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ON TOPOLOGICAL STRUCTURE OF TOPOLOGICAL SEMILATTICES WITH OPEN PRINCIPAL IDEALS

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The compact topological semilattices with open principal ideals are investigated.

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Исследуются компактные топологические полурешетки с открытыми главными идеалами.

In this note we continue investigations of compact topological semilattices with open principal ideals started in [4]. All topological spaces considered in this paper are Hausdorff. Under a *topological semilattice* we understand a topological space S endowed with a continuous associative commutative idempotent operation $\wedge : S \times S \to S$. A subset $\mathcal{I} \subset X$ is called an *ideal* in X if $x \wedge y \in \mathcal{I}$ for every $x \in \mathcal{I}$ and $y \in X$. An ideal $\mathcal{I} \subset X$ is called a *principal ideal* if $\mathcal{I} = \downarrow a = \{x \in X : x \wedge a = x\}$ for some $a \in X$. Clearly, each principal ideal is a closed subsemilattice of X. If every principal ideal is open in X, then we say that X is a topological semilattice with open principal ideals. The semilattice operation of X induces a partial order on X: $x \le y$ if $x \wedge y = x$. A subset $C \subset X$ is called a *chain* (resp. *antichain*) if for every $x, y \in C$ $x \wedge y \in \{x, y\}$ (resp. $x \wedge y \notin \{x, y\}$).

Topological semilattices with open principal ideals are tightly connected with so-called *well-founded* semilattices, i.e., semilattices whose partial order is well-founded. We recall that a partially ordered set $(P, \le)$ is defined to be *well-founded* if every subset of P has a minimal element, equivalently, if P contains no infinite decreasing sequences, see [6, §14.1].

In fact, for a linearly ordered compact topological semilattice S the following conditions are equivalent: (1) S is a topological semilattice with open principal ideals, (2) S is a well-founded semilattice, and (3) S is topologically isomorphic to some non-limit ordinal α endowed with the interval topology and the semilattice min-operation. Note that the implication (1)⇒(2) still holds for any compact topological semilattice, while the converse implication fails in general: the one-point compactification $\alpha D = \{*\} \cup D$ of any infinite discrete space D endowed with the continuous semilattice operation

$$x \wedge y = \begin{cases} x, & \text{if } x = y \\ *, & \text{if } x \ne y \end{cases}$$

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is a well-founded compact topological semilattice which is not a topological semilattice with open principal ideals.

We recall that a topological space X is called *scattered* if every non-empty subspace of X has an isolated point. By $\mathrm{Iso}(X)$ we denote the set of all isolated points of a topological space X. The following structural theorem belongs to O. Gutik [4].

Theorem. (Gutik) *Let S be a compact topological semilattice with open principal ideals. Then*

(1) S is a scattered space with $|\mathrm{Iso}(S)| = |S|$;

(2) Every antichain in $\mathrm{Iso}(S)$ is finite;

(3) S is topologically isomorphic to a non-limit ordinal α if and only if S is linearly ordered, i.e., $x \wedge y \in \{x, y\}$ for any $x, y \in S$.

In this note we show that an isomorphic copy of a sufficiently large non-limit ordinal α can be found in any compact topological semilattice with open principal ideals.

Theorem 1. *Every infinite compact topological semilattice S with open principal ideals contains a subsemilattice topologically isomorphic to some non-limit ordinal α with $|\alpha| = |S|$.*

Proof. By the first statement of the Gutik Theorem, $\mathrm{Iso}(S)$ is a partially ordered infinite set. It follows from the Erdős-Dushnik-Miller Partition Theorem [3] (see also [6, p. 233]) that either $\mathrm{Iso}(S)$ contains an infinite antichain or else $\mathrm{Iso}(S)$ contains a chain $C \subset \mathrm{Iso}(S)$ with $|C| = |\mathrm{Iso}(S)|$. By the second statement of the Gutik Theorem, the first case is not possible, consequently, $\mathrm{Iso}(S)$ contains a chain C of cardinality $|C| = |\mathrm{Iso}(S)|$. Then the closure $\bar{C}$ of C in S is a compact linearly ordered topological semilattice with open principal ideals, which by the third statement of the Gutik Theorem, is topologically isomorphic to some non-limit ordinal α with $|\alpha| = |\bar{C}| \geq |C| = |\mathrm{Iso}(S)| = |S|$. □

Another result proven in [4] states that the one-point compactification of an uncountable discrete space is homeomorphic to no topological semilattice with open principal ideals. We generalize this Gutik's result proving that the scatteredness index $i(S)$ of a compact uncountable topological semilattice S with open principal ideals has cardinality $|i(S)|$ equal to $|S|$.

Let us recall the definition of the scatteredness index $i(X)$ of a scattered topological space X. Let $X^{(0)} = X$ and for an ordinal α define the α-th derivative set $X^{(\alpha)}$ of X by transfinite induction: $X^{(\alpha)} = X^{(\beta)} \setminus \mathrm{Iso}(X^{(\beta)})$ if $\alpha = \beta + 1$ for some ordinal β; and $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ if α is a limit ordinal. Let $i(X)$ be the smallest ordinal α such that $X^{(\alpha)} = \varnothing$. It is well known that $i(X)$ is a non-limit ordinal, provided X is a compact scattered space. If X is a subspace of a scattered topological space Y, then $X^{(\alpha)} \subset Y^{(\alpha)}$ for every ordinal α. This implies $i(X) \leq i(Y)$. On the other hand, if $f : X \to Y$ is a continuous surjective map between scattered *compact* spaces, then $Y^{(\alpha)} \subset f(X^{(\alpha)})$ for every ordinal α, see [1, VI.8.1]. This implies $i(Y) \leq i(X)$ if Y is a continuous image of a compact scattered space X.

It is well known (and can be proven by transfinite induction) that $|i(\alpha)| = |\alpha|$ for every uncountable ordinal α. This observation and Theorem 1 imply

Corollary. *If S is an uncountable compact topological semilattice with open principal ideals, then S is a compact scattered space with $|i(S)| = |S|$.*

Thus no uncountable compact scattered space X with $|i(X)| < |X|$ supports a structure of a topological semilattice with open principal ideals. This concerns also the one-point compactification αD of an uncountable discrete space D, for which $i(\alpha D) = 2$. There exists also a compact scattered space X such that $|i(X)| = |X|$ but nonetheless X is homeomorphic to no topological semilattice with open principal ideals. Just take $X = \alpha(\omega_1 \cup \aleph_1)$ be the one-point compactification of the disjoint topological sum of the ordinal space ω_1 and a discrete space of cardinality $\aleph_1$. Clearly, $|i(X)| = \aleph_1 = |X|$. The following theorem generalizing Corollary shows that this space is homeomorphic to no topological semilattice with open principal ideals.

Theorem 2. *If K is a compact subset of a topological semilattice S with open principal ideals, then K is a scattered compactum. Moreover, $|i(K)| = |K|$, provided K is uncountable and the semilattice S satisfies one of the following conditions:*

(1) S is compact;

(2) S is a scattered space;

(3) S is a well-founded semilattice.

The proof of this theorem requires some preliminary work. We start with the following lemma which in a simplified form reflects the main idea of the proof of Theorem 2 and will be used for the proof of the subsequent Theorem 3.

Lemma 1. *Suppose $(S, \wedge)$ is a topological semilattice and $K \subset S$ is an uncountable compact subset having a unique non-isolated point x_0. If $|K \setminus \uparrow x_0| > \aleph_0$, where $\uparrow x_0 = \{x \in S : x \wedge x_0 = x_0\}$, then S contains a subset $X \subset S$ having neither isolated points nor minimal elements.*

Proof. Under a *supersequence* in S we shall understand an uncountable compact subset $C \subset S$ with a unique non-isolated point $c_0 \in C$ (denoted by $\lim C$) such that $x \le c_0$ (equivalently, $x \wedge c_0 = x$) for every $x \in C$.

Observe that the set $K \wedge x_0 = \{x \wedge x_0 : x \in K\}$ is just a supersequence. Indeed, $K \wedge x_0$, being a continuous image of K, has at most one non-isolated point. Clearly, $x \le x_0$ for every $x \in K \wedge x_0$. Let us show that the set $K \wedge x_0$ is uncountable. Assuming the converse, we would write $(K \wedge x_0) \setminus \{x_0\} = \{y_n : n \in \mathbb{N}\}$. Observe that for every $n \in \mathbb{N}$ the set $Y_n = \{x \in K : x \wedge x_0 = y_n\}$ is finite (since it is closed in K and does not contain the limit point x_0). Consequently, the set $K \setminus \uparrow x_0 = \bigcup_{n=1}^{\infty} Y_n = \{x \in K : x \wedge x_0 \ne x_0\}$ is countable, a contradiction with our assumption. Thus $K \wedge x_0$ is a supersequence in S.

Let $X_0 = \varnothing$ and $X_1 = \{x_0\}$. By induction, for every $n \in \mathbb{N}$ we shall construct a subset $X_{n+1} \supset X_n$ of S such that every point $x \in X_n \setminus X_{n-1}$ is the limit point $\lim C(x)$ of some supersequence $C(x) \subset X_{n+1}$ and every point $y \in X_{n+1} \setminus X_n$ is the limit point $\lim C(y)$ of some supersequence $C(y) \subset S$.

Assuming for a moment that such a sequence (X_n) is constructed, we conclude that the union $X = \bigcup_{n=1}^{\infty} X_n \subset S$ has neither isolated points nor minimal elements.

Inductive Step. Suppose that for some $n \in \mathbb{N}$ subsets $X_0 \subset X_1 \subset \cdots \subset X_n$ of S are constructed so that every point $x \in X_n \setminus X_{n-1}$ is the limit point of some supersequence $C(x) \subset S$. For every $x \in X_n \setminus X_{n-1}$ we shall find a supersequence $C'(x) \subset C(x)$ such that every point $y \in C'(x) \setminus \{x\}$ is the limit point of some supersequence of S and shall take $X_{n+1} = X_n \cup \bigcup_{x \in X_n \setminus X_{n-1}} C'(x)$. Clearly, the so-defined set X_{n+1} will satisfy our requirements.

Now we show how to construct the supersequence $C'(x) \subset C(x)$ for every $x \in X_n \setminus X_{n-1}$. First we find an uncountable antichain $A(x)$ in $C(x)$. Observe that for every $y \in C(x) \setminus \{x\}$ the set $C(x) \cap \downarrow y$ is finite (since it is closed in $C(x)$ and does not contain the limit point x). Then for some $n \in \mathbb{N}$ the set $A(x) = \{y \in C(x) : |C(x) \cap \downarrow y| = n\}$ is uncountable. Clearly, $A(x)$ is an antichain in $C(x)$. We claim that the set $C'(x) = \{x\} \cup A(x)$ is a supersequence satisfying our requirements, i.e., every $y \in C'(x) \setminus \{x\}$ is the limit point of some supersequence in S. Clearly, $C'(x)$ is a supersequence in S. Now fix any $y \in C'(x) \setminus \{x\}$ and consider the compactum $y \wedge C(x)$. We claim that $y \wedge C(x)$ is a supersequence with $\lim y \wedge C(x) = y$. In fact, the only thing we have to verify is the uncountability of $y \cap C(x)$. Assuming the converse, we could write $y \wedge C(x) \setminus \{y\} = \{z_n : n \in \mathbb{N}\}$. Observe that for every $n \in \mathbb{N}$ the set $Z_n = \{z \in C(x) : z \wedge y = z_n\}$ is finite (since it is closed in $C(x)$ and does not contain the limit point x). Consequently, the set $Z = \{x \in C(x) : z \wedge y \neq y\}$ is at most countable. Since the antichain $A(x) \subset C(x)$ is uncountable, there is $a \in A(x) \setminus (Z \cup \{y\})$, i.e., $a \wedge y = y$, a contradiction with the fact that $A(x) \ni a, y$ is an antichain. □

Now we adapt the proof of Lemma 1 to the general case. We shall use the so-called rank function ρ defined on any well-founded partially ordered set X as follows: $\rho(x) = 0$ for a minimal element x of X and $\rho(x) = \sup\{\rho(y) + 1 : y < x\}$ for a non-minimal element $x \in X$, see [6, p.255]. Thus, $\rho : X \to \mathrm{Ord}$ is a monotone map of X onto an initial segment of ordinals (by Ord we denote the class of all ordinals).

Under a (well-founded) *pospace* we understand a topological space X endowed with a (well-founded) partial order $\leq$ which is closed as a subset of $X \times X$. We say that a pospace X is a *pospace with open lower sets* if the set $\downarrow x = \{y \in X : y \leq x\}$ is open in X for every $x \in X$. It is easy to see that every compact pospace K with open lower sets is well-founded, and hence admits a well-defined rank-function $\rho\colon K \to \rho(K) \subset \mathrm{Ord}$ (this function needs not be continuous). According to [5], every compact pospace with open lower sets is scattered.

Lemma 2. *If K is a compact pospace with open lower sets, then $i(\rho(K)) \leq i(K)$.*

Proof. It suffices to verify that $\rho^{-1}(\rho(K)^{(\alpha)}) \subset K^{(\alpha)}$ for every ordinal α (then $\rho^{-1}(\rho(K)^{(i(K))}) \subset K^{(i(K))} = \varnothing$ and thus $\rho(K)^{(i(K))} = \varnothing$ and $i(\rho(K)) \leq i(K)$).

The inclusion $\rho^{-1}(\rho(K)^{(\alpha)}) \subset K^{(\alpha)}$ is trivial for $\alpha = 0$. Assume that it is true for all ordinals $\alpha < \beta$, where β is a fixed ordinal. If β is limit, then

$$\rho^{-1}(\rho(K)^{(\beta)}) = \rho^{-1}\Big(\bigcap_{\alpha<\beta} \rho(K)^{(\alpha)}\Big) = \bigcap_{\alpha<\beta} \rho^{-1}(\rho(K)^{(\alpha)}) \subset \bigcap_{\alpha<\beta} K^{(\alpha)} = K^{(\beta)}.$$

So it rests to verify the case of a non-limit ordinal $\beta = \alpha + 1$. Let $x \in K$ be any point with $\rho(x) \in \rho(K)^{(\alpha+1)}$. By the definition of $\rho(K)^{(\alpha+1)}$, there exists a subset $A \subset \rho(K)^{(\alpha)}$ such that $\rho(x) \notin A$ and $\rho(x) = \sup A$. Since $\rho(\downarrow x) = \{\gamma \in \mathrm{Ord} : \gamma \leq \rho(x)\} \supset A$, for every $a \in A$ we may find a point $x_a \in \downarrow x$ such that $\rho(x_a) = a$. By the inductive assumption, $\{x_a\}_{a \in A} \subset \rho^{-1}(A) \subset \rho^{-1}(\rho(K)^{(\alpha)}) \subset K^{(\alpha)}$. Since $\rho(x) \notin A$, we get $x \notin \{x_a\}_{a \in A}$. Thus, to show that $x \in K^{(\alpha+1)}$, it suffices to verify that x is a cluster point of the net $\{x_a\}_{a \in A}$. By the compactness of the lower set $\downarrow x \subset K$, the net $\{x_a\}_{a \in A}$ has a cluster point in $\downarrow x$, that is a point $x_\infty \in \downarrow x$ such that for every neighborhood $U \subset \downarrow x$ of x_∞ and every $a \in A$ there exists $b \in A$ with $b \geq a$ and $x_b \in U$. We claim that $x_\infty = x$. Assuming the converse, we would get $x_\infty < x$ and thus $\rho(x_\infty) < \rho(x)$. On the other hand, the lower set $\downarrow x_\infty$ is an open neighborhood of x_∞ in K. Consequently, for every $a \in A$ there exists $b \in A$ such that $b \geq a$ and $x_b \in \downarrow x_\infty$. This yields $x_b \leq x_\infty$ and thus $a \leq b \leq \rho(x_b) \leq \rho(x_\infty)$, i.e., $\rho(x_\infty) \geq \sup A = \rho(x)$, a contradiction with $\rho(x_\infty) < \rho(x)$. □

Lemma 3. *If K is an uncountable scattered pospace with open lower sets and $|i(K)| < |K|$, then K contains an antichain A of cardinality $|A| > \aleph_0 \cdot |i(K)|$.*

Proof. By Lemma 2, $i(\rho(K)) \leq i(K)$ and consequently, $|\rho(K)| \leq \aleph_0 \cdot |i(\rho(K))| \leq \aleph_0 \cdot |i(K)| < |K|$. Observe that for every ordinal α, $\rho^{-1}(\alpha)$ is an antichain in K. Assuming that $|\rho^{-1}(\alpha)| \leq \aleph_0 \cdot |i(K)|$ for every α, we would get

$$|K| = \Big|\bigcup_{\alpha\in\rho(K)} \rho^{-1}(\alpha)\Big| \leq |\rho(K)| \cdot \aleph_0 \cdot |i(K)| \leq \aleph_0 \cdot |i(K)| < |K|,$$

a contradiction. □

Lemma 4. *Suppose $(S, \wedge)$ is a topological semilattice with open principal ideals and $K \subset S$ is an uncountable scattered subset with $|i(K)| < |K|$. Then S contains a subset X having neither isolated points nor minimal elements.*

Proof. Let $\mathcal{C}$ be the set of all uncountable scattered compact subsets C of S such that $|i(C)| < |C|$. By our assumption, this set is not empty. Let $\tau = \min\{|C| : C \in \mathcal{C}\}$ and $\lambda = \min\{i(C) : C \in \mathcal{C}$ and $|C| = \tau\}$. Clearly, λ is a non-limit ordinal. Let finally, $m = \min\{|C^{(\lambda-1)}| : C \in \mathcal{C},\ |C| = \tau,\ i(C) = \lambda\}$.

Claim A. $m = 1$.

Proof. Let $C \in \mathcal{C}$ be a compactum with $|C| = \tau$, $i(C) = \lambda$, and $|C^{(\lambda-1)}| = m$. Assuming that $m > 1$, we could write $C^{(\lambda-1)} = A \cup B$, where A, B are disjoint finite subsets of C. Since the space C is zero-dimensional (being hereditarily disconnected, see [2, 1.4.5]), we can find disjoint closed-and-open sets $U, V \subset C$ such that $U \cup V = C$ and $A \subset U$, $B \subset V$. Clearly, $i(A) = i(B) = i(K) = \lambda$ and $\max\{|U|, |V|\} = |K| = \tau$. Without loss of generality, $|U| = \tau$. It is easy to see that $U \in \mathcal{C}$ and $|U^{(\lambda-1)}| < m$, a contradiction with the minimality of m. □

Let $\mathcal{K} = \{C \in \mathcal{C} : |C| = \tau,\ i(C) = \lambda,\ |C^{(\lambda-1)}| = 1\}$. For a compactum $C \in \mathcal{K}$ by $\lim C$ we denote the unique point of $C^{(\lambda-1)}$.

Claim B. *If $C \in \mathcal{K}$ and U is a neighborhood of $\lim C$ in C, then $|C \setminus U| \leq \aleph_0 \cdot |\lambda|$.*

Proof. Assuming the converse, we would find an uncountable closed subset $F \subset C$ with $|\lambda| < |F| \leq \tau$ and $\lim C \notin F$. Observe that $i(F) \leq \lambda - 1$. Consequently, $F \in \mathcal{C}$ and $|F| = \tau$, $i(F) < \lambda$, a contradiction with the choice of the ordinal λ. □

Generalizing the definition of a supersequence from Lemma 1, under a *supersequence* in a topological semilattice S we shall understand any compactum $C \in \mathcal{K}$ such that $\lim C$ is the greatest element of C, i.e., $x \leq \lim C$ for any $x \in C$.

Claim C. *If $C \in \mathcal{K}$, then $C \wedge \lim C$ is a supersequence in S.*

Proof. Since $C \wedge \lim C$ is a continuous image of the scattered compactum C, it is scattered too, moreover, $i(C \wedge \lim C) \leq i(C) = \lambda$, see Lemma 8.1 of [1, Ch.VI]. Next, we verify that $|C \wedge \lim C| > \aleph_0 \cdot |\lambda|$. Observe that

$$C = \{\lim C\} \cup (C \setminus \downarrow \lim C) \cup \bigcup_{\substack{x\in C\wedge\lim C\\ x\neq\lim C}} C_x,$$

where $C_x = \{y \in C : y \wedge \lim C = x\}$ for $x \in C \wedge \lim C$. The sets $C \setminus \downarrow \lim C$ and C_x, $x \in (C \wedge \lim C) \setminus \{\lim C\}$, are closed in C and do not contain the point $\lim C$. By Claim B, these sets have cardinality $\le \aleph_0 \cdot |\lambda|$. Assuming that $|C \wedge \lim C| \le \aleph_0 \cdot |\lambda|$ we would get $|C| \le \aleph_0 \cdot |\lambda| < \tau = |C|$, a contradiction.

Because $i(C \wedge \lim C) \le \aleph_0 \cdot |\lambda| < |C \wedge \lim C|$, we conclude $C \wedge \lim C \in \mathcal{C}$. Since $|C \wedge \lim C| \le |C| = \tau$ and $i(C \wedge \lim C) \le i(C) = \lambda$, by the choice of τ and λ we get $|C \wedge \lim C| = \tau$ and $i(C \wedge \lim C) = \lambda$. Moreover, according to Lemma 8.1 of [1, Ch.VI], $(C \wedge \lim C)^{(\alpha)} \subset C^{(\alpha)} \wedge \lim C$ for every ordinal α. Consequently,

$$(C \wedge \lim C)^{(\lambda-1)} \subset C^{(\lambda-1)} \wedge \lim C = \{\lim C\}$$

and thus $(C \wedge \lim C)^{(\lambda-1)}$ consists of the unique point $\lim C = \lim(C \wedge \lim C)$. Then, $C \wedge \lim C \in \mathcal{K}$ and $\lim C = \lim(C \wedge \lim C)$ is the greatest element of $C \wedge \lim C$, i.e, $C \wedge \lim C$ is a supersequence. □

Claim D. *If $C \in \mathcal{K}$ and A is an antichain in C with $|A| > \aleph_0 \cdot |\lambda|$, then $C \wedge a$ is a supersequence in S for every $a \in A$ with $a \le \lim C$.*

Proof. Fix any $a \in A$ with $a \le \lim C$. Applying Lemma 8.1 of [1, Ch.VI] we conclude that $i(C \wedge a) \le i(C)$ (since $C \wedge a$ is a continuous image of C).

Next, we show that $|C \wedge a| > \aleph_0 \cdot |\lambda|$. Assume the converse: $|C \wedge a| \le \aleph_0 \cdot |\lambda|$. For every $x \in C \wedge a$ let $C_x = \{y \in C : y \wedge a = x\}$. Evidently, C_x is a closed subset of C not containing the point $\lim C$ if $x \ne a$. Consequently, $|C_x| \le \aleph_0 \cdot |\lambda|$ for any $x \in (C \wedge a) \setminus \{a\}$ and thus the cardinality of the set $B = \bigcup_{x \in (C \wedge a) \setminus \{a\}} C_x$ does not exceed $\aleph_0 \cdot |\lambda|$. Since $|A| > \aleph_0 \cdot |\lambda| \ge |B|$, we may find a point $a' \in A \setminus (B \cup \{a\})$. Note that $a' \wedge a = a$, a contradiction with the choice of A as an antichain.

Thus $\tau \ge |C \wedge a| > \aleph_0 \cdot |\lambda| \ge |i(C \wedge a)|$ and $C \wedge a \in \mathcal{C}$. By the choice of τ and λ, $|C \wedge a| = \tau$ and $i(C \wedge a) = \lambda$. Analogously as in the proof of the previous claim we may show that $(C \wedge a)^{(\lambda-1)}$ consists of the unique point $a = \lim C \wedge a$ which implies that $C \wedge a$ is a supersequence. □

Finally, we are able to finish the proof of Lemma 4. Without loss of generality, $K \in \mathcal{K}$. Let $X_0 = \varnothing$ and $X_1 = \{\lim K\}$. By Claim C, $\lim K$ is the limit point of the supersequence $K \wedge \lim K$.

By induction, for every $n \in \mathbb{N}$ we shall construct a subset $X_{n+1} \supset X_n$ of S such that every point $x \in X_n \setminus X_{n-1}$ is a cluster point of some subset $A(x) \subset X_{n+1} \cap (\downarrow x \setminus \{x\})$ and every point $y \in X_{n+1} \setminus X_n$ is the limit point $\lim C(y)$ of some supersequence $C(y) \in S$.

Assuming for a moment that such a sequence (X_n) is constructed, we conclude that the union $X = \bigcup_{n=1}^{\infty} X_n \subset S$ has neither isolated points nor minimal elements.

Inductive Step. Suppose for some $n \in \mathbb{N}$ subsets $X_0 \subset X_1 \subset \cdots \subset X_n$ of S are constructed so that every point $x \in X_n \setminus X_{n-1}$ is the limit point of some supersequence $C(x) \subset S$. By Lemma 3, the supersequence $C(x)$ contains an antichain $A(x)$ of cardinality $|A(x)| > \aleph_0 \cdot |i(C(x))| = \aleph_0 \cdot |\lambda|$. Since each closed subset of $C(x)$ not containing the limit point $x = \lim C(x)$ has cardinality $\le \aleph_0 \cdot |\lambda|$ (see Claim B), the closure of $A(x)$ in $C(x)$ contains the point x. Let $X_{n+1} = X_n \cup \bigcup_{x \in X_n \setminus X_{n-1}} A(x)$ and notice that the so-defined set X_{n+1} satisfies our requirements. Indeed, every point $a \in X_{n+1} \setminus X_n$ is the limit point of a supersequence $C(x) \wedge a$, where $x \in X_n$ is such that $a \in A(x) \subset C(x)$, see Claim D. □

Proof of Theorem 2. Suppose K is a compact subset of a topological semilattice S with open principal ideals. Then K endowed with the induced partial order is a compact pospace with open lower sets. We show that K is a scattered compactum (cf. [5]). Let A be any subset of K. Observe that the upper set $\uparrow A = \{x \in K : \exists a \in A \text{ with } a \le x\}$ is a closed subset in K as the complement to the open set $\bigcup_{x \in K \setminus \uparrow A} \downarrow x$. Consequently, $\uparrow A$, being a compact pospace has a minimal element a. Clearly, $a \in A$. Since $A \cap \downarrow a = \{a\}$, the point a is isolated in A. Thus every subset of K has an isolated point and K is a scattered space.

If K is uncountable and $|i(K)| < |K|$, then by Lemma 4, S contains a subset X having neither isolated point nor minimal elements. Consequently S is neither scattered space nor a well-founded semilattice. Also S can not be compact since otherwise, it would be a scattered space according to the Gutik Theorem [4]. □

Remark that unlike to Lemma 4, in Lemma 1 there is no requirement on S to have open principal ideals. This fact allows us to apply Lemma 1 to prove that the scattered topological lattices contain no uncountable compacta with finite scatteredness index.

We recall that a *topological lattice* is a topological space L endowed with two continuous semilattice operations $\wedge, \vee : L \times L \to L$ connected by the distributivity laws: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Theorem 3. *Every uncountable compact subset of a scattered topological lattice has infinite index of scatteredness.*

Proof. Suppose on the contrary that C is a compact subspace of a scattered topological lattice L such that $|i(C)| < \aleph_0 < |C|$. It is easy to prove (by induction on $i(C)$) that C contains an uncountable compact subset K with a unique non-isolated point x_0.

It is well known that the partial orders induced by the semilattice operations on a lattice are compatible in the following sense: $x \wedge y = x$ if and only if $x \vee y = y$ for every $x, y \in L$, see [7, p.193]. Consider the partial order $\le$ on L defined as: $x \le y$ if $x \wedge y = x$ (equivalently, $x \vee y = y$) for $x, y \in L$. Let $\downarrow x_0 = \{x \in L : x \le x_0\}$ and $\uparrow x_0 = \{x \in L : x \ge x_0\}$.

Then Lemma 1 applied to the topological semilattice $(L, \wedge)$ implies that $|K \setminus \uparrow x_0| \le \aleph_0$, while applied to the semilattice $(L, \vee)$ yields $|K \setminus \downarrow x_0| \le \aleph_0$. Consequently, the set $K = \{x_0\} \cup (K \setminus \uparrow x_0) \cup (K \setminus \downarrow x_0)$, being a union of at most countable sets, is at most countable, a contradiction. □

Theorems 2 and 3 suggest the following

Conjecture. *If K is an uncountable compact subset of a scattered topological lattice, then* $|i(K)| = |K|$.

Let us remark that the requirement of the openness of principal ideals of the semilattice S in Theorems 1 and 2 is essential and cannot be replaced by the compactness and well-foundedness of S: as we remarked in the beginning of the paper, the one-point compactification αD of any discrete space D carries the structure of a well-founded compact topological semilattice. Unlike to compact topological semilattices with open principal ideals, well-founded compact topological semilattices need not be scattered.

Example. There exists a zero-dimensional metrizable compact well-founded topological semilattice S having no isolated point. Let $S = \bigcup_{n \le \omega} \mathbb{N}^n$ be the set of sequences (both finite and infinite) of positive integers. For a sequence $x = (x_i)_{i<n}$ let $l(x) = n$ be the length of x. A semilattice operation $\wedge$ on S is defined as follows: for two sequences $x, y \in S$ let $x \wedge y = z$, where $l(z) = \sup\{i + 1 : i < \min\{l(x), l(y)\} \text{ and } x_i = y_i\}$ and $z_i = x_i$ for $i < l(z)$. It is easy to see that S endowed with the operation $\wedge$ is a well-founded semilattice. Next, we introduce a metrizable compact topology τ on S, compatible with the operation $\wedge$. This topology is generated by the base

$$\langle x, m\rangle = \{x\} \cup \{y \in S : y \wedge x = x,\ l(y) > l(x),\ y_{l(x)} \geq m\},$$

where x runs over all finite sequences in S and $m \in \mathbb{N}$. It can be easily shown that τ is a metrizable separable topology without isolated points on S, compatible with the semilattice operation $\wedge$.

Nonetheless, the well-foundedness imposes some restrictions on the topology of a topological semilattice. We recall that a topological space X is called *totally disconnected* if for any distinct points $x, y \in X$ there exists an open-and-closed subset $U \subset X$ such that $x \in U$ but $y \notin U$. It is known that a locally compact topological space is totally disconnected if and only if it is zero-dimensional [2, 1.4.5]. On the other hand, there exist totally disconnected strongly infinite-dimensional separable complete-metrizable spaces, see [2, 6.2.4].

Theorem 4. *Every well-founded topological semilattice is totally disconnected.*

Proof. Assume on the contrary that some points $a \neq b$ of a well-founded topological semilattice cannot be separated by a closed-and-open subset. Without loss of generality, $a \wedge b \neq b$. To get a contradiction, we shall construct inductively a decreasing sequence $(a_n)_{n=0}^{\infty}$ in S such that $a_n \wedge b \neq a_n$ for every n.

Let $a_0 = a$ and assume that for some $n \geq 0$ points $a_0 > a_1 > \cdots > a_n$ such that $a_n \wedge b \neq a_n$ have been constructed. Let U and V be disjoint open neighborhoods of the points a_n and $a_n \wedge b$, respectively. Evidently, the set $W = \{x \in S : x \wedge a_n \in U,\ x \wedge a_n \wedge b \in V\}$ is an open set in S such that $a \in \uparrow a_n \subset W \subset S \setminus \{b\}$. Since the points a and b cannot be separated by an open-and-closed set, we conclude that $\uparrow a_n \neq W$ and thus there exists a point $x \in W \setminus \uparrow a_n$. Then the point $a_{n+1} = x \wedge a_n$ satisfies the conditions $a_{n+1} < a_n$ and $a_{n+1} \wedge b \neq a_{n+1}$ (because $a_{n+1} \wedge b = x \wedge a_n \wedge b \in V$ while $a_{n+1} \notin V$). □

Question. Is every well-founded topological semilattice zero-dimensional?

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