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P. V. Filevych, S. I. Fedynyak

ON BELONGING OF ENTIRE DIRICHLET SERIES TO CONVERGENCE CLASS

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Let $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers increasing to $+\infty$ and $S(\Lambda)$ be the class of entire Dirichlet series $F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s = \sigma + it$. Put $M(\sigma, F) = \max\{|F(s)| : \operatorname{Re} s = \sigma\}$ and let $\mu(\sigma, F) = \max\{|a_n|\exp\{\sigma\lambda_n\} : n \ge 0\}$ be the maximal term of the function $F \in S(\Lambda)$. We prove that in order that $\int_0^{+\infty} e^{-\sigma\rho} \ln \mu(\sigma, F) d\sigma < \infty$ imply $\int_0^{+\infty} e^{-\sigma\rho} \ln M(\sigma, F) d\sigma < \infty$ for every $F \in S(\Lambda)$, it is necessary and sufficient that $\ln n = O(\lambda_n)$, $n \to \infty$.

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Пусть $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ — возрастающая к $+\infty$ последовательность неотрицательных чисел, а $S(\Lambda)$ — класс целых рядов Дирихле $F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s = \sigma + it$. Положим $M(\sigma, F) = \max\{|F(s)| : \operatorname{Re} s = \sigma\}$ и пусть $\mu(\sigma, F) = \max\{|a_n|\exp\{\sigma\lambda_n\} : n \ge 0\}$ — максимальный член функции $F \in S(\Lambda)$. Доказано, что для того, чтобы из неравенства $\int_0^{+\infty} e^{-\sigma\rho} \ln \mu(\sigma, F) d\sigma < \infty$ следовало неравенство $\int_0^{+\infty} e^{-\sigma\rho} \ln M(\sigma, F) d\sigma < \infty$ для любой $F \in S(\Lambda)$, необходимо и достаточно, чтобы $\ln n = O(\lambda_n)$, $n \to \infty$.

Let $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers increasing to $+\infty$ and $S(\Lambda)$ be the class of entire Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \qquad s = \sigma + it. \tag{1}$$

For $F \in S(\Lambda)$ let $M(\sigma, F) = \max\{|F(s)| : \operatorname{Re} s = \sigma\}$ and let $\mu(\sigma, F) = \max\{|a_n|\exp\{\sigma\lambda_n\} : n \ge 0\}$ be the maximal term of series (1).

As in [1] we say that entire Dirichlet series (1) belongs to convergence class if and only if the condition

$$\int_0^{+\infty} \frac{\ln M(\sigma, F)}{e^{\sigma\rho}} d\sigma < \infty, \tag{2}$$

is valid, where $0 < \rho < +\infty$. According to the Cauchy inequality, $\mu(\sigma, F) \le M(\sigma, F)$ and therefore (2) implies the inequality

$$\int_0^{+\infty} \frac{\ln \mu(\sigma, F)}{e^{\sigma\rho}} d\sigma < \infty. \tag{3}$$

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In [2] it is noted that if for series (1) the condition

$$\ln n = O(\lambda_n), \qquad n \to \infty, \tag{4}$$

holds then (3) implies (2).

At the Lviv regional workshop in Mathematical Analysis M. M. Sheremeta formulated a problem to determine a necessary and sufficient condition on the sequence Λ under which for every entire Dirichlet series $F \in S(\Lambda)$ inequalities (2) and (3) are equivalent. It turns out that (4) is just such a condition, i.e. the following theorem is true.

Theorem. *Let $\rho \in (0; +\infty)$. In order that for every Dirichlet series $F \in S(\Lambda)$ inequality (3) implies inequality (2) it is necessary and sufficient that the sequence Λ satisfies condition (4).*

Proof. It is sufficient to prove that if condition (4) is not valid then there exists Dirichlet series $F \in S(\Lambda)$ such that (3) holds and (2) fails.

Using Lemma 1 from [3] it is easy to show that from every increasing to $+\infty$ sequence of nonnegative numbers $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ such that $\overline{\lim\limits_{n\to\infty}}(\ln n/\lambda_n) = +\infty$, we may choose a subsequence $\Lambda^* = \{\lambda_k^*\}_{k=0}^{\infty}$ for which $\overline{\lim\limits_{k\to\infty}}(\ln k/\lambda_k^*) = +\infty$ and

$$\lim_{k\to\infty} \frac{\ln k}{\lambda_k^* \ln \lambda_k^*} = 0. \tag{5}$$

Therefore we may find increasing to $+\infty$ sequences $\{k_p\}_{p=0}^{\infty}$ and $\{l_p\}_{p=0}^{\infty}$ of positive integers and positive numbers respectively such that for every $p \geq 0$ the relations $\ln k_p = l_p \lambda_{k_p}^*$;

$$\lambda_{m_p}^* > 2\lambda_{k_p}^*, \qquad m_p \stackrel{\text{def}}{=} [k_{p+1}/2]; \tag{6}$$

$$0 \leq \varkappa_p \stackrel{\text{def}}{=} \rho^{-1} \ln(l_{p+1}\lambda_{k_{p+1}}^*) \uparrow +\infty; \tag{7}$$

$$\sum_{p=0}^{\infty} \frac{1}{l_p} < \infty; \qquad \sum_{p=0}^{\infty} \frac{\varkappa_p \lambda_{k_p}^*}{e^{\varkappa_p \rho}} < \infty \tag{8}$$

hold.

We put $a_{k_0}^* = 1$,

$$a_{k_{p+1}}^* = \left(\prod_{i=0}^{p} e^{\varkappa_i(\lambda_{k_{i+1}}^* - \lambda_{k_i}^*)}\right)^{-1}, \qquad p \geq 0. \tag{9}$$

Let also

$$a_k^* = a_{k_p}^* \exp\{-\varkappa_p(\lambda_k^* - \lambda_{k_p}^*)\}, \qquad \text{if} \quad k \in [m_p; k_{p+1}) \quad \text{and} \quad p \geq 0. \tag{10}$$

If the value a_k^* for some $k \geq 0$ is not defined yet, then we put $a_k^* = 0$. We remark that $a_k^* \leq 1$ for all $k \geq 0$.

Let us consider the Dirichlet series

$$F^*(s) = \sum_{k=0}^{\infty} a_k^* e^{s\lambda_k^*} = \sum_{p=0}^{\infty} \left(a_{k_p}^* e^{s\lambda_{k_p}^*} + \sum_{k=m_p}^{k_{p+1}-1} a_k^* e^{s\lambda_k^*}\right). \tag{11}$$

From (9) and (7) it follows that

$$\frac{\ln|a^*_{k_p}| - \ln|a^*_{k_{p+1}}|}{\lambda^*_{k_{p+1}} - \lambda^*_{k_p}} = \varkappa_p \uparrow +\infty, \qquad p \to \infty. \tag{12}$$

From (12) and (10) we easily obtain the inequality $a^*_k \le \exp\{-\varkappa_p(\lambda^*_k - \lambda^*_{k_p}\}$ if $k \in [m_p; k_{p+1}]$ and $p \ge 0$, and therefore according to (6)

$$a^*_k \le \exp\left\{-\varkappa_p\left(\lambda^*_k - \frac{1}{2}\lambda^*_{m_p}\right)\right\} \le \exp\left\{-\frac{1}{2}\varkappa_p\lambda^*_k\right\}, \qquad k \in [m_p; k_{p+1}], \quad p \ge 0. \tag{13}$$

Using (13), (7) and (5) for all $k \in [m_p; k_{p+1}]$ and $p \ge 0$ we have

$$\frac{\ln k}{-\ln a^*_k} \le \frac{\ln k_{p+1}}{\varkappa_p \lambda^*_k/2} \le \frac{2\rho\ln(2m_p+2)}{\lambda^*_k \ln(l_{p+1}\lambda^*_{k_{p+1}})} \le \frac{2\rho\ln(2m_p+2)}{\lambda^*_{m_p}\ln(l_{p+1}\lambda^*_{m_p})} = o(1), \qquad p \to \infty,$$

hence $\ln k = o(\ln|a^*_k|)$, $k \to \infty$. In view of the latter inequality the abscissa of absolute convergence of the series (11) can be found [4] by the formula $\sigma_a = \varliminf\limits_{k\to\infty} \frac{1}{\lambda^*_k}\ln\frac{1}{|a^*_k|}$. From (13) we obtain $\sigma_a = +\infty$ and thus $F^* \in S(\Lambda^*)$.

Further we remark that (12) implies the equalities

$$\mu(\sigma, F^*) = |a^*_{k_{p+1}}|e^{\sigma\lambda^*_{k_{p+1}}}, \qquad \text{if} \quad \sigma \in [\varkappa_p; \varkappa_{p+1}) \quad \text{and} \quad p \ge 0, \tag{14}$$

and also $\mu(\varkappa_p, F^*) = |a^*_{k_p}|\exp\{\varkappa_p\lambda^*_{k_p}\}$ if $p \ge 0$. Therefore from (13) for all $p \ge 0$ we have:

$$\begin{aligned}\int_{\varkappa_p}^{\varkappa_{p+1}} \frac{\ln\mu(\sigma,F^*)}{e^{\sigma\rho}}d\sigma &= -\frac{1}{\rho}\int_{\varkappa_p}^{\varkappa_{p+1}} \ln\mu(\sigma,F^*)de^{-\sigma\rho} = \\ &= -\frac{\ln\mu(\sigma,F^*)}{\rho e^{\sigma\rho}}\Bigg|_{\varkappa_p}^{\varkappa_{p+1}} + \frac{\lambda^*_{k_{p+1}}}{\rho}\int_{\varkappa_p}^{\varkappa_{p+1}}\frac{d\sigma}{e^{\sigma\rho}} \le \\ &\le \frac{\ln\mu(\varkappa_p,F^*)}{\rho e^{\varkappa_p\rho}} + \frac{\lambda^*_{k_{p+1}}}{\rho^2 e^{\varkappa_p\rho}} \le \frac{\varkappa_p\lambda^*_{k_p}}{\rho e^{\varkappa_p\rho}} + \frac{1}{\rho^2 l_{p+1}}.\end{aligned}$$

Hence and from (8) we conclude that for $F = F^*$ (3) holds.

On the other hand, (11), (10) and (12) give:

$$M(\varkappa_p, F^*) \ge \sum_{k=m_p}^{k_{p+1}-1} a^*_k e^{\varkappa_p\lambda^*_k} = \sum_{k=m_p}^{k_{p+1}-1} a^*_{k_p} e^{\varkappa_p\lambda^*_{k_p}} = (k_{p+1} - m_p)\mu(\varkappa_p, F^*) \ge k_{p+1}, \quad p \ge p_0.$$

Thus $\ln M(\varkappa_p, F^*) \ge \ln k_{p+1} = l_{p+1}\lambda^*_{k_{p+1}} = e^{\rho\varkappa_p}$, $p \ge p_0$, and therefore for $F = F^*$ relation (2) does not hold.

In order to complete the proof of the theorem it is enough to put $a_n = a^*_k$ if $\lambda_n = \lambda^*_k \in \Lambda^*$ and $a_n = 0$ if $\lambda_n \notin \Lambda^*$ and consider Dirichlet series (4) with coefficients defined in such a way. This series belongs to $S(\Lambda)$ and (3) is valid for it, while (2) is not valid. $\square$

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Faculty of Mechanics and Mathematics, Lviv National University

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