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M. I. Ivanchov

INVERSE PROBLEM FOR A MULTIDIMENSIONAL HEAT EQUATION WITH AN UNKNOWN SOURCE FUNCTION

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We establish existence and uniqueness conditions for the inverse problem consisting of the determination of unknown functions $f_1(x)$ and $f_2(t)$ in the equation $u_t = \Delta u + g_0(x,t) + f_1(x)g_1(t)+f_2(t)g_2(x)$ in the case of the Dirichlet boundary condition and the overdetermination conditions of the form $u(x_0,t) = \varkappa(t),\ x_0 \in D \in R^n,\ t \in [0,T],\ \int_0^T u(x,t)dt = \psi(x),\ x \in \overline{D}$.

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Установлены условия существования и единственности решения обратной задачи, состоящей в нахождении неизвестных функций $f_1(x)$ и $f_2(t)$ в уравнении $u_t = \Delta u+g_0(x,t)+f_1(x)g_1(t) + f_2(t)g_2(x)$ в случае граничного условия типа Дирихле и условий переопределения вида $u(x_0,t) = \varkappa(t),\ x_0 \in D \in R^n,\ t \in [0,T],\ \int_0^T u(x,t)dt = \psi(x),\ x \in \overline{D}$.

In coefficient inverse problems for parabolic equations that include the problems of determination of the source function, unknown coefficients are supposed to depend, as usual, only on a part of arguments. On the other hand, research methods for these problems are determinated essentially by the fact on which arguments the unknown functions depend, namely, spatial variables or time-like ones (see [1-7]). That is why the inverse problems for finding functions of various variables are of the great interest. Such problems were investigated in the papers [8,9].

In the present paper an inverse problem for the heat equation with a source function depending on two unknown functions of various arguments is studied.

In the bounded domain $Q_T = \{(x,t) : x \in D \in \mathbb{R}^n, 0 < t < T\}$, consider an inverse problem for the equation

$$u_t = \Delta u + g_0(x,t) + f_1(x)g_1(t) + f_2(t)g_2(x), \quad (x,t) \in Q_T, \tag{1}$$

with unknown functions $f_1(x)$ i $f_2(t)$, the initial condition

$$u(x,0) = \varphi(x), \quad x \in \overline{D}, \tag{2}$$

the boundary condition

$$u(x,t)\big|_{S_T} = \mu(x,t), \quad (x,t) \in S_T \equiv S \times [0,T],\ S = \partial D, \tag{3}$$

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and the overdetermination conditions

$$u(x_0,t) = \varkappa(t), \quad x_0 \in D, \quad t \in [0,T], \tag{4}$$

$$\int\limits_0^T u(x,t)dt = \psi(x), \quad x \in \overline{D}. \tag{5}$$

By the solution of problem (1)–(5) we mean a triple of functions $(f_1(x),\ f_2(t), u(x,t))$ from the class $H^{\gamma}(\overline{D}) \times H^{\gamma/2}[0,T] \times H^{2+\gamma,1+\gamma/2}(\overline{Q}_T), 0<\gamma<1$, (see [10]) verifying conditions (1)–(5).

Theorem 1. *Suppose that the following conditions are fulfilled:*

(A1) $\varphi(x) \in H^{2+\gamma}(\overline{D}), \psi(x) \in H^{2+\gamma}(\overline{D}), \mu(x,t) \in H^{2+\gamma,1+\gamma/2}(\overline{Q}_T), \varkappa(t) \in H^{1+\gamma/2}[0,T]$, $g_0(x,t) \in H^{\gamma,\gamma/2}(\overline{Q}_T)$, $g_1(t) \in H^{\gamma/2}[0,T]$, $g_2(x) \in H^{\gamma}(\overline{D})$, $S \in H^{2+\gamma}$, $0<\gamma<1$;

(A2) $g_2(x_0) \neq 0,\ \int_0^T g_1(t)dt \neq 0, g_1(t) \Big/ \int_0^T g_1(t)dt \geq 0$;

(A3) $\varphi(x)\big|_S = \mu(x,0)\big|_S$, $\varphi(x_0) = \varkappa(0)$, $\int_0^T \varkappa(t)dt = \psi(x_0)$, $\psi(x)\big|_S = \int_0^T \mu(x,t)dt\big|_S$.

Then for sufficiently small $T>0$ there exists a unique solution of problem (1)–(5).

Proof. Finding the solution of direct problem (1)–(3) by means of the Green function $G(x,t,\xi,\tau)$ and using overdetermination conditions (4), (5), we reduce inverse problem (1)–(5) to the following system of equations

$$f_1(x) = f_{01}(x) + \frac{1}{\int_0^T g_1(t)dt} \int\limits_0^T \int\limits_D G(x,T,\xi,\tau)(f_1(\xi)g_1(\tau) + f_2(\tau)g_2(\xi))d\xi d\tau, \quad x \in \overline{D}, \tag{6}$$

$$f_2(t) = f_{02}(t) + \frac{1}{g_2(x_0)} \int\limits_0^t \int\limits_D \Delta G(x_0,t,\xi,\tau)(f_1(\xi)g_1(\tau) + f_2(\tau)g_2(\xi))d\xi d\tau, \quad t \in [0,T], \tag{7}$$

in which the first equation is a Fredholm integral equation, the other one a Volterra integral equation of the second kind, and $f_{01}(x), f_{02}(t)$ are known function which are determinated by the problem data.

Consider the first equation of the system. We shall show that the corresponding to (6) homogeneous equation

$$f(x) = \frac{1}{\int_0^T g_1(t)dt} \int\limits_0^T \int\limits_D G(x,T,\xi,\tau) f(\xi) g_1(\tau) d\xi d\tau, \quad x \in \overline{D}, \tag{8}$$

has the only trivial solution. For this, consider the auxiliary inverse problem with respect to unknown functions $(f(x), v(x,t))$

$$v_t = \Delta v + f(x) g_1(t), \quad (x,t) \in Q_T, \tag{9}$$

$$v(x,0) = 0, \quad x \in \overline{D}, \quad v(x,t)\Big|_{S_T} = 0, \tag{10}$$

$$\int\limits_0^T v(x,t)dt = 0, \quad x \in \overline{D}. \tag{11}$$

It is easy to see that problem (9)–(11) is reduced to the integral equation

$$f(x) = v(x,T) \left(\int\limits_0^T g_1(t)dt \right)^{-1}, \tag{12}$$

where $v(x,T) = \int_0^T \int_D G(x,T,\xi,\tau) f(\xi) g_1(\tau)\, d\xi d\tau$.

If the condition $\int_0^T g_1(t)dt \neq 0$ is satisfied then inverse problem (9)–(11) and integral equation (12) are equivalent in the class of solutions $(f(x), v(x,t)) \in H^{\gamma}(\overline{D}) \times H^{2+\gamma, 1+\gamma/2}(\overline{Q}_T)$. Indeed, if $(f(x), v(x,t))$ is a solution of problem (9)–(11) from the indicated class then $f(x)$ is evidently a solution of integral equation (12). On the other hand, if $f(x) \in H^{\gamma}(\overline{D})$ is a solution of integral equation (12) then we put it in equation (9) and we find a solution of problem (9), (10). Show that condition (11) will be satisfied as well. For this we integrate equation (9) with respect of t from 0 to T:

$$v(x,T) = \int\limits_0^T \Delta v(x,t)dt + f(x) \int\limits_0^T g_1(t)dt.$$

Taking into account that $f(x)$ is a solution of equation (12), we get

$$\Delta \left(\int\limits_0^T v(x,t)dt \right) = 0, \quad x \in \overline{D}. \tag{13}$$

From the boundary condition for the function $v(x,t)$ we obtain

$$\int\limits_0^T v(x,t)dt \Bigg|_S = 0. \tag{14}$$

Problem (13), (14) with respect to the unknown function $\int_0^T v(x,t)dt$ has the only trivial solution. Hence, condition (11) is verified and the equivalence of problem (9)–(11) and integral equation (12) is established.

Substitute $f(x)$ in equation (9) by its expression according to equation (12):

$$v_t = \Delta v + q(t)v(x,T), \quad (x,t) \in Q_T, \tag{15}$$

where $q(t) = g_1(t)\left(\int_0^T g_1(t)dt\right)^{-1}$, and instead of inverse problem (9)–(11) we come to direct problem (15), (10) for the equation containing the trace of the unknown function on the plane $t = T$. Note that similar problems are considered in [11].

We reduce problem (15), (10) to the integral equation

$$v(x,T) = \int_0^T \int_D G(x,T,\xi,\tau)q(\tau)v(\xi,\tau)\, d\xi d\tau, \tag{16}$$

where $G(x,t,\xi,\tau)$ is the Green function for the heat equation

$$u_t = \Delta u, \quad (x,t) \in Q_T,$$

with conditions (10).

We show that integral equation (16) has a unique solution $v(x,T) \equiv 0$. If this is not true and the function $v(x,T)$ takes, for example, positive values then in the point of its positive maximum we obtain

$$v(\tilde{x},T) = \int_0^T \int_D G(\tilde{x},T,\xi,\tau)q(\tau)v(\xi,\tau)\, d\xi d\tau \le v(\tilde{x},T) \int_0^T \int_D G(\tilde{x},T,\xi,\tau)q(\tau)\, d\xi d\tau. \tag{17}$$

Show that for $\tilde{x} \in D, 0 < \tau < t < T$ the following inequality holds:

$$\int_D G(\tilde{x},T,\xi,\tau)d\xi < 1. \tag{18}$$

Indeed, the function $w(x,t) \equiv 1$ is a solution of the problem

$$w_t = \Delta w, \ (x,t) \in Q_T, \quad w\big|_{t=0} = 1, \ x \in \overline{D}, \quad w\big|_{S_T} = 1,$$

for which the expression $\int_D G(\tilde{x},T,\xi,0)d\xi$ is the only one of its positive terms.

Taking into account estimate (18) and the fact that from the definition of the function $q(t)$ we have $\int_0^T q(t)dt = 1$, we come to contradiction in (17). This means that the function $v(x,T)$ cannot attain positive values. By the same way we conclude that $v(x,T) \le 0$, and, hence, $v(x,T) \equiv 0, x \in \overline{D}$. It implies that equation (16) together with equation (8) has the only trivial solution. By the Fredholm alternative there exists a unique solution $f_1(x)$ of equation (6). It may be given by the aid of the resolvent $\Gamma(x,\xi)$ in the form

$$f_1(x) = \int_D \Gamma(x,y)\left(f_{01}(y) + \frac{1}{\int_0^T g_1(t)dt}\int_0^T \int_D G(y,T,\xi,\tau)f_2(\tau)g_2(\xi)d\xi d\tau\right) dy, \quad x \in \overline{D}. \tag{19}$$

We substitute (19) into equation (7) and reduce it to the form

$$f_2(t) = \tilde{f}_{02}(t) + \int\limits_0^t K_1(t,\tau) f_2(\tau) d\tau + \int\limits_0^t d\tau \int\limits_0^T K_2(t,\tau,\sigma) f_2(\sigma) d\sigma, \quad t \in [0,T], \tag{20}$$

where $\tilde{f}_{02}(t)$ is a known function, the kernels $K_1(t,\tau), K_2(t,\tau,\sigma)$ are expressed in the evident way by means of the given data and the following estimates hold [10]:

$$|K_1(t,\tau)| \leq \frac{C_1}{(t-\tau)^{1-\gamma/2}}, \quad |K_2(t,\tau,\sigma)| \leq \frac{C_2}{(t-\tau)^{1-\gamma/2}}, \quad 0 \leq \tau < t \leq T,\ 0 \leq \sigma \leq T.$$

Applying the iteration method to equation (20) it is easy to establish the existence of a unique solution of equation (18) for sufficiently small $T : 0 < T \leq T_0$, where the number $T_0 > 0$ is determinated by the given data. Using the properties of the heat potentials [12], it is easy to verify the corresponding smoothness of the obtained solution $f_1(x), f_2(t)$ of integral equations system (6), (7). The solution of direct problem (1)–(3) constructed by the use of the obtained functions $f_1(x), f_2(t)$ has the necessary smoothness as well [10]. Thus, the theorem is proved. □

Note that this result can be used for study of the uniqueness of solution of inverse problems for parabolic equations with two unknown coefficients, one depending on the spatial variables and another on the time variable [10].

REFERENCES

1. Прилепко А. И., Костин А. Б. *Об обратных задачах определения коэффициента в параболическом уравнении. I* // Сиб. мат. журн. – 1992. – Т. 33, №3. – С. 146–155.
2. Прилепко А. И., Костин А. Б. *Об обратных задачах определения коэффициента в параболическом уравнении. II*// Сиб. мат. журн. – 1993. – Т. 34, №5. – С. 147–162.
3. Прилепко А. И., Соловьев В. В. *Теоремы разрешимости и метод Ротэ в обратных задачах для уравнения параболического типа. II*// Дифференц. уравн. – 1987. – Т. 23, №11. – С. 1971–1980.
4. Прилепко А. И., Соловьев В. В. *Теоремы разрешимости и метод Ротэ в обратных задачах для уравнения параболического типа. I*// Дифференц. уравн. – 1987. – Т. 23, №10. – С. 1791–1799.
5. Majchrowski M. *On inverse problem with nonlocal condition for parabolic systems of partial differential equations and pseudoparabolic equations* // Demonstr. Math. – 1993. – V. 26, №1. – P. 255–275.
6. Іванчов М. І. *Обернена задача визначення потужності джерел тепла для параболічного рівняння при довільних крайових умовах* // Мат. методи і фіз.-мех. поля. – 1997. – Вип. 40, №1. – С. 125–129.
7. Смирнов Г. П., Фатыхов М. А. *О задаче определения правой части уравнения теплопроводности* // Дифференц. уравн. – 1988. – Т. 24, №4. – С. 711–716.
8. Ivantchov M. I. *Détermination simultanée de deux coefficients aux variables diverses dans une équation parabolique* // Мат. студії. – 1998. – Т.10, №2. – С. 173–187
9. Саватеев Е. Г. *О задаче идентификации коэффициента в параболическом уравнении* // Сиб. мат. журн. – 1995. – Т. 36, №1. – С. 177–185.
10. Ладыженская О. А., Солонников В. А., Уральцева Н. Н. Линейные и квазилинейные уравнения параболического типа. – М.: Наука, 1967. – 736 с.

11. Cannon J. R., Yin H. M. *On a class of nonlinear parabolic equations with nonlinear trace type functionals* // Inverse Problems – 1991. – V. 7. – P. 149–161.

12. Фридман А. Уравнения с частными производными параболического типа. – М.: Мир, 1968. – 428 с.

Faculty of Mechanics and Mathematics, Lviv National University

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