(\sigma, F).$$

It remains to note that for the set $E = \bigcup_{k=1}^{+\infty} [\varkappa_k, \varkappa_k + c_k^*/h'(\varkappa_k)]$ we have

$$m_h(E\cap[\varkappa_{k_1}, +\infty)) \geqslant \sum_{k=k_0}^{+\infty} c_k^* = +\infty,$$

and Proposition 2 is proved. □

2. Equivalence of the maximum modulus and the maximum term. It is natural to ask whether estimates of exceptional sets outside which relations of the Wiman-Valiron theory hold are sharp concerning the relations

$$M(x, F) = (1+o(1))\mu(x, F), \quad M(x, F) = (1+o(1))M_0(x, F). \tag{18}$$

It is known (see [4]) that (18) holds as $x\to+\infty$ outside some set E of a finite Lebesgue measure (so of a finite h-measure $m_hE < +\infty$, for some continuously differentiable function $h(x)$ such that $h'(x)\nearrow+\infty$ $(x\to+\infty)$), when the condition is fulfilled

$$\sum_{n=0}^{+\infty}\frac{1}{\lambda_{n+1}-\lambda_n} < +\infty \tag{19}$$

is fulfilled.Let us remark that if

$$\varkappa_n \equiv (\ln|a_{n-1}| - \ln|a_n|)/(\lambda_n - \lambda_{n-1}) \uparrow +\infty, \tag{20}$$

for function $F\in H(\lambda)$, then $\mu(\sigma, F) = |a_n|e^{\sigma\lambda_n}$ for $\sigma\in[\varkappa_n, \varkappa_{n+1}]$ and the inequality $|a_{n+1}|e^{\sigma\lambda_{n+1}} \geqslant e^{-q}\mu(\sigma, F)$ is valid on this segment if and only if $\sigma\in\left[\varkappa_{n+1}-\frac{q}{\lambda_{n+1}-\lambda_n}, \varkappa_{n+1}\right] \equiv I_n$; similarly $|a_{n-1}|e^{\sigma\lambda_n} \geqslant e^{-q}\mu(\sigma, F)$ if and only if $\sigma\in\left[\varkappa_n, \varkappa_n+\frac{q}{\lambda_n-\lambda_{n-1}}\right] \equiv J_n$, where $q>0$.

Thus, if $\varkappa_{n+1}-\varkappa_n \geqslant q\max\{\frac{1}{\lambda_{n+1}-\lambda_n}; \frac{1}{\lambda_n-\lambda_{n-1}}\}$ i $a_n>0$ $(n\geqslant 0)$, then for all $\sigma\in E\equiv \bigcup_{n=1}^{+\infty}(I_n\cup J_n)$

$$F(\sigma) \geqslant (1+e^{-q})\mu(\sigma, F). \tag{21}$$

Now, if $h(x)$ is continuously differentiable function such that $h'(x)\nearrow$, then

$$m_hE \geqslant \int\limits_{[\varkappa_n, \varkappa_n+q/(\lambda_n-\lambda_{n-1})]} dh(x) \geqslant q\sum_{n=1}^{+\infty}\frac{h'(\varkappa_n)}{\lambda_n-\lambda_{n-1}},$$

and we obtain the following proposition.

Proposition 3. *Let $h(x)$ be a continuously differentiable function such that $h'(x)\nearrow$. If for a function $F\in H(\lambda)$ condition (20) is fulfilled and*

$$(\exists q>0)\ :\ \varkappa_{n+1}-\varkappa_n \geqslant q\max\left\{\frac{1}{\lambda_{n+1}-\lambda_n}; \frac{1}{\lambda_n-\lambda_{n-1}}\right\}, \tag{22}$$

then inequality (21) is valid for all $\sigma \in E$ *and*

$$m_h E \geqslant q \sum_{n=1}^{+\infty} \frac{h'(\varkappa_n)}{\lambda_n - \lambda_{n-1}}. \tag{23}$$

It is easy to obtain the following consequence.

Corollary. *For every positive continuously differentiable function* $h(x)$ *such that* $h'(x) \nearrow +\infty$ $(x \to +\infty)$, *for every sequence* (λ_n) *satisfying (19), there exist a function* $F \in H(\lambda)$, *a constant* $d > 0$, *and a set* $E \subset [0, +\infty)$ *such that for every* $\sigma \in E$

$$F(\sigma) \geqslant (1 + d)\mu(\sigma, F)$$

and $m_h E = +\infty$.

Proof. By Proposition 3, it is enough to choose sequence $\varkappa_n \uparrow +\infty$ $(n \to +\infty)$, for which condition (22) is fulfilled and series (23) is divergent simultaneously. Hence the function

$$F(z) = \sum_{n=1}^{\infty} a_n e^{z\lambda_n}$$

defined by equalities (20) and $a_0 = 1$ belongs to the class $H(\lambda)$. Let $c_n \uparrow +\infty$ $(n \to +\infty)$ be a sequence satisfying $\sum_{n=1}^{+\infty} c_n/(\lambda_n - \lambda_{n-1}) = +\infty$. Remark that condition (19) implies that $\lambda_n - \lambda_{n-1} \to +\infty$ $(n \to +\infty)$, therefore it is enough to choose $c_n = \max\{\lambda_k - \lambda_{k-1} : 1 \leqslant k \leqslant n\}$. A sequence $\varkappa_n$ is chosen to satisfy conditions

$$h'(\varkappa_n) = C_1, \quad h'(\varkappa_{n+1}) \geqslant C_{n+1}, \quad \varkappa_{n+1} \geqslant \varkappa_n + q \max\{\frac{1}{\lambda_n - \lambda_{n-1}}; \frac{1}{\lambda_{n+1} - \lambda_n}\} \quad n \geqslant 1.$$

It is clear that conditions of the choice of $(\varkappa_n)$ are not contradictive. Now we put $a_0 = 1$. Since (19) implies $n = o(\lambda_n)$ $(n \to +\infty)$, and by the construction $\varkappa \uparrow +\infty$ $(n \to +\infty)$, Stolz's theorem ([5, p.25]) yields $-\ln a_n/\lambda_n \to +\infty$ $(n \to +\infty)$. Therefore [6, p.85] the function F defined by the sequence $(\varkappa_n)$ by equalities (20) belongs to $H(\lambda)$. The corollary is proved. □

Now let

$$F_1(z) = \sum_{n=0}^{\infty} (-1)^n a_n e^{z\lambda_n},$$

where (a_n) is defined in the proof of the corollary. Evidently, $F_1 \in H(\lambda)$. In the case when $\varkappa_n \uparrow +\infty$ $(n \to +\infty)$ one can immediately check that for all $\sigma \in [\varkappa_n, \varkappa_{n+1}]$ we have $a_m e^{\sigma\lambda_m} \leqslant a_{m+1} e^{\sigma\lambda_{m+1}}$ $(0 \leqslant m \leqslant n-1)$ and $a_m e^{\sigma\lambda_m} \geqslant a_{m+1} e^{\sigma\lambda_{m+1}}$ $(m \geqslant n)$. Then for all $\sigma \geqslant \varkappa_1$

$$M_0(\sigma, F_1) \leqslant |F_1(\sigma)| \leqslant \mu(\sigma, F_1) = \mu(\sigma, F),$$

where F is the function from the proof of the corollary.

By Parseval's equality

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |F_1(\sigma + iy)| dy = \sum_{n=0}^{\infty} |a_n|^2 e^{2\sigma\lambda_n}$$

for all $\sigma \in E$ (E is the set from the corollary)

$$|M(\sigma, F_1)|^2 \geqslant (1 + e^{-2q})(\mu(\sigma, F_1))^2.$$

Thus, for all $\sigma \in E$

$$M_0(\sigma, F_1)(1 + e^{-2q})^{1/2} \leqslant M(\sigma, F_1),$$

i. e. we have obtained the following proposition.

Proposition 4. *For every sequence $\lambda = (\lambda_n)$ satisfying condition (19), for every continuously differentiable function $h'(x) \nearrow +\infty$ $(x \to +\infty)$, there exist a function $F \in H(\lambda)$ and a set E such that $m_h E = +\infty$, but relations (18) do not hold simultaneously on the set E.*

From Proposition 4 it follows immediately that the estimate of an exceptional set in similar relations for entire functions defined by gap power series [7] is sharp.

3. Final conclusions. Hence, for every sequence $\lambda = (\lambda_n)$ the theorem from [4] yields a sharp estimate of an exceptional set E in relations (18), in the class $H(\lambda)$. On the other hand, in [8] it is proved, that under the condition

$$\ln \mu(\sigma, F) \geqslant \sigma \Phi(\sigma), \sigma \geqslant \sigma_0, \tag{24}$$

where $\Phi(\sigma)$ is a positive continuous increasing to $+\infty$ function the relation

$$(\exists b > 0) : \lim_{\sigma \to +\infty} h_0(\sigma) \sum_{\lambda_n > b\Phi(\sigma)} \frac{1}{\lambda_{n+1} - \lambda_n} = 0, \tag{25}$$

implies that relations (18) are valid as $x \to +\infty$ outside a set E satisfying

$$D_h E \equiv \lim_{\sigma \to +\infty} h_0(\sigma) m(E \cap [\sigma, +\infty)),$$

where $h_0(\sigma)$ is a positive continuous increasing to $+\infty$ function such that

$$\frac{h_0(\sigma) \ln \Phi(\sigma)}{\Phi(\sigma)} \to 0, \quad \sigma \to +\infty.$$

At the same time condition (25) in the subclass $H(\lambda)$ defined by (24) is necessary for fulfillment of relations (18) when $\sigma \to +\infty$ $(\sigma \notin E, D_h E = 0)$ for every function F from this subclass.

The questions arises:

1. What is the sharp estimate of an exceptional set in relations (18) in the subclass $H(\lambda)$ defined by condition (24)?

The proof of the corollary allows us to suppose that the following conjecture is true.

Conjecture 2. *If $h'(x) \nearrow +\infty$ $(x \to +\infty)$ and conditions (24) and*

$$(\exists b > 0) : \sum_{n=1}^{+\infty} \frac{h'(b\varphi(b\lambda_n))}{\lambda_n - \lambda_{n-1}} < +\infty$$

are fulfilled for a function $F \in H(\lambda)$, where $\varphi(t)$ is the inverse to $\Phi(\sigma)$ function, then for an exceptional set E in relations (18) we have $m_h E < +\infty$, and in the same time, the last estimate of the exceptional set is sharp in some sense.

Concerning relation (1), probably, the next proposition is true.

Conjecture 3. *If $h'(x) \nearrow +\infty$ $(x \to +\infty)$, and conditions (24) and*

$$(\exists b > 0) : \quad \sum_{n=1}^{+\infty} \frac{h'(b\varphi(b\lambda_n))}{n\lambda_n} < +\infty$$

are true for a function $F \in H(\lambda)$, then for an exceptional set E in relation (1) $m_h E < +\infty$, and the estimate of the exceptional set is sharp.

According to what was exposed, the following conjecture seems to be true.

Conjecture 4. *Let $\omega(t)$ be a positive continuous increasing to $+\infty$ on $[0, +\infty)$ function. In the theorems, giving sufficient conditions for validity of the relation*

$$\omega(\ln M(x, F)) \sim \omega(\ln \mu(x, F)), \quad x \to +\infty, x \notin E,$$

for every function $F \in H(\lambda)$ the sharp estimate of the measure of an exceptional set E is the condition $mE < +\infty$ (see also [9,10]).

Concerning sharpness of estimates of exceptional sets in some other relations obtained in Wiman-Valiron theory see [3,11–14].

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