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SPACES OF NONEXPANDING MAPS: CATEGORICAL PROPERTIES

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The contravariant functor of the spaces of nonexpanding maps into a given bounded metric space acts in the category of metric spaces and nonexpanding maps. We consider the problem of extension of this functor onto the Kleisli categories of some monads.

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Контравариантный функтор пространств нерастягивающих отображений в заданное ограниченное метрическое простарнство действует в категории метрических пространств и нерастягивающих отображений. Рассматривается задача продолжения этого функтора на категории Клейсли некоторых монад.

1. Introduction

Every monad in a category generates the category of algebras (the Eilenberg-Moore category) and the Kleisli category (which is known to be equivalent to the category of free algebras). General problems of lifting functors to the categories of algebras and extensions of (covariant) functors to the Kleisli categories were considered in [1], [3], [7], [12]. Different examples of solution of these problems for monads generated by functors close to being normal in the category of compact Hausdorff spaces are collected in the monograph [9].

The Kleisli categories of monads have applications in different areas of mathematics and computer science. In particular, they appear in a general categorical setting for modeling program composition (see, e. g. [10], [11]). The problem of extension of functors to the Kleisli categories was considered in [12] in connection with the problem of interpretation for abstract programming languages.

In [6] the first-named author considered the general problem of extension of contravariant functors onto the Kleisli categories of a monad. A criterion of existence of such extension was obtained and applied to the case of the contravariant functor of continuous functions in the pointwise convergence topology acting in the category of Tychonov spaces and continuous maps. The monads under consideration were: the monad generated by the second iteration

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of the functor of continuous functions in the pointwise convergence, the hyperspace monad and its finite hyperspace submonad [8].

The aim of the present article is to obtain counterparts of these results in the metric categories. We deal with the category of (compact) metric spaces and nonexpanding maps, but the analogous results can be obtained also for the category of metric spaces and Lipschitz maps.

2. Monads, algebras, Kleisli categories

A *monad* on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$ consisting of an endofunctor $T: \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta: 1_{\mathcal{C}} \to T$, $\mu: T^2 \to T$ such that $\mu \circ \eta T = \mu \circ T\eta = 1_T$ and $\mu \circ T\mu = \mu \circ \mu T$.

A pair (X, α) is called a $\mathbb{T}$-*algebra* if $\alpha: TX \to X$ is a morphism such that $\alpha \circ \eta X = 1_X$ and $\alpha \circ \mu X = \alpha \circ T\alpha$. A *morphism* of a $\mathbb{T}$-algebra (X, α) into a $\mathbb{T}$-algebra (X', α') is a morphism $f: X \to X'$ in $\mathcal{C}$ such that $f \circ \alpha = \alpha' \circ Tf$. The $\mathbb{T}$-algebras and their morphisms form a category.

The *Kleisli category* of $\mathbb{T}$ is the category $\mathcal{C}_{\mathbb{T}}$ defined as follows: $|\mathcal{C}_{\mathbb{T}}| = |\mathcal{C}|$, $\mathcal{C}_{\mathbb{T}}(X, Y) = \mathcal{C}(X, TY)$, and the composition of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y, Z)$ is given by the formula $g * f = \mu Z \circ Tg \circ f$.

Define the functor $I: \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ by $IX = X$, $X \in |\mathcal{C}|$ and $If = \eta Y \circ f$ for $f \in \mathcal{C}(X, Y)$. A (contravariant) functor $\bar{F}: \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ is called an *extension* of a (contravariant) functor $F: \mathcal{C} \to \mathcal{C}$ if $\bar{F}I = IF$.

The following result is a counterpart of a result of Vinárek [7] concerning extensions of covariant functors onto the Kleisli categories; see [6] for its proof.

Proposition 2.1. *There exists a bijective correspondence between the extensions of a contravariant functor F onto the category $\mathcal{C}_{\mathbb{T}}$ and the natural transformations $\xi: F \to TFT$ satisfying the conditions: (i) $TF\eta \circ \xi = \eta F$; (ii) $TF\mu \circ \xi = \mu FT^2 \circ T\xi T \circ \xi$.*

Suppose that contravariant functors $F, F': \mathcal{C} \to \mathcal{C}$ have extensions $\bar{F}$, $\bar{F}'$ onto the category $\mathcal{C}_{\mathbb{T}}$ and $\xi: F \to TFT$, $\xi': F' \to TF'T$ are the natural transformations that correspond, by Proposition 2.1, to these extensions.

Proposition 2.2. *A natural transformation $t: F \to F'$ is also a natural transformation of $\bar{F}$ to $\bar{F}'$ if and only if $TtT \circ \xi = \xi' \circ t$.*

3. Monads in the category of metric spaces and nonexpanding maps

Denote by NE the category of metric spaces and non-expanding maps and by CNE the full subcategory of NE whose objects are compact metric spaces.

3.1. Hyperspace monad. For a metric space (X, d) we denote by $\exp X$ the space of nonempty compact subsets of X endowed with the Hausdorff metric:

$$d_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}.$$

If a map $f: (X, d) \to (Y, \varrho)$ is nonexpanding, then so is the map $\exp f: (\exp X, d_{\mathrm{H}}) \to (\exp Y, \varrho_{\mathrm{H}})$. We thus obtain the *hyperspace functor* exp in NE. The natural transformations $s: 1_{\exp} \to \exp$, $u: \exp^2 \to \exp$ are defined as follows: $sX(x) = \{x\}$, $uX(\mathcal{A}) = \cup\mathcal{A}$. It is a folklore (and easy to see) that the maps sX and uX are non-expanding and the triple $\mathbb{H} = (\exp, s, u)$ is a monad on NE (the *hyperspace monad*). Note that exp preserves the subcategory CNE and we preserve the denotation $\mathbb{H}$ for the restriction of the hyperspace monad onto CNE.

3.2. Inclusion hyperspace monad. A nonempty closed subset $\mathcal{A}$ of $\exp X$ is called an *inclusion hyperspace* if for every $A \subset B \in \exp X$ implies $B \in \mathcal{A}$. If $X \in |\mathrm{CNE}|$ then the set GX of all inclusion hyperspaces is a closed subset of $\exp^2 X$ and is endowed with the Hausdorff metric d_{HH}.

If $f\colon (X,d) \to (Y,\varrho)$ is a morphism in CNE, define the map $Gf\colon GX \to GY$ by the formula

$$Gf(\mathcal{A}) = \{B \in \exp Y \mid B \supset f(A) \text{ for some } A \in \mathcal{A}\}.$$

Proposition 3.1. *The map $Gf\colon GX \to GY$ is a morphism in* CNE.

Proof. Suppose $\mathcal{A}, \mathcal{B} \in GX$ and $d_{\mathrm{HH}}(\mathcal{A},\mathcal{B}) < \varepsilon$. Consider $A' \in Gf(\mathcal{A})$, then there exists $A \in \mathcal{A}$ such that $A' \supset f(A)$. There is $B \in \mathcal{B}$ such that $d_{\mathrm{H}}(A,B) < \varepsilon$. Obviously, $B' = A' \cup f(B) \in Gf(\mathcal{B})$ and $\varrho_{\mathrm{H}}(A',B') < \varepsilon$.

Similarly, for every $B' \in \mathcal{B}$ there exists $A' \in \mathcal{A}$ such that $\varrho_{\mathrm{H}}(A',B') < \varepsilon$. This implies that $\varrho_{\mathrm{HH}}(Gf(\mathcal{A}), Gf(\mathcal{B})) < \varepsilon$. □

A corollary of this proposition is that G is an endofunctor in CNE (the *inclusion hyperspace functor*).

The natural transformation $\eta\colon 1 \to G$ is defined by the formula $\eta X(x) = \{A \in \exp X \mid x \in A\}$. It is easy to see that ηX is an isometric embedding for every $X \in |\mathrm{CNE}|$. Besides, define the natural transformation $\mu\colon G^2 \to G$ by the formula $\mu X(\mathfrak{A}) = \cup\{\cap\mathcal{A} \mid \mathcal{A} \in \mathfrak{A}\}$. It was proved in [13] that the maps μX are morphisms of CNE. It is essentially due to T. Radul [15] that the triple $\mathbb{G} = (G, \eta, \mu)$ is a monad on the category CNE (the *inclusion hyperspace monad*).

3.3. Probability measure monad For a compact Hausdorff space X denote by PX the space of all probability measures on X endowed with the weak∗-topology. It is well-known that P is a covariant functor on the category of compact Hausdorff spaces and continuous maps. If (X,d) is a compact metric space, then the weak∗-topology on PX is generated by the Uspenskij metric,

$$d_{\mathrm{U}}(\mu_1,\mu_2) = \inf\{\lambda(d) \mid \lambda \in P(X \times X),\ P\mathrm{pr}_1(\lambda) = \mu_1,\ P\mathrm{pr}_2(\lambda) = \mu_2\}$$

(here pr_i denotes the projection of $X \times X$ onto the i-th factor). There exists a natural transformation $\eta\colon 1_{\mathrm{NE}} \to P$; for every $x \in X$ the probability measure $\eta X(x)$ is the Dirac measure δ_x, $\delta_x(\varphi) = \varphi(x)$, $\varphi \in C(X)$.

By P_ω we denote the subfunctor of probability measures with finite supports of P. The space $P_\omega X$ consists of the probability measures of the form $\sum_{i=1}^n \alpha_i \delta_{x_i}$, where $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$.

Define the natural transformation $\mu\colon P_\omega^2 \to P_\omega$ by the following formula

$$\mu X\Big(\sum_{i=1}^n \alpha_i \delta_{m_i}\Big) = \sum_{i=1}^n \alpha_i m_i, \quad \alpha_i \geq 0, \quad \sum_{i=1}^n \alpha_i = 1, \quad m_i \in P_\omega X.$$

It is proved in [18] that the maps

$$\eta X\colon (X,d) \to (P_\omega X, d_{\mathrm{U}}), \quad \mu X\colon (P^2_\omega X, d_{\mathrm{UU}}) \to (P_\omega X, d_{\mathrm{U}})$$

are morphisms of NE. The triple (P_ω, η, μ) is a monad in NE. Similarly, the triple (P, η, μ) is a monad in CNE (here μX is naturally extended onto P^2X, $X \in |\mathrm{CNE}|$).

3.4. Monad of hyperspace of compact convex sets of probability measures. Suppose X is a compact convex subset of a locally convex space. We denote by $\mathrm{cc}X$ the subset $\exp X$ consisting of all nonempty compact convex subsets of X. Obviously, cc is a functor on the category of compact convex subset of a locally convex spaces and affine maps. Actually, the probability measure functor P can be considered as a functor into the category of compact convex subset of a locally convex spaces and affine maps and therefore the composition $\mathrm{cc}P$ is defined. If (X,d) is a compact metric space, we endow $\mathrm{cc}PX$ with the metric d_{UH} induced by the Hausdorff metric on the hyperspace of (PX, d_{U}).

We define the natural transformation $\eta\colon 1_{\mathrm{CNE}} \to ccP$ by the formula $\eta X(x) = \{\delta_x\}$.

O. Nykyforchyn [14] defined a natural transformation of $(\mathrm{cc}P)^2$ to $\mathrm{cc}P$ as follows. Suppose X is a compact convex subspace of a locally convex space and $M \in P\mathrm{cc}X$ is of the form $M = \sum_{i=1}^n \alpha_i \delta_{B_i}$, then let

$$\varphi X(M) = \left\{ \sum_{i=1}^n \alpha_i x_i \mid x_1 \in B_1, \dots, x_n \in B_n \right\}.$$

It is proved in [14] that φX extends continuously over $P\mathrm{cc}X$. Now let $\mathcal{B} \in \mathrm{cc}P\mathrm{cc}PX$, denote $\mu X(\mathcal{B}) = \bigcup_{M\in\mathcal{B}} \varphi PX(M) \in \mathrm{cc}PX$.

Proposition 3.2. *The triple* $(\mathrm{cc}P, \eta, \mu)$ *is a monad on the category* CNE.

Proof. We need only to prove that the map

$$\mu X\colon (\mathrm{cc}P\mathrm{cc}PX, d_{\mathrm{UHUH}}) \to (\mathrm{cc}PX, d_{\mathrm{UH}})$$

(we use a self-explaining denotation for metrics) is a morphism in CNE for every (X,d). Let $\mathcal{A}, \mathcal{B} \in \mathrm{cc}P\mathrm{cc}PX$ and $d_{\mathrm{UHUH}}(\mathcal{A}, \mathcal{B}) < \varepsilon$. It is sufficient to prove that for every $M = \sum_{i=1}^s \alpha_i \delta_{A_i} \in \mathcal{A}$ and every $m \in \varphi PX(M)$ there is $n \in \mu X(\mathcal{B})$ with $d_{\mathrm{U}}(m,n) < \varepsilon$.

There exists $N = \sum_{j=1}^t \beta_j \delta_{B_j} \in \mathcal{B}$ such that $d_{\mathrm{UHU}}(M,N) < \varepsilon$. Then there exist $\gamma_{ij} \ge 0$, $1 \le i \le s$, $1 \le j \le t$, such that

$$\sum_{i=1}^s \sum_{j=1}^t \gamma_{ij} = 1 \quad \text{and} \quad \sum_{i=1}^s \sum_{j=1}^t \gamma_{ij} d_{\mathrm{UH}}(A_i, B_j) < \varepsilon.$$

Now, if $m = \sum_{i=1}^s a_i \in \varphi PX(M)$, where $a_i \in A_i$, $1 \le i \le s$, find $b_{ij} \in B_j$ such that $d_{\mathrm{U}}(a_i, b_{ij}) \le d_{\mathrm{UH}}(A_i, B_j)$, $1 \le i \le s$, $1 \le j \le t$.

Since $\sum_{i=1}^s \left(\gamma_{ij} \big/ \sum_{i=1}^s \gamma_{ij}\right) b_{ij} \in B_j$, for every $j = 1, \dots, t$, we see that

$$n = \sum_{i=1}^s \sum_{j=1}^t \gamma_{ij} b_{ij} = \sum_{j=1}^t \left(\sum_{i=1}^s \gamma_{ij} b_{ij} \right) \left(\sum_{i=1}^s \left(\gamma_{ij} \Big/ \sum_{i=1}^s \gamma_{ij} \right) b_{ij} \right) \in \varphi PX(N).$$

Show that $d_{\mathrm{U}}(m,n)<\varepsilon$. Let $c_{ij}\in P(X\times X)$ be such that $P\mathrm{pr}_1(c_{ij})=a_i$, $P\mathrm{pr}_2(c_{ij})=b_{ij}$, and $c_{ij}(d)=d_{\mathrm{U}}(a_i,b_{ij})$. Then

$$l=\sum_{i=1}^{s}\sum_{j=1}^{t}\gamma_{ij}c_{ij}\in P(X\times X),\quad P\mathrm{pr}_1(l)=m,$$

$P\mathrm{pr}_2(l)=n$, and

$$l(d)=\sum_{i=1}^{s}\sum_{j=1}^{t}\gamma_{ij}c_{ij}(d)\le\sum_{i=1}^{s}\sum_{j=1}^{t}\gamma_{ij}d_{\mathrm{UH}}(A_i,B_j)<\varepsilon,$$

and therefore $d_{\mathrm{U}}(m,n)<\varepsilon$.

□

4. Monad generated by the functor F_K^2

Let K be a fixed metric space. If necessary, we denote the metric on K by ϱ. For every $X\in|\mathrm{NE}|$ denote by F_KX the set of bounded nonexpanding maps from X to K (a map $f\colon X\to K$ is *bounded* if its image $f(X)$ is a bounded subset of K) The set F_KX is endowed with the sup-metric for which we preserve the denotation ϱ. For any morphism $f\in\mathrm{NE}(X_1,X_2)$ define the map $f_*=F_KX_2\to F_KX_1$ by the formula

$$f_*(\varphi)=\varphi\circ f,\ \ \varphi\in F_KX_2.$$

The proof of the following proposition is obvious.

Proposition 4.1. *The map f_* is nonexpanding.*

We obtain the contravariant functor $F_K\colon\mathrm{NE}\to\mathrm{NE}$ by putting $F_Kf=f_*$ on morphisms.

Let $X\in\mathrm{NE}$. For every $x\in X$ denote by $\mathrm{ev}_x\colon F_KX\to K$ the *evaluation map* which acts by the formula: $\mathrm{ev}_x(\varphi)=\varphi(x)$. Obviously, $\mathrm{ev}_x\colon F_KX\to K$ is a nonexpanding map, i. e. $\mathrm{ev}_x\in F_K^2X$.

Proposition 4.2. *The map $x\mapsto\mathrm{ev}_x\colon X\to F^2X$ is an isometric embedding.*

Proof. For $x,y\in X\in|\mathrm{NE}|$ we have

$$\begin{aligned}\varrho(\mathrm{ev}_x,\mathrm{ev}_y)&=\sup\{\varrho(\mathrm{ev}_x(\varphi),\mathrm{ev}_y(\varphi))\mid\varphi\in F_KX\}=\\&=\sup\{\varrho(\varphi(x),\varphi(y))\mid\varphi\in F_KX\}\le d(x,y).\end{aligned}$$

□

We denote the map $x\mapsto\mathrm{ev}_x$ by ηX. It is easy to see that $\eta=(\eta X)_{X\in|\mathrm{NE}|}$ is a natural transformation from the identity functor 1_{NE} to the functor F_K^2.

Suppose $C\colon\mathcal{C}\to\mathcal{C}$ is a contravariant functor on a category $\mathcal{C}$ such that there exists a natural transformation $\eta\colon 1\to C^2$ satisfying the property: $C\eta\circ\eta C=1_C$. Put $T=C^2$ and define the natural transformation $\mu\colon T^2=C^4\to C^2=T$ by putting $\mu=C\eta C$. It is known that in this case the triple $\mathbb{T}=(T,\eta,\mu)$ is a monad on the category $\mathcal{C}$ (see, e. g., [6]).

We apply this to the functor F_K^2. Analogously to the case of the contravariant functor C_p in the category of Tychonov spaces and continuous maps, we obtain

$$F_K\eta X\circ\eta F_KX(\varphi)(x)=F_K\eta X(\mathrm{ev}_\varphi)(x)=$$
$$=\mathrm{ev}(\varphi(\eta X(x))=\eta X(x)(\varphi)=\mathrm{ev}_x(\varphi)=\varphi(x),$$

for $\varphi\in F_KX$, $x\in X$, i. e. $F_K\eta\circ\eta F_K=1_{F_K}$.

We obtain the monad $\mathbb{F}_K^2=(F_K^2,\eta,\mu=F_k\eta F_K)$ in the category NE. See also [16] and [17] for related monads.

4.1. Functorial extension operators. Suppose $T\colon \mathrm{NE}\to\mathrm{NE}$ is a covariant functor for which there exists a natural transformation $\eta\colon 1_{\mathrm{NE}}\to T$. A *functorial extension operator* is a natural transformation $E_K\colon F_K\to F_KT$ such that $E_K\circ\eta=1_{F_K}$.

Now let T be the functorial part of a monad $\mathbb{T}=(T,\eta,\mu)$ on NE. We say that E_K is $\mathbb{T}$-*associated* if $E_KT\circ E_K=F_K\mu\circ e_K$.

Proposition 4.3. *Let E_K be a $\mathbb{T}$-associated functorial extension operator. Then the natural transformation $\xi\colon F_K\to TF_KT$, $\xi=\eta FT\circ E_K$, satisfies conditions (i), (ii) of Proposition 2.1.*

Proof. (i) Obvious.

(ii) We have

$$TF_K\mu\circ\xi=TF_K\mu\circ\eta F_KT\circ E_K=\eta F_KT^2\circ F_K\mu\circ E_K=$$
$$=\eta FT^2\circ E_KT\circ E_K=TE_KT\circ\eta F_KT\circ E_K=$$
$$=\mu FT^2\circ T\eta FT^2\circ TE_KT\circ\eta F_KT\circ E_K$$
$$=\mu\circ F_KT^2\circ T\xi T\circ\xi.$$

□

Let $\mathbb{T}=(T,\eta,\mu)$ be a monad in NE. Suppose that (K,α) is a $\mathbb{T}$-algebra. For $X\in|\mathrm{NE}|$ define the map $\xi X: F_KX\to TF_KTX$ as follows:

$$\xi X(\varphi)=\eta F_KTX(\alpha\circ T\varphi),\ \varphi\in F_KX.$$

The following counterpart of Lemma 1 from [8] holds.

Lemma 4.4. *Suppose the map ξX is a morphism in $|\mathrm{NE}|$ for every $X\in|\mathrm{NE}|$. Then $\xi=(\xi X)_{x\in|\mathrm{NE}|}$ is a natural transformation satisfying conditions (i) and (ii) from Proposition 2.1.*

Proof. Despite the complete similarity to the proof of the mentioned lemma from [8], we present alternative arguments. Note that $E_K=(E_KX)_{X\in\mathrm{NE}}$, $E_KX(\varphi)=\alpha\circ T\varphi$ is a $\mathbb{T}$-associated functorial extension operator and apply Proposition 4.3. □

Define the map $\alpha\colon F_K^2K\to K$ by the formula: $\alpha(\varphi)=\varphi(1_K)$, $\varphi\in F_K^2K$. Obviously, α is a morphism in NE.

Proposition 4.5. *The pair (K,α) is an $\mathbb{F}_K^2$-algebra.*

Proof. We have

$$\alpha \circ \eta K(x) = \alpha(\mathrm{ev}_x) = \mathrm{ev}_x(1_K) = 1_K(x) = x.$$

We are going to prove that $\alpha \circ F_K^2\alpha = \alpha \circ \mu K$. Let $\Phi \in F_K^4 K$, then

$$(\alpha \circ \mu_K)(\Phi) = \alpha(\mu K(\Phi)) = \mu K(\Phi)(1_K) = \Phi(\mathrm{ev}_{1_K})$$

and, on the other hand,

$$\alpha \circ F_K^2\alpha(\Phi) = F_K^2\alpha(\Phi)(1_K) = (\Phi \circ F_K\alpha)(1_K) = \Phi(\alpha \circ 1_K) = \Phi(\alpha) = \Phi(\mathrm{ev}_{1_K}).$$

□

Proposition 4.6. *The monad $\mathbb{F}_k^2$ satisfies the conditions of Lemma 4.4.*

Proof. Suppose $\varphi_1, \varphi_2 \in F_K X$, then

$$\begin{aligned}\varrho(\xi X(\varphi_1), \xi X(\varphi_2)) &= \sup\{\varrho(\xi X(\varphi_1)(\Phi), \xi X(\varphi_2)(\Phi)) \mid \Phi \in F_K^4 X\} = \\ &= \sup\{\varrho(\mathrm{ev}_{\alpha\circ F_K^2\varphi_1}(\Phi), \mathrm{ev}_{\alpha\circ F_K^2\varphi_2}(\Phi) \mid \Phi \in F_K^4 X\} = \\ &= \sup\{\varrho(\Phi(\alpha \circ F_K^2\varphi_1), \Phi(\alpha \circ F_K^2\varphi_2)) \mid \Phi \in F_K^4 X\} \leq \\ &\leq \varrho(\alpha \circ F_K^2\varphi_1, \alpha \circ F_K^2\varphi_2) \leq \\ &\leq \varrho(F_K^2\varphi_1, F_K^2\varphi_2) \leq \varrho(\varphi_1, \varphi_2),\end{aligned}$$

and thus the map ξX is a morphism of NE. □

The following theorem is a consequence of Propositions 5.1, 2.1, and Lemma 4.4.

Theorem 4.7. *The contravariant functor F_K has an extension onto the Kleisli category of the monad $\mathbb{F}_K^2$.*

Let us compare this result with that for the contravariant functor C_p in the category of Tychonov spaces and continuous maps see [8]. Since, for every $\psi \in F_K^2 X$,

$$(\alpha \circ F_K^2\varphi)(\psi) = \alpha(\psi \circ F\varphi) = (\psi \circ F\varphi)(1_K) = \psi(F\varphi(1_K)) = \psi(\varphi) = \mathrm{ev}_\varphi(\psi),$$

we see that $\xi = \eta F_K^3 \circ \eta F_K$. This demonstrates the complete analogy between the considered case and that of functor C_p.

Suppose now that K is compact, i. e. $K \in \mathrm{CNE}$. By the Arzela-Ascoli Theorem, for every $X \in |\mathrm{CNE}|$ the space $F_K X$ is compact.

5. Extension of the functor F_K onto the Kleisli categories of monads

Proposition 5.1. *Suppose $\mathbb{T}$ is a submonad of the following monad on the category* CNE*: (a) the hyperspace monad; (b) the probability measure monad; (c) the inclusion hyperspace monad; (d) the monad of compact convex sets of probability measures. Then $\mathbb{T}$ satisfies the conditions of Lemma 4.4.*

Proof. (a) We have $\xi X(\varphi) = \{\alpha \circ \exp\varphi\}$, $\varphi \in F_K X$. If $\varphi_1, \varphi_2 \in F_K X$, then

$$\begin{aligned} \varrho(\xi X(\varphi_1), \xi X(\varphi_2)) &= \varrho(\alpha \circ \exp\varphi_1, \alpha \circ \exp\varphi_2) \leq \\ &\leq \varrho_{\mathrm{H}}(\exp\varphi_1, \exp\varphi_2) = \varrho(\varphi_1, \varphi_2). \end{aligned}$$

(b) In this case $\xi X(\varphi) = \delta_{\alpha\circ P\varphi}$, $\varphi \in F_K X$.
If $\varphi_1, \varphi_2 \in F_K X$, then

$$\begin{aligned} &\varrho(\xi X(\varphi_1), \xi X(\varphi_2)) = \varrho(\alpha \circ P\varphi_1, \alpha \circ P\varphi_2) \leq \\ &\leq \varrho_{KR}(P\varphi_1, P\varphi_2) = \sup\{\varrho_{KR}(P\varphi_1(\mu), P\varphi_2(\mu)) \mid \mu \in PX\} = \\ &= \sup\{|P\varphi_1(\mu)(f) - P\varphi_2(\mu)(f)| \mid \mu \in PX,\ f \in \mathrm{NE}(K, \mathbb{R})\} = \\ &= \sup\{|\mu(f \circ \varphi_1) - \mu(f \circ \varphi_2)| \mid \mu \in PX,\ f \in \mathrm{NE}(K, \mathbb{R})\} \leq \\ &\leq \varrho(\varphi_1, \varphi_2). \end{aligned}$$

□

The proof of the following result is analogous.

Proposition 5.2. *Suppose $\mathbb{T}$ is a submonad of the following monad on the category* NE*: (a) the hyperspace monad; (b) the monad (P_ω, η, μ). Then $\mathbb{T}$ satisfies the conditions of Lemma 4.4.*

The following theorem is a consequence of Propositions 5.1, 2.1, and Lemma 4.4.

Theorem 5.3. *Suppose $\mathbb{T}$ is one of the following monad from the formulation of Propositions 5.1 and 5.2. Then there is an extension of the functor F_K onto the category* $\mathrm{NE}_{\mathbb{T}}$.

Every morphism $f\colon K \to L$ in NE determines a natural transformation $f^*\colon F_K \to F_L$ by the formula $f^*X(\varphi) = f \circ \varphi$.

Proposition 5.4. *Suppose $f\colon (K, \alpha) \to (L, \alpha')$ is a morphism of $\mathbb{T}$-algebras in* NE*. The natural transformation $f^*\colon F_K \to F_L$ is also a natural transformation of the extensions of the functors F_K and F_L onto the Kleisly category of the monads $\mathbb{T}$ from Theorem 5.3.*

Proof. For arbitrary M, we denote by $\xi_M\colon F_M \to TF_MT$ the natural transformation defined by the formula $\xi_M(\varphi) = \eta F_M TX(\alpha \circ T\varphi)$. Then

$$\begin{aligned} &Tf^*TX \circ \xi_K X(\varphi) = Tf^*TX \circ \eta F_K TX(\alpha \circ T\varphi) = \\ &= \eta F_L TX \circ f^*TX(\alpha \circ T\varphi) = \eta F_L TX(f \circ \alpha \circ T\varphi) = \\ &= \eta F_L TX(\alpha' \circ Tf \circ T\varphi) = \eta F_L TX(\alpha' \circ T(f^*(\varphi)) = \\ &= \xi_L X \circ f^*X(\varphi). \end{aligned}$$

□

One could conjecture that all the extensions of the functor F_K onto the Kleisli categories are of the same algebraic nature, in particular, that the natural transformations corresponding to these extensions by Proposition 2.1 factor through $\eta F_K T$. We prove this in one special case.

Theorem 5.5. *Suppose $|K| = 2$ and the functor F_K has an extension onto the category $\mathbb{H} = (\exp, s, u)$. Then the natural transformation $\xi\colon F_K \to \exp F_K \exp$ that corresponds to this extension by Proposition 2.1 factors through $sF_K \exp$.*

Proof. Put $K = \{0, 1\}$. Suppose that there exist $X \in |\mathrm{NE}|$ and $\varphi \in F_K X$ such that $|\xi X(\varphi)| \geq 2$. Then there exist $A \in \exp X$ and $f_0, f_0 \in F_K \exp X$ such that $f_0(A) \neq f_1(A)$. Since $\varphi = F_K\varphi(1_K)$ and $\xi X \circ F_K\varphi = \exp F_K \exp\varphi \circ \xi K$, there exist $g_0, g_1 \in \xi K(1_K)$ such that $g_0(K) \neq g_1(K)$. We assume that $g_i(K) = i$, $i = 1, 2$.

It is easy to see that $\xi \exp K(g_i) = \{h_{i0}, h_{i1}\}$, where $h_{ij}(\mathcal{A}) = j$ if and only if either $g_i(\mathcal{A}) = \{j\}$ or $g_i(\mathcal{A}) = K$. Then

$$uF_K \exp^2 K \circ \exp \xi \exp K \circ \xi K(1_K) = \{h_{00}, h_{01}, h_{10}, h_{11}\}.$$

It is easy to verify that

$$h_{01}(\{\{a\}, \{a, b\}\}) = 0, \ h_{01}(\{\{b\}, \{a, b\}\}) = 1.$$

This contradicts to the fact that all h_{ji} must factor through uK. □

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