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GRADIENT LIKE MORSE-SMALE DYNAMICAL SYSTEMS ON 4-MANIFOLDS

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A complete invariant for gradient like Morse-Smale dynamical system (vector field) on closed 4-manifolds is constructed. It coincides with the Kirby diagram in the case of polar vector field without fixed points of index 3.

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Построен полный инвариант градиентно подобной динамической системы Морса-Смейла на замкнутом 4-мерном многообразии. В случае полярного векторного поля без особых точек индекса 3 он совпадает с диаграммой Кирби.

1. Introduction

A smooth dynamical system (vector field) is a Morse-Smale dynamical system if:

1) It has finite number of critical elements (fixed points and closed orbits) and all of them are nondegenerated (hyperbolic);
2) The stable and unstable manifolds of critical elements have transversal intersections;
3) The limit set of each trajectory is a critical element.

In the papers [1]–[4] a topological classification of Morse-Smale dynamical systems on 2-manifolds and in [4]–[8] on 3-manifolds is obtained. A Morse-Smale dynamic system is a gradient like if it does not contain closed trajectories [9].

Two vector fields are topologically equivalent, if there is a homeomorphism of the manifold to itself which maps integral trajectories into integral trajectories keeping their orientations. This homeomorphism is called conjugated.

The purpose of this paper is to obtain a topological classification of gradient like Morse-Smale dynamical systems on 4-manifolds.

In Section 2, we prove the criterion of topological equivalence of vector fields on 4-manifolds. We construct an invariant of a field that is 3-manifolds N with imbedded 2-spheres, framed links and surfaces. In Section 3 and 4, we use it for construction of the diagram of vector fields and prove the criteria of vector field equivalence using these diagrams.

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In Section 5, we investigate when the diagram can be realized as a diagram of a dynamical system.

2. Criterion of topological equivalence of vector fields. Let M be a closed 4-manifold, X and X' Morse-Smale vector fields.

Let $a_1, ..., a_k$ be fixed points of index 0 of the field X, and $a'_1, ..., a'_k$ of the field X'; $b_1 ...$, b_n and $b'_1, ..., b'_n$ are fixed points of index 1. Let K be the union of stable manifolds of the fixed points of index 0 and 1. We consider a tubular neighborhood $U(K)$ of this union which does not contain other fixed points and such that each trajectory has no more than one point of intersection with it. By $N = \partial U(K)$ we denote the boundary of this neighborhood for the field X and by N' for the field X'. Then these boundaries are 3-manifolds.

We denote the stable and unstable manifolds of the fixed point X by $v(x)$ and $u(x)$, respectively. Let S_i be surfaces which are intersections of unstable manifolds of the fixed points of index 1 with the manifold N. Then S_i is a set of not crossed spheres on the manifold N, the supplement to which in N are three-dimensional spheres with removed disks.

If $c_1, .., c_m$ are fixed points of index 2, then the intersections $v_i = v(c_i) \cap N$ form a set of the closed curves on the manifold N. Similarly for a field X' on manifold N' there is a set of the imbedded spheres and closed curves.

A handle of index k (k-handle) $h^k = D^k \times D^{n-k}$ attached to a manifold M with boundary is the union $M \cup_f h^k$ along the boundary of M according to an embedding $f\colon S^{k-1} \times D^{n-k} \to \partial M$. Then $D^k \times \{0\}$ and $\{0\} \times D^{n-k}$ are called the core and cocore of the handle. The sets $S^{k-1} \times \{0\}$ and $\{0\} \times S^{n-k-1}$ are called the attached and belt spheres of the handle h^k. Handle decomposition is a sequence of imbeddings $M_0 \subset M_1 \subset ... \subset M_N = M$ such that M_0 is a union of n-disks (0-handles), M_{i+1} is obtained from M_i by gluing handle. Handle decompositions are isomorphic, if there exists a homeomorphism between the manifolds which maps the handles onto the handles, the cores onto the cores and the cocores onto the cocores.

In another way, manifold $N = \partial U(K)$ together with the imbedded spheres and closed curves can be obtained if we consider the handle decomposition that is associated with the given vector field. This is a handle decomposition in which to each handle there corresponds exactly one fixed point (and vice versa), and the core of each handle lays on the stable manifold of the appropriate fixed point, and the cocore lays on the unstable one. Then $U(K)$ can be considered as the union of the handles of index 0 and 1, the imbedded spheres as the belt spheres of the handles of index 1, and a closed curve as the attached spheres of the handles of index 2. The gluing of each 2-handle is defined as a solid torus imbedded (neighborhood of the closed curve) in the manifold N. It can be given by a closed curve and a parallel on the torus or simply by an integer (in the case if the manifold N is a three-dimensional sphere). This parallel is called a framing.

We consider the intersections of the stable manifolds of the fixed points of index 3 with the manifold N. These intersections are surfaces spanned on the closed curves, that is surfaces with boundaries whose interiors are imbedded in N, and the boundary components coincide with the closed curves (one closed curve can be contained in several component of the boundary). Thus the closed curve so many times coincides with boundary components of a surface, as many trajectories are in the intersections of unstable and stable manifolds of the appropriate fixed points of index 2 and 3.

Theorem 1. *A vector field X is topologically equivalent to a field X' if and only if there is a homeomorphism of manifolds $f\colon N \to N'$ which maps the spheres onto the spheres,* *the closed curves onto the closed curves keeping their framing, and the surfaces onto the surfaces.*

Proof. Necessity. Let φ be a conjugated homeomorphism between the fields X and X'. We set a homeomorphism f as follows. For each point $x \in N$ we shall consider a trajectory $g(x)$ which contains it. We define the image $f(x) = \varphi(\gamma(x)) \cap N'$.

Since the unstable manifolds of the fixed points of index 1 map onto unstable manifolds of the fixed points of index 1 by the homeomorphism φ, the spheres map onto the spheres by the homeomorphism f. Similarly, the closed curves map onto the closed curves, and the surfaces onto the surfaces. It follows from existence of the homeomorphism φ that the appropriate framings of the closed curve are equal.

Sufficiency. Suppose that there is a homeomorphism $f\colon N \to N'$. We consider disks D_i which lay on unstable integrated manifolds $u(b_i)$, contain the points b_i and are bounded by the spheres S_i. Then we can extend homeomorphisms from the boundaries S_i of these disks up to homeomorphisms of disks that map integrated trajectories onto integrated trajectories (because each integrated trajectory which has the intersections with a disk D_i, except the fixed points, must intersect the boundary of the disk). Then the disks D_i decompose a neighborhood $U(K)$ into 4-disks D^4_j each of which contains one fixed point of index 0.

Let us extend homeomorphisms from boundaries of these disks to the interior. For this we consider the three-dimensional spheres S^3_j which are the boundaries of tubular neighborhoods of the fixed points of index 0 such that each trajectory from an interior D^4_j crosses the sphere S^3_j in a unique point. On each arc of the trajectory which crosses D^4_j we introduce a new parameterization in such a way that a point on S^3_j corresponds to value of the parameter $t = 0$, the point on the boundary D^4_j to $t = 1$ and the fixed point of index 0 to $t = -1$. We demand that on each of the intervals $(-1; 0)$ and $(0; 1)$ the parameter t is proportional to the length of the arc in the Riemannian metric that is fixed for all trajectories. The homeomorphism of the boundaries of the disks D^4_j sets a correspondence between the trajectories inside this disk. A required homeomorphism maps a point of each integrated trajectory to a point of the appropriate integrated trajectory with the same parameter t.

For each closed curve v_i we shall consider its tubular neighborhood $U(v_i)$. Let W_i be a neighborhood of the appropriate fixed point c_i which does not contain other fixed points and such that the boundary $\partial W_i = U(v_i) \cup V_i$ is the union of two solid tori. Thus the trajectories that have the intersection with $W_i \backslash c_i$ go in W_i through the solid torus $U(v_i)$ and leave it through the solid torus V_i.

We remove the solid torus $U(v_i)$ from the manifold N and glue the solid torus V_i to obtain the torus $\partial U(v_i) = \partial V$. Such the operation is called a spherical surgery along v_i. Fulfilling the spherical surgeries along all v_i, we denote the received manifold by L. Thus between the solid tori $U(v_i)$ and V_i without middle circles (closed curves v_i and w_i on the manifolds N and L) there is a homeomorphism which maps a point $x \in N \backslash v_i$ to $\gamma(x) \in L$. Then the homeomorphism of solid tori $U(v_i)$ and $U(v'_i)$ induces a homeomorphism between $V_i \backslash w_i$ and $V'_i \backslash w'_i$. Using equality of framings of the closed curves v_i and v'_i we can extend this homeomorphism to the homeomorphism of the solid tori V_i and V'_i. We actually have constructed a homeomorphism between the manifolds L and L'. Thus, surfaces from N map onto imbedded 2-spheres in L, and 2-spheres map onto surfaces. If a homeomorphism from N to N' maps surfaces onto surfaces then a homeomorphism from L to L' maps 2-spheres onto 2-spheres.

We extend the homeomorphism from the boundaries W_i to their interior, and from the manifold L to the other part of the manifold M, just as we did with extension of a homeomorphism from N to $U(K)$. The constructed homeomorphism of manifold M will be as required. □

3. Diagram of a polar vector field. Let a vector field be polar (i. e. with one source and one sink). In this case, K is a bouquet of circles and the manifold N is a connected sum $\#_n S^1 \times S^2$. We cut the manifold N by 2-spheres. The obtained manifold is a 3-sphere without disks. Thus the closed curves are divided into arcs, and surfaces into surfaces with these arcs and arc in 2-spheres as its boundaries.

So constructed three-dimensional spheres, together with the imbedded pairs of 2-spheres, with the given homeomorphisms of the spheres of one pair, the arcs and the closed curves with the framing and the surfaces are called a diagram of the vector field. Two diagrams are equivalent if there is a homeomorphism of the three-dimensional spheres of one diagram onto the three-dimensional spheres of other diagram which maps

1) the pairs of 2-spheres to the pairs of 2-spheres and commutes with the given homeomorphisms of these spheres,
2) the arcs and the closed curves to the arcs and the closed curves and preserves the framing,
3) the surfaces to the surfaces.

If instead of the surfaces we consider the number of the fixed points (or handles) of the index 3, we obtain the Kirby diagram of the manifold M[10].

Proposition 1. *Two polar gradient like Morse-Smale vector fields on 4-manifold are topologically equivalent if and only if their diagrams are equivalent.*

We will prove Proposition 1 in general case.

4. Diagram of vector field in general case. As above, we construct the manifold N with the imbedded 2-dimensional spheres, the framing circles and surfaces with boundary on this circles. Cutting the manifold N by the spheres, we obtain 3-dimensional spheres with the imbedded pairs of 2-dimensional spheres, as the boundaries of the deleted disks, the arcs which connect them and the closed curves with the framing and the surfaces with the boundaries on this arc and closed curves.

The three-dimensional spheres which are constructed in such a way, together with the pairs of the embedded 2-dimensional spheres with the given homeomorphisms of the spheres of one pair, the arcs and the closed curve with the framing and the surfaces with the boundaries on them are called a diagram of the vector field. Two diagrams are equivalent if there is a homeomorphism of three-dimensional spheres of one diagram onto three-dimensional spheres of other diagram which maps the pairs of 2-dimensional spheres onto the pairs of spheres, commuting with the given homeomorphisms of these spheres, the arcs and the closed curves onto the arcs and the closed curves keeping the framing, and the surfaces onto the surfaces.

Proposition 2. *Two gradient like Morse-Smale vector fields on 4-manifold are topologically equivalent if and only if their diagrams are equivalent.*

Proof. Necessity. A required homeomorphism of three-dimensional spheres can be obtained as a result of restriction of the homeomorphism $f\colon N \to N'$ on the corresponding three-dimensional spheres with deleted disks and extension of it on these disks.

Sufficiency. If there exists a homeomorphism between three-dimensional spheres, we consider its restriction on these spheres with deleted three-dimensional disks which are bounded by pairs of 2-dimensional spheres. Using the fact that this homeomorphism commutes with given homeomorphisms of 2-dimensional spheres, we can glue it into a homeomorphism of three-dimensional manifolds $f\colon N \to N'$. Using Theorem 1 we shall the receive necessary conditions of the proposition. □

5. Realization of dynamical system with given invariant. In this section we describe when the three-dimensional spheres, together with the pairs of imbedded spheres, with given homeomorphisms of spheres of one pair, arcs and closed curves with framing and surfaces with boundary on its are the diagram of some vector field.

We start with the necessary conditions on the surfaces. Since, after the spherical surgery along the curves and the arcs with the framing, the surfaces should became the spheres, they are homeomorphic to the spheres with deleted disks. Their intersections with the tori, which are either the boundary of the curve or the arcs with framing, consist of the circles that set the framing. A diagrams that satisfy these conditions will be called admissible.

1) We shall consider a diagram, in which there are no pairs of imbedded spheres and surfaces. Each such diagram is the Kirby diagram. If it is the diagram of a vector field, then this field has one source and sink and does not have the fixed points of index 1 and 3. Then after spherical surgery along the framing link given by this diagram, the three-dimensional sphere should turn out. Therefore, the Kirby diagram will be the diagram of a vector field in the only case when it is the diagram of three-dimensional sphere. From [10] it follows that it will be in the only case when from this diagram it is possible to receive the empty diagram by adding a closed curve and addings or removals from the diagram of the trivial knot with the framing $+1$ or -1. Fenn and Rourke proved that these two movements can be replaced by one, blow-up and its reverse, blow-down [11].

Nevertheless, these criteria do not give an algorithm which check whether the Kirby diagram is the diagram of three-dimensional sphere or not. Such methods of the recognition of three-dimensional sphere having the Heegaard diagram are in [12] and [13]. Using Rolfsen's idea, Christian Mercat showed how the Heegaard diagrams can be constructed from the Kirby diagrams.

2) We show how from any admissible diagram it is possible to obtain the Kirby diagram (without embedded 2-dimensional spheres). In the beginning we get rid from superfluous 1 and 3 handles (that is pairs of embedded spheres and surfaces) which can be reduced together with additional 0 and 4 handles. If the diagram consists of several three-dimensional spheres, then using the connectedness of the manifold M we can find a pair of 2-dimensional spheres (1-handle) laying in different three-dimensional spheres. If we take the connected sum of three-dimensional spheres along this pair, we obtain the diagram with one three-dimensional sphere fewer than the initial diagram. We repeat this procedure until there will be one three-dimensional sphere.

Similarly it is possible to get rid from superfluous surfaces, consistently deleting them. Thus at each stage the surface will be superfluous, if there not exists a simple closed curve, which has transversal intersections with given surfaces and does not have with others.

We replace the remained pairs of spheres and surfaces with 2-handles. Thus we shall receive the diagram from 1).

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