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BOUNDEDNESS OF l-INDEX FOR ENTIRE FUNCTIONS OF ZERO GENUS

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We investigate conditions on zeros of an entire function f of zero genus under which f is of bounded l-index.

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Исследуются условия на нули целой функции f нулевого рода, при которых f является функцией ограниченного l-индекса.

1^0. Introduction. Let Λ be the class of positive continuous functions l on $[0,+\infty)$ and Q be the class of functions $l\in\Lambda$ such that $l(r+O(1/l(r)))=O(l(r))$ $(r\to+\infty)$. By Q_* we denote the class of nonincreasing functions $l\in Q$. Remark that a nonincreasing function $l\in\Lambda$ belongs to Q provided that $rl(r)\nearrow+\infty$ as $r\to+\infty$. In fact, if $rl(r)$ nondecreases to $+\infty$, then for any $q>0$ we have

$$l\Big(r-\frac{q}{l(r)}\Big)\le\frac{r}{r-q/l(r)}l(r)=\frac{1}{1-q/(rl(r))}l(r)=(1+o(1))l(r),\quad r\to+\infty.$$

The inequality $l\big(r+q/l(r)\big)\le l(r)$ is trivial. Thus, $l\in Q_*\subset Q$.

For $l\in\Lambda$ an entire function f is said to be of bounded l-index [1], [2, p. 3] if there exists $N\in\mathbb{Z}_+$ such that $\dfrac{|f^{(n)}(z)|}{n!l^n(|z|)}\le\max\left\{\dfrac{|f^{(k)}(z)|}{k!l^k(|z|)}:\,0\le k\le N\right\}$ for all $n\in\mathbb{Z}_+$ and $z\in\mathbb{C}$. For $l(x)\equiv1$ we obtain the definition [3] of an entire function of bounded index.

Let

$$f(z)=\prod_{n=1}^{\infty}\Big(1-\frac{z}{a_n}\Big),\quad \sum_{n=1}^{\infty}\frac{1}{|a_n|}<+\infty, \tag{1}$$

be an entire function of zero genus. If $a_k>0$ and $a_1\le a_k-a_{k-1}\nearrow+\infty\,(2\le k\to\infty)$ then [4] f is an entire function of bounded index. The result is improved in [5], where it is proved

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that if $|a_1| \le |a_k| - |a_{k-1}| \nearrow +\infty\,(2 \le k \to \infty)$ then there exists a decreasing to 0 function $l \in \Lambda$ such that f is a function of bounded l-index.

If $a_n = n^{1/\rho}$, $0 < \rho < 1$, then [6] f is of bounded l-index with $l(r) = r^{\rho-1} \sim \dfrac{n(r)}{r}$ $(r \to +\infty)$, where $n(r)$ is the counting function of (a_n). The Mittag-Leffler function E_ρ, $0 < \rho < 1$, is [7] also of bounded l-index with $l(r) \sim \dfrac{n(r)}{r}$ $(r \to +\infty)$. In [8] the following result is announced.

Theorem 1. *If zeros a_k of function (1) are positive and $(1+\eta)a_n \le a_{n+1}$, $\eta > 0$, for all $n \ge 1$ then there exists a function $l \in Q_*$ such that $l(r) \sim \dfrac{n(r)}{r}$ for $r \to \infty$, and f is of bounded l-index.*

In virtue of these results in [8] it is formulated the following

Conjecture. *If $a_n > 0\,(n \ge 1)$ and $n/a_n \searrow 0\,(n \to \infty)$ then there exists a function $l \in Q_*$ such that $l(r) \sim \dfrac{n(r)}{r}\,(r \to +\infty)$ and function (1) is of bounded l-index.*

We prove Theorem 1 and disprove the conjecture.

2^0. Preliminary results. We put $M_f(r) = \max\{|f(z)| : |z| = r\}$. It is known [2, p. 71] that if $l \in Q$ and an entire function f is of bounded l-index then

$$\ln M_f(r) = O(L(r)),\ r \to +\infty, \quad L(r) = \int_0^r l(t)dt. \tag{2}$$

If $a_k \in \mathbb{C}$ are zeros of an entire function f then we put $n(r, z_0, 1/f) = \sum\limits_{|a_k - z_0| \le r} 1$, and $G_q(f) = \bigcup_k \{z : |z - a_k| \le q/l(|a_k|)\}$ for $l \in \Lambda$, $q \in (0, +\infty)$.

Lemma 1. [6; 2, p. 27] *If $l \in Q$ then an entire function f is of bounded l-index if and only if 1) for every $q > 0$ there exists $P(q) > 0$ such that $|f'(z)/f(z)| \le P(q)l(|z|)$ for all $z \in \mathbb{C} \setminus G_q(f)$ and 2) for every $q > 0$ there exists $n^*(q) \in \mathbb{N}$ such that $n(q/l(|z_0|), z_0, 1/f) \le n^*(q)$ for each $z_0 \in \mathbb{C}$.*

Lemma 2. *Let $l \in Q_*$ and a sequence (a_k) satisfy the following conditions:*

a) $l(|a_n|) = O(l(|a_{n+1}|)),\quad n \to \infty;$

b) $|a_{n+1}| - |a_n| > \dfrac{2q_0}{l(|a_{n+1}|)}$ for some $q_0 > 0$ and all $n \ge 1$;

c) $\displaystyle\sum_{k=1}^{n-1} \frac{1}{|a_n| - |a_k|} = O(l(|a_n|)),\quad n \to \infty;$

d) $\displaystyle\sum_{k=n+2}^{\infty} \frac{1}{|a_k| - |a_n|} = O(l(|a_n|)),\quad n \to \infty.$

Then function (1) is of bounded l-index.

Proof of Lemma 2. Since $l \in Q_*$, choosing $q_1 \in (0, q_0)$ to satisfy $l(r - q_0/l(r)) < \frac{q_0}{q_1} l(r)$ we obtain $n\big(\frac{q_1}{l(|z_0|)}, z_0, \frac{1}{f}\big) \le 1$ for arbitrary z_0. Indeed, if $|z_0| - \frac{q_1}{l(|z_0|)} \le a_j \le |z_0| + \frac{q_1}{l(|z_0|)}$ for $j = n, n+1$ and some $n \in \mathbb{N}$, then

$$\frac{2q_1}{l(|z_0|)} \ge a_{n+1} - a_n \ge \frac{2q_0}{l(|z_0| - \frac{q_1}{l(|z_0|)})} > \frac{2q_1}{l(|z_0|)},$$

a contradiction. Further, we can cover each closed disk of radius $q/l(|z_0|)$, by a finite number $m(q_1,q)$ of closed disks of radius $q_1/l(|z_0|)$. Therefore, $n(q/l(|z_0|),z_0,1/f)\le 2m(q_1,q)$, i. e. condition 2) of Lemma 1 holds.

It is sufficient to show that condition 1) of Lemma 1 holds with $q\le q_0$. Denote

$$A_n=\{z:\ ||z|-|a_n||\le q/l(|a_n|),\quad |z-a_n|\ge q/l(|a_n|)\}\,,\quad n\ge 1,$$

$$B_n=\{z:\ |a_n|+q/l(|a_n|)\le |z|\le |a_{n+1}|-q/l(|a_{n+1}|)\}\,,\quad n\ge 1.$$

From (1) it follows that

$$\left|\frac{f'(z)}{f(z)}\right|\le\sum_{k=1}^{\infty}\frac{1}{|z-a_k|}. \tag{3}$$

Condition b) and nonincrease of l imply that $||a_k|-|a_n||\ge 2q_0/l(|a_n|)$, $k\ne n$. Thus, for $z\in A_n$ we have

$$\left|\frac{f'(z)}{f(z)}\right|\le\sum_{k=1}^{n-1}\frac{1}{|z|-|a_k|}+\frac{1}{|z-a_n|}+\sum_{k=n+1}^{\infty}\frac{1}{|a_k|-|z|}\le$$
$$\le\sum_{k=1}^{n-1}\frac{1}{|a_n|-|a_k|-q/l(|a_n|)}+\frac{l(|a_n|)}{q}+\sum_{k=n+1}^{\infty}\frac{1}{|a_k|-|a_n|-q/l(|a_n|)}\le$$
$$\le 2\sum_{k=1}^{n-1}\frac{1}{|a_n|-|a_k|}+2\frac{l(|a_n|)}{q}+2\sum_{k=n+2}^{\infty}\frac{1}{|a_k|-|a_n|}.$$

From conditions $l\in Q$ and $z\in A_n$ it follows that $l(|a_n|)=O(l(|z|))$ $(n\to\infty)$. Therefore, in view of conditions c) and d) for $z\in A_n$ we have

$$|f'(z)/f(z)|=O(l(|z|)),\quad n\to\infty. \tag{4}$$

If $z\in B_n$, then using conditions c), d), a) and $l\in Q_*$ we obtain

$$\left|\frac{f'(z)}{f(z)}\right|\le\sum_{k=1}^{n-1}\frac{1}{|z|-|a_k|}+\frac{1}{|z|-|a_n|}+\frac{1}{|a_{n+1}|-|z|}+\frac{1}{|a_{n+2}|-|z|}+\sum_{k=n+3}^{\infty}\frac{1}{|a_k|-|z|}\le$$
$$\le\sum_{k=1}^{n-1}\frac{1}{|a_n|-|a_k|-q/l(|a_n|)}+\frac{l(|a_n|)}{q}+2\frac{l(|a_{n+1}|)}{q}+\sum_{k=n+3}^{\infty}\frac{1}{|a_k|-|a_{n+1}|+q/l(|a_{n+1}|)}\le$$
$$\le\sum_{k=1}^{n-1}\frac{1}{|a_n|-|a_k|}+\frac{l(|a_n|)}{q}+2\frac{l(|a_{n+1}|)}{q}+\sum_{k=n+3}^{\infty}\frac{1}{|a_k|-|a_{n+1}|}=$$
$$=O(l(|a_n|))+O(l(|a_{n+1}|))=O(l(|a_{n+1}|))=O(l(|z|)),\quad n\to\infty. \tag{5}$$

From (3)–(5) it follows that there exists a number $P_1(q)>0$ such that $|f'(z)/f(z)|\le$ $\le P_1(q)l(|z|)$ for all $z\in\mathbb{C}\setminus G_q(\pi)$ and $|z|\ge R_1=|a_1|-q/l(|a_1|)$. On the other hand, if $|z|\le R_1$, $z\notin G_q(\pi)$, then $|f'(z)|/(|f(z)|l(|z|))\le P_2(q)$, where $P_2(q)$ is a positive constant. Therefore, there exists a positive constant $P(q)$ such that inequality $|f'(z)/f(z)|\le P(q)l(|z|)$ holds for all $z\in\mathbb{C}\setminus G_q(\pi)$, thus condition 1) of Lemma 1 holds. By Lemma 1, f is of bounded l-index. Lemma 2 is proved. □

3^0. Proof of Theorem 1. From condition $a_{n+1} \geq (1+\eta)a_n$ it follows that $a_{n+1}/a_n > > 1 + 1/n$ for $n > 1/\eta$, i. e. $n/a_n \downarrow 0$ as $1/\eta < n \to \infty$. We put $n_1(r) = r/a_1$ for $0 \leq r \leq a_1$ and $n_1(r) = n + \dfrac{r-a_n}{a_{n+1}-a_n}$ for $a_n \leq r \leq a_{n+1}$. Then function $n_1(r)$ is continuous, $n(r) \leq n_1(r) \leq n(r)+1$, $n_1(r)/r \sim n(r)/r$ and $n_1(r)/r \downarrow 0$ as $r_0 \leq r \to \infty$, because for $a_n < r < a_{n+1}$, $n > 1/\eta$, we have $\left(\dfrac{n_1(r)}{r}\right)' = \dfrac{1}{r^2}\left(\dfrac{a_n}{a_{n+1}-a_n} - n\right) < 0$. Hence, if we put $l(r) = n_1(r)/r$, $r \geq r_0$, then $l(r) \downarrow 0$ and $l(r) \sim n(r)/r$ as $r_0 \leq r \to \infty$. It is easy to show also that $l \in Q$.

Let $z \in \mathbb{C} \setminus G_q(f)$ and $a_n \leq |z| < a_{n+1}$ for some $n \in \mathbb{N}$. The condition $a_{n+1} \geq (1+\eta)a_n$ implies that

$$\sum_{k=1}^{n-1} \frac{1}{|z|-a_k} \leq \frac{n-1}{|z|-a_{n-1}} \leq \frac{n(|z|)}{|z|(1-1/(1+\eta))} \leq \frac{1+\eta}{\eta} l(|z|), \quad z \to \infty, \tag{6}$$

and

$$\sum_{k=n+2}^{\infty} \frac{1}{a_k - |z|} \leq \frac{1}{|z|} \sum_{k=n+2}^{\infty} \frac{1}{(1+\eta)^{k-n-1}-1} \leq$$
$$\leq \frac{1}{|z|} \sup_{m\geq 1} \frac{(1+\eta)^m}{(1+\eta)^m - 1} \sum_{m=1}^{+\infty} (1+\eta)^{-m} = \frac{1+\eta}{\eta^2 |z|} = o(l(|z|)), \quad z \to \infty. \tag{7}$$

If $|a_n - z| \geq q/l(|z|)$ and $|a_{n+1} - z| \geq q/l(|z|)$, then $1/|z-a_n| + 1/|z-a_{n+1}| \leq \frac{2}{q} l(|z|)$. Otherwise, either i) $|a_n - z| < q/l(|z|)$ or $|a_{n+1} - z| < q/l(|z|)$.

Since $l \in Q_*$ in case i) we have

$$l(|a_n|) \leq l\Big(|z| - \frac{q}{l(|z|)}\Big) = O(l(|z|)), \quad n \to \infty,$$

and using the relation $l(|z|) = o(|z|)$ $(z \to \infty)$, we get for $z \in \mathbb{C} \setminus G_q(f)$

$$\frac{1}{|z-a_n|} + \frac{1}{|z-a_{n+1}|} \leq \frac{l(|a_n|)}{q} + O\Big(\frac{1}{|z|}\Big) = O(l(|z|)), \quad z \to \infty, z \notin G_q(f). \tag{8}$$

Similarly, in case ii) we obtain $l(|a_{n+1}|) = O(l(|z|))$, and consequently, $\frac{1}{|z-a_n|} + \frac{1}{|z-a_{n+1}|} = O(l(|z|))$. Thus, for $z \in \mathbb{C} \setminus G_q(f)$ we have $1/|z-a_n| + 1/|z-a_{n+1}| = O(l(|z|))$ $(z \to \infty)$. Using (6)–(8), we deduce that for such z

$$\Big|\frac{f'(z)}{f(z)}\Big| \leq \sum_{k=1}^{+\infty} \frac{1}{|z-a_k|} \leq \sum_{k=1}^{n-1} \frac{1}{|z|-a_k} + \frac{1}{|z-a_n|} + \frac{1}{|z-a_{n+1}|} + \sum_{k=n+2}^{+\infty} \frac{1}{a_k - |z|} = O(l(|z|)).$$

and condition 1) of Lemma 1 is satisfied. Further, $a_{n+1} - a_n \geq a_{n+1}(1 - 1/(1+\eta)) \geq \frac{\eta}{(1+\eta)l(a_{n+1})}$, i.e. condition b) of Lemma 2 holds. Similarly to that in the proof of Lemma 2 we obtain that $n(q/l(|z|), z, 1/f) \leq n^{**}(q)$ for each $z \in \mathbb{C}$. Therefore, by Lemma 1 function (1) is of bounded l-index, and Theorem 1 is proved.

We remark that if $l \in Q$ i $l_1(r) \geq cl(r)$, $c = \text{const} > 0$, then [2, p. 23] the l-index boundedness implies the l_1-index boundedness. Therefore, from Theorem 1 it follows that if $a_{n+1} \geq (1+\eta)a_n$ then for every function $l \in Q$ such that $n(r)/r = O(l(r))$ $(r \to \infty)$, function (1) is of bounded l-index and of unbounded l-index for every function $l \in Q$ such that $l(r) = o(n(r)/r)$ $(r \to \infty)$. In fact, otherwise from (2) we would have $\ln M_f(r) = o(N(r))$ $(r \to +\infty)$, where $N(r) = \int_0^r n(t)t^{-1}dt$. The last relation is impossible, because $n(r) = O(\ln r)$ $(r \to +\infty)$, and, hence [9], $\ln M_f(r) \sim N(r)\,(r \to +\infty)$.

We remark also that if $a_{n+1} = O(a_n)$ $(n \to \infty)$, $\{a_n\} = \bigcup_{j=1}^m \{a_{j,k}\}$, $m < \infty$, and $a_{j,k+1} \geq$ $\geq (1+\eta)a_{j,k}$ for all $k \geq 1$ and $1 \leq j \leq m$, then by Theorem 1 and the Multiplication theorem [2, p. 34] the conclusion of Theorem 1 holds.

Using Lemma 2, we prove the following

Theorem 2. *Let $l \in Q_*$ and (a_k) be a convex sequence such that $l(|a_n|) = O(l(|a_{n+1}|))$ as $n \to \infty$, $n(r)\ln n(r) = O(rl(r))$ and $\sum_{a_k \geq r}(1/a_k) = O(l(r))$ as $r \to +\infty$. Then function (1) is of bounded l-index.*

Proof. The convexity of a_n implies

$$\frac{a_n - a_k}{n-k} \geq \frac{a_n - a_1}{n-1}, \quad 1 \leq k \leq n-1. \tag{9}$$

Therefore, in view of condition $n \ln n = O(a_n l(a_n))$ $(n \to \infty)$, we have $a_{n+1} - a_n \geq$ $\geq (1+o(1))a_n/n \geq \ln n/(Kl(a_n))\,(n \to \infty)$, $K = \mathrm{const} > 0$, that is condition b) of Lemma 2 holds.

Further, using (9) we obtain

$$\sum_{k=1}^{n-1} \frac{1}{a_n - a_k} \leq \sum_{k=1}^{n-1} \frac{n-1}{(n-k)(a_n - a_1)} = O\left(\frac{n \ln n}{a_n}\right) = O(l(a_n)), \quad n \to \infty.$$

Inequality (6) also implies the inequality $a_{3n} \geq 2a_n$ $(n \to +\infty)$. Therefore,

$$\sum_{a_{n+2} \leq a_k \leq 2a_n} \frac{1}{a_k - a_n} \leq \sum_{a_{n+2} \leq a_k \leq 2a_n} \frac{k-1}{(a_k - a_1)(k-n)} \leq \frac{1}{a_n - a_1} \sum_{k=n+2}^{3n} \frac{k}{k-n} =$$
$$= O\left(\frac{n \ln n}{a_n}\right) = O(l(a_n)), \quad n \to \infty.$$

Finally,

$$\sum_{a_k \geq 2a_n} \frac{1}{a_k - a_n} \leq 2 \sum_{a_k \geq 2a_n} \frac{1}{a_k} = O(l(a_n)), \quad n \to \infty.$$

Hence, conditions c) and d) of Lemma 2 hold and function (1) is of bounded l-index. □

Remark. The conclusions of Theorems 1 and 2 are valid also for canonical products (1) with complex zeros, but in all conditions it is necessary replace to a_n by $|a_n|$.

4^0. Disproof of the conjecture.

Theorem 3. *Given $\rho \in (0,1]$ there exists an entire function f_ρ of zero genus of the form (1) with the following properties: i) $(\forall n \in \mathbb{N}) : a_n > 0$; ii) $\dfrac{n}{a_n} \searrow 0$ as $n \to +\infty$; iii) $\rho[f_\rho] = \varlimsup\limits_{r \to +\infty} \ln\ln M_{f_\rho}(r)/\ln r = \rho$; iv) f_ρ is of unbounded l-index for any $l \in Q_*$ such that $l(r) \sim n(r)/r$ as $r \to +\infty$, where $n(r)$ is the number of zeros f_ρ in $\{z : |z| \leq r\}$.*

Proof. Let $b_k = 2^{2^k}$, $k \in \mathbb{N}$, and $\rho \in (0,1]$. Define a nondecreasing function $\psi\colon \mathbb{N} \to [1,+\infty)$ by the equality $\ln\psi(n) = \sum\limits_{k>0, b_k<n} k^{-2}$. Then $1 \le \psi(n) \nearrow \exp\{\pi^2/6\}$ as $n \uparrow +\infty$.

Let $\varphi_\rho(x)$ be an arbitrary differentiable on $[1,+\infty)$ regularly growing function with the order $1/\rho - 1$ if $\rho \in (0,1)$ and slowly growing function to $+\infty$ if $\rho = 1$, i.e. $\varphi_\rho(cx) \sim c^{1/\rho-1}\varphi_\rho(x)$ and $\varphi_\rho(x) \nearrow +\infty$ as $x \uparrow +\infty$. In particular, $x\varphi_\rho'(x)/\varphi_\rho(x) \le C_1(\rho)$ for $x \ge 1$ and some positive constant $C_1(\rho)$. If $\rho = 1$ we require, in addition, that $\int_1^{+\infty} \frac{dx}{x\varphi_1(x)} < +\infty$.

Put $a_m = m\varphi_\rho(m)\psi(m)$, $m \in \mathbb{N}$. Evidently, f is of form (1), and properties i) and ii) hold. Further, by the definition of a_m, for every $\varepsilon > 0$ we have $m^{1/\rho-\varepsilon} < a_m < m^{1/\rho+\varepsilon}$ $(m \ge m_0(\varepsilon))$, thus $\rho[f_\rho] = \rho[n(r)] \overset{\text{def}}{=} \varlimsup\limits_{r\to+\infty} \ln^+ n(r)/\ln r = \rho$.

It remains to prove iv). Obviously, $a_m \sim a_{m+1}$ $(m \to +\infty)$, so for $r \in [a_m, a_{m+1})$

$$\frac{n(r)}{r} \sim \frac{m}{a_m} = \frac{1}{\varphi_\rho(m)\psi(m)} \sim \frac{e^{-\pi^2/6}}{\varphi_\rho(n(r))} \equiv \tilde{l}(r), \quad r \to +\infty. \tag{10}$$

We modify $\tilde{l}(r)$ slightly preserving monotonicity to get a continuous function $l(r) = \tilde{l}(r) + O(1)$ with $l(r) \searrow 0$ and $l(r)r \nearrow +\infty$ as $r \uparrow +\infty$, $l \in Q_*$. To prove iv) it is enough to show that condition 1) of Lemma 1 does not hold for $\tilde{l}$ defined by (10).

Let us estimate the distance between a_{b_k} and a_{b_k+1}:

$$\begin{aligned} a_{b_k+1} - a_{b_k} &= (b_k+1)\varphi_\rho(b_k+1)\psi(b_k+1) - b_k\varphi_\rho(b_k)\psi(b_k) \ge \\ &\ge b_k\varphi_\rho(b_k)(\psi(b_k+1) - \psi(b_k)) = b_k\varphi_\rho(b_k)\psi(b_k)(e^{k^{-2}} - 1) \sim \\ &\sim \frac{e^{\pi^2/6}}{k^2} b_k\varphi_\rho(b_k) \sim \frac{C_2 b_k\varphi_\rho(b_k)}{(\ln\ln b_k)^2}, \quad k \to +\infty, \end{aligned} \tag{11}$$

for some positive constant C_2. Put $x_k = a_{b_{k+1}} - c\varphi_\rho(b_k+1)$, where c is a fixed positive number. According to (10) there exist $k_0 \in \mathbb{N}$ and $q > 0$ such that $x_k \notin \{\zeta : |\zeta - a_{b_{k+1}}| \le q/l(a_{b_{k+1}})\}$ for $k \ge k_0$.

By (11) $x_k \notin \{\zeta : |\zeta - a_{b_k}| \le q/l(a_{b_k})\}$ for $k \ge k_1$, and consequently $x_k \notin G_q(f)$ for all sufficiently large k.

For $m \le b_k$ we have $x_k - a_m \le x_k - a_{b_k} = (1 + o(1))(a_{b_k+1} - a_{b_k})$. Thus, using (11) we obtain

$$\sum_{m=1}^{b_k} \frac{1}{x_k - a_m} \le \frac{b_k}{x_k - a_{b_k}} \le C_3 \frac{\ln\ln b_k}{\varphi_\rho(b_k)}, \quad k \to +\infty. \tag{12}$$

If m such that $b_k + 1 \le m \le 2b_k$, then $\psi(m) = \psi(b_k+1)$, and using the definition of ψ, the Lagrange theorem and properties of φ_ρ we get for some $\xi \in [b_{k+1}, m]$

$$\begin{aligned} a_m - a_{b_k+1} &= m\varphi_\rho(m)\psi(m) - (b_k+1)\varphi_\rho(b_k+1)\psi(b_k+1) = \\ &= \psi(b_k+1)\,(x\varphi_\rho(x))'\big|_{x=\xi}\,(m - b_k - 1) \le \\ &\le e^{\pi^2/6}(\varphi_\rho(\xi) + \xi\varphi_\rho'(\xi))(m - b_k - 1) \le C_4(\rho)\varphi_\rho(b_k+1)(m - b_k - 1). \end{aligned}$$

Hence,

$$
\begin{aligned}
\Big|\sum_{m=b_k+1}^{+\infty}\frac{1}{x_k-a_m}\Big| &\ge \sum_{m=b_k+2}^{2b_k}\frac{1}{a_m-a_{b_k+1}+a_{b_k+1}-x_k}\ge\\
&\ge \sum_{m=b_k+2}^{2b_k}\frac{1}{C_4(\rho)\varphi_\rho(b_k+1)(m-b_k-1)+c\varphi_\rho(b_k+1)}=\\
&=\frac{1}{\varphi_\rho(b_k+1)}\sum_{\varkappa=1}^{b_k-1}\frac{1}{C_2(\rho)\varkappa+c}\ge C_5(\rho)\frac{\ln b_k}{\varphi_\rho(b_k)},\quad k\to+\infty. \qquad (13)
\end{aligned}
$$

(12) and (13) imply that

$$
\Big|\frac{f'(x_k)}{f(x_k)}\Big|=\Big|\sum_{m=1}^{+\infty}\frac{1}{x_k-a_m}\Big|\ge\sum_{m=1}^{b_k}\frac{1}{a_m-x_k}+\sum_{m=b_k+1}^{2b_k}\frac{1}{a_m-x_k}\ge\frac{C_5}{2}\frac{\ln x_k}{\varphi_\rho(x_k)},\quad k\to+\infty.
$$

□

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