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PARATOPOLOGICAL GROUPS I

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We consider properties of paratopological groups, in particular, related to cardinal invariants, metrization, and minimality. An example of a regular paratopological group admitting no weaker Hausdorff group topology is presented.

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Рассматриваются свойства паратопологических групп, в частности, связанные с кардинальными инвариантами, метризацией и минимальностью. Приводится пример регулярной партопологической группы, не допускающей более слабой хаусдорфовой топологии.

1. General properties

A group G with topology τ is called a *paratopological group* if the multiplication on the group G is continuous. In this case the topology τ is called a *paratopology*. If the inversion on the group G is continuous then (G, τ) is a *topological group*. Hence, all translations and interior automorphisms of a paratopological group are homeomorphisms.

Proposition 1.1. *For a group with topology (G, τ) the following conditions are equivalent:*

I. *(G, τ) is a paratopological group.*

II. *The following Pontrjagin conditions [1] are satisfied for basis $\mathcal{B} = \mathcal{B}_\tau$ at the unit e of G.*
   1. $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B}) : W \subset U \cap V$.
   2. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^2 \subset U$.
   3. $(\forall U \in \mathcal{B})(\forall x \in U)(\exists V \in \mathcal{B}) : xV \subset U$.
   4. $(\forall U \in \mathcal{B})(\forall x \in G)(\exists V \in \mathcal{B}) : x^{-1}Vx \subset U$.
   The paratopological group G is Hausdorff if and only if
   5. $\bigcap\{UU^{-1} : U \in \mathcal{B}\} = \{e\}$.
   The paratopological group G is a topological group if and only if
   6. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^{-1} \subset U$.

III. *Let $\mathcal{S}$ be a subbase of the topology τ and for every points $x, y \in G$ and every neighborhood $U \in \mathcal{S}$ of the point xy there exist neighborhoods $V, W \in \mathcal{S}$ of the points x, y respectively such that $VW \subset U$.*

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For a group G, by $\mathcal{P}(G)$ we denote the set of all paratopologies on the group G. For paratopologies $\tau_1, \tau_2 \in \mathcal{P}(G)$ put $\tau_1 \wedge \tau_2 = \sup\{\tau \in \mathcal{P}(G) : \tau \subset \tau_1 \cap \tau_2\}$, $\tau_1 \vee \tau_2 = \inf\{\tau \in \mathcal{P}(G) : \tau \supset \tau_1 \cup \tau_2\}$.

Proposition 1.2. *Let τ_1, τ_2 be paratopologies on a group G with bases at the unit $\mathcal{B}_1, \mathcal{B}_2$ respectively. Then $\mathcal{B}_1 \vee \mathcal{B}_2 = \{U_1 \cap U_2 : U_i \in \mathcal{B}_i\}$ is a base at the unit of the paratopology $\tau_1 \vee \tau_2$.*

A paratopological group G with a base at the unit $\mathcal{B}$ is a *SIN-group* (Small Invariant Neighborhoods), if $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})(\forall x \in G) : x^{-1}Vx \subset U$.

Proposition 1.3. *Let τ_1, τ_2 be paratopologies on a group G with bases at the unit $\mathcal{B}_1, \mathcal{B}_2$ respectively. If τ_1 is a SIN-paratopology then $\mathcal{B}_1 \wedge \mathcal{B}_2 = \{U_1U_2 : U_i \in \mathcal{B}_i\}$ is a base at the unit of the paratopology $\tau_1 \wedge \tau_2$. Moreover, if τ_2 is a SIN-paratopology then $\tau_1 \wedge \tau_2$ is a SIN-paratopology.*

For $\tau \in \mathcal{P}(\mathcal{G})$ put $\tau^{-1} = \{U^{-1} : U \in \tau\}$. Then $\tau^* = \tau \wedge \tau^{-1}$ is the finest group topology weaker than τ on the group G, which is called the *corresponding to* τ group topology. Proposition 1.3 implies that if τ is a Hausdorff SIN-paratopology then τ^* is a Hausdorff SIN-group topology.

Remark. If τ is a Hausdorff paratopology then τ^* is not necessarily a Hausdorff paratopology (see Section 4).

For a topology τ, by $\overline{A}^{\tau}$ we denote the closure of a set A in the topology τ.

Proposition 1.4. *Let τ_1 and τ_2 be paratopologies on a group G and τ_1 a SIN-paratopology such that τ_1^*, τ_2^* are Hausdorff and $\tau_1^* \vee \tau_2^*$ is discrete. Then the topology $\tau_1 \wedge \tau_2$ is Hausdorff. Moreover, if τ_1 and τ_2 are regular then $\tau_1 \wedge \tau_2$ is regular.*

Proof. We show that the topology $\tau_1^* \wedge \tau_2^*$ is Hausdorff. Let $U_i \in \mathcal{B}_{\tau_i^*}$ and $U_1^{-1}U_1 \cap U_2U_2^{-1} = \{e\}$. Let $x \in U_1U_2 \backslash \{e\}$. Then $x = u_1u_2$, where $u_i \in U_i$. Then there exist neighborhoods $W_i \in \mathcal{B}_{\tau_i^*}, W_i \subset U_i$ such that $u_1 \notin W_1$ or $u_2 \notin W_2$. Suppose that $x \in W_1W_2$ then $x = w_1w_2$, where $w_i \in W_i$. Hence $U_1^{-1}U_1 \cap U_2U_2^{-1} \ni u_1^{-1}w_1 = u_2w_2^{-1} = e$. Thus $u_1 = w_1$ and $u_2 = w_2$, a contradiction. Hence the topology $\tau_1 \wedge \tau_2$ is Hausdorff.

Suppose that the paratopologies τ_1 and τ_2 are regular. Let $U_i \in \mathcal{B}_{\tau_i}$ and $(U_1U_1^{-1})^2 \cap (U_2U_2^{-1})^2 = \{e\}$. It suffices to show that $\overline{U_1U_2}^{\tau_1\tau_2} \subset \overline{U_1}^{\tau_1}\overline{U_2}^{\tau_2}$. Suppose that $x \in \overline{U_1U_2}^{\tau_1\tau_2}$. Then $x \in \bigcap\{U_1U_2V_2^{-1}V_1^{-1} : V_i \in \mathcal{B}_{\tau_i}\} \subset \bigcup\{U_1V_1^{-1}U_2V_2^{-1} : V_i \in \mathcal{B}_{\tau_i}\}$, since τ_1 is a SIN-paratopology. If $x \notin \overline{U_1}^{\tau_1}\overline{U_2}^{\tau_2}$ then there exist two different representations $x = u_1u_2 = w_1w_2$, where $u_i, w_i \in U_iU_i^{-1}$ Hence $(U_1^{-1}U_1)^2 \cap (U_2U_2^{-1})^2 \ni u_1^{-1}w_1 = u_2w_2^{-1} = e$. Thus $u_1 = w_1$ and $u_2 = w_2$, a contradiction. Hence the topology $\tau_1 \wedge \tau_2$ is regular. □

Question 1.1. Let τ_1 and τ_2 be Hausdorff group topologies on a group G such that $\tau_1 \vee \tau_2$ is discrete. Is the topology $\tau_1 \wedge \tau_2$ Hausdorff?

A subsemigroup S of a group G is said to be *normal* if $x^{-1}Sx \subset S$ for every $x \in G$. For every normal submonoid S of the group G by τ_S we denote the paratopology with the base $\{xS : x \in G\}$. Then $\tau_S = \inf\{\tau \in \mathcal{P}(G) : S \in \tau\}$.

Example 1.1. Sorgenfrey arrow. $(\mathbb{R}, \tau_s) = (\mathbb{R}, \tau_{\mathbb{R}_+} \wedge \tau)$, where τ is the standard topology on $\mathbb{R}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

Example 1.2. Let $G = (\mathbb{R}^2, +)$. Put $\mathcal{B} = \{\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : (x - 1/n)^2 + y^2 < 1/n^2\} : n \in \mathbb{N}\}$. Then $\mathcal{B}$ is a base at the zero of a paratopology on the group G.

Example 1.3. The p-arrow. Let p be a natural number. Put $(\mathbb{Z},\tau)=(\mathbb{Z},\tau_{Z_+}\wedge\tau_p)$, where τ_p has a base at the unit $\mathcal{B}_p=\{p^n\mathbb{Z}:n\in\mathbb{N}\}$ and $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$.
Example 1.4. The set-set topology. Let (X,τ) be a topological space, $H(X)$ be the group of all homeomorphisms of the space X. Let $\mathcal{U},\mathcal{V}$ be families of subsets of the space X. For $U\in\mathcal{U},V\in\mathcal{V}$ put $(U,V)=\{f\in H(X):f(U)\subset V\}$. Put $\mathcal{S}(\mathcal{U},\mathcal{V})=\{(U,V):U\in\mathcal{U},V\in\mathcal{V}\}$. If $\mathcal{S}(\mathcal{U},\mathcal{V})$ is a subbase for some paratopology $T(\mathcal{U},\mathcal{V})$ on the group $H(X)$ then $T(\mathcal{U},\mathcal{V})$ is called a *set-set topology.*

Proposition 1.5. *Let $(H(X),T(\mathcal{U},\tau))$ is a paratopological group. If $\mathcal{U}$ is a π-network of the space (X,τ) then for every $U\in\mathcal{U},V\in\tau$ we have $\overline{(U,V)}\subset(U,\overline{V})$.*

Proof. Let $U\in\mathcal{U},V\in\tau$ and $f\notin(U,\overline{V})$. Then $f^{-1}(X\backslash\overline{V})\cap U\neq\varnothing$ and therefore there exists a set $W\in\mathcal{V}$ such that $W\subset f^{-1}(X\backslash\overline{V})\cap U$. Then $f\in(W,X\backslash\overline{V})$ and $(W,X\backslash\overline{V})\cap(U,V)=\varnothing$. □

We shall use the following definitions. Let $\mathcal{U},\mathcal{V}$ be families of subsets of a set X. Let $f\colon X\to Y$, $g\colon Z\to X$ be maps then $f(\mathcal{U})=\{\{(\mathcal{U}):\mathcal{U}\in\mathcal{U}\}$, $g^{-1}(\mathcal{U})=\{g^{-1}(U):U\in\mathcal{U}\}$.

Proposition 1.6. *If one of the following conditions is satisfied then $(H(X),T(\mathcal{U},\mathcal{V}))$ is a paratopological group.*

1. *$\mathcal{U}=\mathcal{V}$ and $f(\mathcal{U})\subset\mathcal{U}$ for every $f\in H(X)$. Moreover, if X is T_1-space and $\mathcal{U}=\tau$ then the topology $T(\mathcal{U},\mathcal{V})$ is zero dimensional.*
2. *The space X is normal and $\mathcal{U}$ is the family $\exp X$ of all closed subsets in (X,τ), $\mathcal{V}=\tau$. The topology $T(\mathcal{U},\mathcal{V})$ is regular.*
3. *The space X is locally compact and $\mathcal{U}$ is the family $\exp_c X$ of all compact sets in (X,τ), $\mathcal{V}=\tau$. Moreover, if X is either compact or locally connected then $(H(X),T(\exp_c X,\tau))$ is a topological group [2].*

Proof. By Proposition 1.1 it suffices to show that for every maps $f_1,f_2\in H(X)$ and for every neighborhood $(U_1,V_2)\in\mathcal{S}(\mathcal{U},\mathcal{V})$ of f_1f_2 there exist sets $V_1\in\mathcal{V},U_2\in\mathcal{U}$ such that $f_i\in(U_i,V_i)$ and $(U_1,V_1)(U_2,V_2)\subset(U_1,V_2)$.

1. Put $V_1=U_2=f_2^{-1}(V_2)$. Let $\mathcal{U}=\tau$, $f\notin(U,V)\in\mathcal{S}(\tau,\tau)$. Let $x\in U$ be such that $f(x)\notin V$. Then $f\in(X\backslash\{x\},X\backslash\{f(x)\})$ and $(U,V)\cap(X\backslash\{x\},X\backslash\{f(x)\})=\varnothing$.

2. There exists a neighborhood V_1 of $f_1(U_1)$ such that $f_2(\overline{V}_1)\subset V_2$. Put $U_2=\overline{V}_1$. Let $U\in\mathcal{U},\mathcal{V}\in\tau$ and $f\in(U,V)$. There exists a neighborhood $W\in\tau$ such that $f(U)\subset W\subset\overline{W}\subset V$. Then Proposition 1.4 implies that $f\in(U,f(U))\subset(U,W)\subset\overline{(U,W)}\subset(U,\overline{W})\subset(U,V)$.

3. For every point $t\in f_1(U_1)$ there exists a neighborhood V_t of t such that $\overline{V}_t$ is compact and $f_2(\overline{V}_t)\subset V_2$. Thus there exist $t_1,\ldots,t_m\in f_1(U_1)$ such that $V_1=\bigcup V_{t_i}\supset f_1(U_1)$. Therefore $U_2=\overline{V}_1$ is compact and $f_2(U_2)\subset V_2$. □

Example 1.5. Let τ be the standard topology on $\mathbb{R}$. Consider a set-set topology $T(\exp_1\mathbb{R},\tau)$ on $H(\mathbb{R})$ where $\exp_1\mathbb{R}$ is the family of all singletons in $\mathbb{R}$. Then $T(\exp_1\mathbb{R},\tau)=T(\exp_c\mathbb{R},\tau)$.

It is well known that every T_0 topological group is completely regular. For paratopological groups the situation with the separation axioms is worse.

Example 1.6. $T_0\not\Rightarrow T_1$. Put $G=(\mathbb{R},+)$ and $\tau=\{x+[0;\infty):x\in\mathbb{R}\}$.

Example 1.7. $T_1\not\Rightarrow T_2$. Put $G=(\mathbb{R},+)$ and $\tau=\{x+[y;\infty):x,y\in\mathbb{R}\}$.

Example 1.8. $T_2 \not\Rightarrow T_3$ Put $G = \mathbb{R}^2$. Define a base $\mathcal{B}$ at the unit of the group G putting $\mathcal{B} = \{\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : 0 < x, y < 1/n\} : n \in \mathbb{N}\}$. Also the space of Example 1.2 is Hausdorff non regular.

Question 1.2. Is every regular paratopological group completely regular?

Proposition 1.7. *For every disjoint compact subsets K_1, K_2 of a Hausdorff paratopological group G there exists a neighborhood U of the unit such that $UK_1 \cap UK_2 = \varnothing$.*

Proof. For every points $x \in K_1, y \in K_2$ there exists a neighborhood $V(x,y)$ of the unit such that $V(x,y)x \cap V(x,y)y = \varnothing$. Let $U(x,y)$ be a neighborhood of the unit such that $U(x,y)^2 \subset V(x,y)$. For every $x \in K_1$ choose a finite family $\mathcal{Y}(x)$ such that $K_2 \subset \bigcup\{U(x,y)y : y \in \mathcal{Y}(x)\}$. Put $U(x) = \bigcap\{U(x,y) : y \in \mathcal{Y}(x)\}$. There exists a finite family $\mathcal{X}$ such that $K_1 \subset \bigcup\{U(x)x : x \in \mathcal{X}\}$. Put $U = \bigcap\{U(x) : x \in \mathcal{X}\}$. Then $UK_1 \cap UK_2 \subset U\bigcup\{U(x)x : x \in \mathcal{X}\} \cap UK_2 \subset \bigcup\{U(x)^2x : x \in \mathcal{X}\} \cap UK_2 = \bigcup\{U(x)^2x \cap UK_2 : x \in \mathcal{X}\} \subset \bigcup\{U(x)^2x \cap U\bigcup\{U(x,y)y : y \in \mathcal{Y}(x)\} : x \in \mathcal{X}\} \subset \bigcup\{U(x,y)^2x \cap U(x,y)^2y : x \in \mathcal{X}, y \in \mathcal{Y}(x)\} = \varnothing$. □

Proposition 1.8. *Let G be a paratopological group, $K \subset G$ be a compact subspace, $F \subset G$ be a closed set and $K \cap F = \varnothing$. Then there exists a neighborhood U of the unit such that $UK \cap F = \varnothing$.*

Proof. For every point $x \in K$ there exists a neighborhood $V(x)$ of the unit such that $V(x)x \cap F = \varnothing$. Let $U(x)$ be a neighborhood of the unit such that $U(x)^2 \subset V(x)$. There exists a finite family $\mathcal{X}$ such that $K \subset \bigcup\{U(x)x : x \in \mathcal{X}\}$. Put $U = \bigcap\{U(x) : x \in \mathcal{X}\}$. Then $UK \cap F \subset U\bigcup\{U(x)x : x \in \mathcal{X}\} \cap F \subset \bigcup\{U(x)^2x : x \in \mathcal{X}\} \cap F = \varnothing$. □

Example 1.9. Let G be the Sorgenfrey arrow, $F = \{-1/n : n \in \mathbb{N}\}$. Then F is a closed subset of G but $U \cap U + F \neq \varnothing$ for every neighborhood U of the zero.

A *subgroup* of a paratopological group G is a subgroup of G endowed with the induced from G topology. Clearly, any subgroup of a paratopological group is again a paratopological group.

Proposition 1.9. *Every open subgroup of a paratopological group is closed.*

Proposition 1.10. *The center of a Hausdorff paratopological group is a closed normal subgroup.*

Proposition 1.11. *The component of the unit of a paratopological group is a closed normal subgroup.*

Proof. Let C be the component of the unit of a paratopological group G. Since a connected component is closed in every topological space, C is closed. Let $x \in C$. Then $x^{-1}C \subset C$, because $x^{-1}C$ is a connected set containing the unit of the group G. Then $\bigcup\{x^{-1}C : x \in C\} = C^{-1}C \subset C$ hence C is a group. The subgroup C is normal, because for every $x \in G$ a set $x^{-1}Cx$ is a connected set containing the unit of the group G. □

Remark. The closure of a subgroup of a paratopological group in general is not a paratopological group [3].

Let H be a subgroup of a paratopological group (G, τ). Define a topology $\tilde{\tau}$ on the space of left cosets G/H of the group G in the following way. A set U is open in G/H if and only if $\pi^{-1}(U)$ is open in G, where $\pi\colon G \to G/H$ is the natural projection, $\pi(x) = xH$.

Proposition 1.12. *The map π is continuous and open, the space $(G/H, \tilde{\tau})$ is homogeneous. Moreover, if the subgroup H is normal then the multiplication $xHyH = xyH$ in G/H is continuous and $(G/H, \tilde{\tau})$ is a paratopological group.*

Proof. The continuity of the map π is obvious. If $U \subset G$ is an open set then $\pi^{-1}\pi(U) = UH$ and hence $\pi(U)$ is open. The space G/H is homogeneous, because the translations $l_a\colon xH \to axH$ are homeomorphisms.

Now let H be a normal subgroup of the group G. If $\tilde{U}$ is a neighborhood of the point $\tilde{c} = \tilde{a}\tilde{b}$ then $c = ab$ for some representatives a, b, c from the classes $\tilde{a}, \tilde{b}, \tilde{c}$ respectively. For a neighborhood $U = \pi^{-1}(\tilde{U}) \ni c$ there exist neighborhoods $V_1(a)$ and $V_2(b)$ such that $V_1(a)V_2(b) \subset U$. Thus $\pi(V_1(a))\pi(V_2(b)) \subset \tilde{U}$ and G/H is a paratopological group. □

Proposition 1.13. *If H is compact then the map π is closed. If the space (G, τ) is Hausdorff then the space $(G/H, \tilde{\tau})$ is Hausdorff. If the space (G, τ) is regular then the space $(G/H, \tilde{\tau})$ is regular.*

Proof. Let F be a closed subset of the group G. Let $\tilde{x} \in G/H \backslash \pi(F)$. Consider an arbitrary point $x \in \pi^{-1}(\tilde{x})$. Then $xH \cap F = \varnothing$. By Proposition 8 there exists on open neighborhood U of the unit such that $UxH \cap F = \varnothing$. Then $\tilde{x} \in \pi(Ux)$ and $\pi(Ux) \cap \pi(F) = \varnothing$ thus the map π is closed.

Let G be Hausdorff and $\tilde{x}_1, \tilde{x}_2 \in G/H$. Consider arbitrary points $x_i \in \pi^{-1}(\tilde{x}_i)$. Then $x_1H \cap x_2H = \varnothing$. By Proposition 1.7 there exists on open neighborhood U of the unit such that $Ux_1H \cap Ux_2H = \varnothing$. Then $\tilde{x}_i \in \pi(Ux_i)$ and $\pi(Ux_1) \cap \pi(Ux_2) = \varnothing$ thus the space G/H is Hausdorff.

Let G be regular, $\tilde{F}$ be a closed subset of G/H and $\tilde{x} \in G/H \backslash \tilde{F}$. Consider an arbitrary point $x \in \pi^{-1}(\tilde{x})$. Then $x \notin \pi^{-1}(\tilde{F})$. Proposition 8 and regularity of G imply that there exists on open neighborhood U of the unit such that $\overline{U}x \cap \pi^{-1}(\tilde{F}) = \varnothing$. Then $\tilde{x} \in \pi(Ux)$ and $\overline{\pi(Ux)} \cap \tilde{F} = \varnothing$, thus the space G/H is regular. □

Corollary 1.1. *Let H be a compact subgroup of a paratopological group G, F be a closed subset of G. Then FH is a closed subset of G.*

Proof. Let $\pi : G \to G/H$ be the standard projection. Then $FH = \pi^{-1}\pi(F)$ is a closed subset of G. □

The *product* of a family of paratopological groups $\{G_\alpha : \alpha \in A\}$ is the product $\prod G_\alpha$ endowed with the Tychonov product topology. The *box product* of a family of paratopological groups $\{G_\alpha : \alpha \in A\}$ is the product $\prod G_\alpha$ endowed with the box product topology. It is easy to see that both the product and the box product of a family of paratopological groups is again a paratopological group.

Example 1.11. Let $G = (\mathbb{Q}, \tau_s|\mathbb{Q})^2$, where (R, τ_s) is the Sorgenfrey arrow. Let Q be an arbitrary dense proper subgroup of $\mathbb{Q}$. Then $H = \{(x, -x) : x \in Q\}$ is a closed subgroup of G. Let $\pi\colon G \to G/H$ be the standard projection. If $x \in \mathbb{Q}\backslash Q$ then the points $\pi(x, -x)$ and $\pi(0, 0)$ have no disjoint neighborhoods in the quotient topology of G/H.

Let $K = \{(0, 0)\} \cup \{(1/n, 0) : n \in \mathbb{N}\}$. Then K is a compact subset of G but $H + K$ is not closed.

Proposition 1.14. *Let G_1, G_2 be paratopological groups, $f\colon G_1 \to G_2$ be a homeomorphism. Then f is continuous (open) if and only if f is continuous (open) at the unit of the group G_1.*

Theorem on continuous epimorphism. *Let G and H be paratopological groups, $\varphi\colon G \to H$ be a continuous epimorphism, $N = \ker\varphi$. Let a map $\sigma\colon G/N \to H$ be defined as $\sigma(xN) = \varphi(x)$. Then the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & H \\ {\scriptstyle\pi}\big\downarrow & \nearrow{\scriptstyle\sigma} & \\ G/N & & \end{array}$$

is commutative and σ is a continuous isomorphism. Moreover, if the map φ is open then σ is a topological isomorphism.

Proof. The map φ is well-defined, because $\{\varphi(y)\} = \varphi(x)\varphi(N) = \{\varphi(x)\}$ for every $y \in xN$. It is clear that σ is a bijection. If U is a neighborhood of the unit in H then $\varphi^{-1}(U)$ is a neighborhood of the unit in G and $\sigma\pi\varphi^{-1}(U) \subset U$. Hence σ is a continuous isomorphism. Similarly we can show that the map σ is open provided the map φ is open. □

We shall write $G_1 \simeq G_2$ if paratopological groups G_1 and G_2 are topologically isomorphic.

Corollary 1.2. *Let N and L be normal subgroups of a paratopological group G and $L \subset N$. Then $G/N \simeq (G/L)/(N/L)$.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\ \pi_N\ } & G/N \\ {\scriptstyle\pi_L}\big\downarrow & \nearrow{\scriptstyle\varphi} & \big\uparrow{\scriptstyle\sigma} \\ G/L & \xrightarrow{\ \pi_{N/L}\ } & (G/L)/(N/L) \end{array}$$

Define a map $\varphi\colon G/L \to G/N$ putting $\varphi(xL) = xN$. The Theorem on continuous epimorphism implies that φ is a continuous open epimorphism with the kernel G/L. Using once again the Theorem on continuous epimorphism we obtain that $G/N \simeq (G/L)/(N/L)$. □

Theorem on isomorphism. *Let G be a paratopological group, H be a subgroup of G and N be a normal subgroup of G. Then HN is a subgroup of G, N is a normal subgroup of G and the map $\sigma\colon H/(H \cap N) \to (NH)/N$ defined as $\sigma(h(H \cap N)) = hN$ is a homomorphic compression.*

Proof. Let $\varphi = \pi|H$. Then $\varphi(H) = (HN)/N$ and $\ker\varphi = \{h \in H : hN = N\} = H \cap N$. Therefore $H \cap N$ is a normal subgroup of the group H. It remains to apply to H and φ the Theorem on continuous epimorphism. □

Example 1.12. Let $G = (\mathbb{R}^2, +)$. Let τ_1 be the topology on the group G from Example 1.2, τ_2 has the base at the zero $\{\{(x, y) \in \mathbb{R}^2 : 0 \le x < 1/n, |y| \le x\} : n \in \mathbb{N}\}$. Let $H = (\mathbb{R}, 0)$. Then $\tau_1 \subset \tau_2$, $\tau_1|H = \tau_2|H$ and $\tau_1/H = \tau_2/H$ but $\tau_1 \ne \tau_2$.

Example 1.13. For a paratopological group G by $\mathrm{Aut}(G)$ we denote the group of automorphisms of the group G. Let H be a subgroup of the group $\mathrm{Aut}(G)$ and the map $\sigma\colon H \times G \to G$, $(h, x) \mapsto h(x)$ is continuous. Then the topological product $G \times_\sigma H$ with multiplication $(x_1, h_1)(x_2, h_2) = (x_1 h_1(x_2), h_1 h_2)$ for every $(x_1, h_1), (x_2, h_2) \in G \times H$ is a paratopological group. This group is called the *semidirect product* of the groups G and H.

2. Cardinal invariants

A function φ defined on the class $\mathcal{C}$ of paratopological groups is called a *cardinal function* if it assigns to each member $G \in \mathcal{C}$ an infinite cardinal number $\varphi(G)$. Now we shall list the cardinal functions to be examined in what follows. Remark that in the below definitions $\min^*\{\cdot\} = \omega \cdot \min\{\cdot\}$ and $\sup^*\{\cdot\} = \omega \cdot \sup\{\cdot\}$.
Boundedness: Let λ be an ordinal. A paratopological group is left (right) λ-bounded if for every open set U there exists a set $A \subset G$ such that $|A| \leq \lambda$ and $AU = G$ ($UA = G$). A paratopological group is left (right) totally bounded if for every open set U there exists a finite set $A \subset G$ such that $AU = G$ ($UA = G$). $bn(G) = \min^*\{\lambda \in \text{Card} : G \text{ is } \lambda\text{-bounded}\}$.

Question 2.1. (I. Guran) Is every left totally (ω-)bounded paratopological group a right totally (ω-)bounded?

Cellurarity: $c(G) = \sup^*\{|\mathcal{U}| : \mathcal{U} \text{ is a disjoint family of open subsets of } G\}$.
Character: $\chi(G) = \min^*\{|\mathcal{B}| : \mathcal{B} \text{ is a neighborhood base at unit of } G\}$.
Density: $d(G) = \min^*\{|S| : S \subset G, \overline{S} = G\}$.
Network weight: $nw(G) = \min^*\{|\mathcal{U}| : \mathcal{U} \text{ is a network for } G\}$.
Pseudocharacter: $\psi(G) = \min^*\{|\mathcal{U}| : \mathcal{U} \text{ is a family of open sets and } \bigcap\mathcal{U} = \{e\}\}$.
Spread: $s(G) = \sup^*\{|S| : S \subset G,\ S \text{ is discrete as a subspase}\}$.
Weakly Lindelöf degree: $wl(G) = \min^*\{\lambda \in \text{Card} : \text{in every open cover } \mathcal{V} \text{ there exists a subfamily } \mathcal{U} \subset \mathcal{V} \text{ such that } \overline{\mathcal{U}} = G \text{ and } |\mathcal{U}| \leq \lambda\}$.
Weight: $w(G) = \min^*\{|\mathcal{U}| : \mathcal{U} \text{ is an open base for } G\}$.

Proposition 2.1. (I. Guran) *Let G be a paratopological group such that* $\operatorname{int} U^{-1} \neq \varnothing$ *for every open set $U \subset G$. Then $bn(G) \leq wl(G)$.*

Question 2.2. (I. Guran) Let G be a paratopological group and $c(G) \leq \omega$ (respectively $wl(G) \leq \omega$). Is G ω-bounded?

Proposition 2.2. *Let G be a paratopological group, H be a subgroup of G. Then $\varphi(G) \leq \varphi(H)\varphi(G/H)$ where $\varphi \in \{d, \psi\}$.*

Proof. Let $\varphi = \psi$. Let $\{e\} = H \cap \bigcap\mathcal{U}$ and $\pi(e)\bigcup\tilde{\mathcal{U}}$, where $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are families of open sets of H and G/H respectively and $|\mathcal{U}| \leq \psi(H), |\tilde{\mathcal{U}}| \leq \psi(G/H)$. Then $\{e\} = \overline{\bigcap\mathcal{U} \cap \bigcap \pi^{-1}}(\tilde{\mathcal{U}})$.

Let $\varphi = d$. Let $D \subset H$, $\overline{D} = H$ and $|D| = d(H)$. Let $\tilde{D} \subset G$, $\overline{\pi(\tilde{D})} = G/H$ and $|\tilde{D}| = d(G/H)$. Consider an arbitrary nonempty open set $U \subset G$. There exists a point $x \in \tilde{D}$ such that $\pi(x) \in \pi(U)$. Then $x^{-1}U \cap H \neq \varnothing$ and hence there exists a point $y \in D$ such that $y \in x^{-1}U \cap H$. Therefore the set $\tilde{D}D$ is dense in G and $|\tilde{D}D| \leq d(H)d(G/H)$. □

If $\mathcal{U}, \mathcal{V}$ are families of subsets of a semigroup then we put $\mathcal{U}\mathcal{V} = \{UV : U \in \mathcal{U}, V \in \mathcal{V}\}$.

Proposition 2.3. *If M is a monoid with continuous multiplication and open translations then $w(M) = nw(M)\chi(M)$.*

Proof. Clearly, $w(M) \geq nw(M)\chi(M)$. Let $\mathcal{B}$ be a base at the unit of the monoid M, $|\mathcal{B}| \leq \chi(M)$. Let $\mathcal{N}$ be a network of the monoid M, $|\mathcal{N}| \leq nw(M)$. Then $\mathcal{B}\mathcal{N}$ is a base of the monoid M and $|\mathcal{B}\mathcal{N}| \leq nw(M)\chi(M)$. □

Example 2.1. Let $G = (\mathbb{R}, \tau_s)$ be the Sorgenfrey arrow. Then $s(G) = \omega$ but $s(G^2) = 2^\omega$.

3. Memorization

Let X be a topological space, $d\colon X \times X \to \mathbb{R}_+$ be a function. Consider the following conditions

M1. $(\forall x \in X) : d(x,x) = 0.$

M2. $(\forall x, y \in X) : d(x,y) = 0 \iff x = y.$

M3. $(\forall x, y \in X) : d(x,y) = d(y,x).$

M4. $(\forall x, y, z \in X) : d(x,z) \le d(x,y) + d(y,z).$

If the function d satisfies all the conditions then it is called a *metric*. If the function d satisfies all the conditions but M2 then it is called a *pseudometric*. If the function d satisfies all the conditions but M3 then it is called a *quasimetric*. If the function d satisfies all the conditions but M2 and M3 then it is called a *pseudoquasimetric*.

Let $x \in X$ and $\varepsilon > 0$. The set $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ is called a *ball* with center at the point x and radius ε. A topological space (X,τ) is *pseudoquasimetrizable* if there is a metric on X such that the balls $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$ form a base of the topology τ. The notions of *metrizable, quasimetrizable and pseudometrizable* spaces are defined similarly.

Let G be a group with the unit e. A function $d\colon G \times G \to \mathbb{R}_+$ is *left (two-side) invariant* if for arbitrary elements x, y, a, b of the group G we have $d(ax, ay) = d(x,y)$ $(d(axb, ayb) = d(x,y))$.

Let $d\colon G \times G \to \mathbb{R}_+$ be a left invariant function. Define a function $f_d\colon G \to \mathbb{R}_+$ putting $f_d(x) = d(e,x)$. Conversely, every function $f\colon G \to \mathbb{R}_+$ defines a left invariant function $d_f : G \times G \to \mathbb{R}_+$ as $d_f(x,y) = f(x^{-1}y)$. The function $d = d_f$ is two-side invariant if and only if $f_d(y^{-1}xy) = f_d(x)$ for every $x, y \in G$. The function $d = d_f$ satisfies one of the conditions M1–M4 if and only if $f = f_d$ satisfies the respective condition.

N1. $f(e) = 0.$

N2. $(\forall x \in X) : f(x) = 0 \iff x = e.$

N3. $(\forall x \in X) : f(x) = f(x^{-1}).$

N4. $(\forall x, y \in X) : f(xy) \le f(x) + f(y).$

If $d = d_f$ is a left (two-side) invariant metric then the function $f = f_d$ is an *(invariant) norm*. The notions of *pseudonorm, quasinorm and pseudoquasinorm* are defined similarly. The topology on the group G generated by the pseudoquasinorm d_f is denoted by τf. The family $\{\{x \in G : f(x) < \varepsilon\} : \varepsilon > 0\}$ is a base at the unit of the group $(G, \tau f)$.

Lemma 3.1. *Let $\{U_k | k \in \mathbb{N}\}$ be a sequence of neighborhoods of the unit e of a paratopological group G such that $U_{k+1}^2 \subset U_k, k \in \mathbb{N}$, and let $H = \bigcap\{U_k | k \in \mathbb{N}\}$. Then there exists a pseudoquasinorm g on the group G such that*

1. *$g(x) = 0 \iff x \in H$.*
2. *$g(x) \le 2^{-k+2}$, if $x \in U_k$.*
3. *$g(x) \ge 2^{-k}$, if $x \notin U_k$.*
4. *If $(\forall x \in G)(\forall k \in \mathbb{N}) : U_k^x = U_k$, then g is invariant.*

Proof. Put $V_{2^{-k}} = U_k, k \in \mathbb{N}$. Define sets V_r for every binary rational number r, $0 < r < 1$ as follows. If $r = 2^{-l_1} + 2^{-l_2} + \dots + 2^{-l_n}$, $0 < l_1 < l_2 < \dots < l_n$, then put $V_r = V_{2^{-l_1}} + V_{2^{-l_2}} + \dots + V_{2^{-l_n}}$. If $r \geq 1$ then put $V_r = G$. Remark that $V_r \subset V_s$ for every binary rational $r < s$. Indeed if $s \geq 1$ then $V_r \subset G = V_s$. Suppose that $r, s < 1$. Let $r = \sum_{i=1}^n 2^{-l_i}$, $s = \sum_{j=1}^p 2^{-m_j}$, where $0 < m_1 < m_2 < \dots < m_p$. There exists a unique number k such that $l_j = m_j$ for $j < k$ and $l_k > m_k$. Let $W = V_{2^{-l_1}} \cdots V_{2^{-l_{k-1}}}$. Then

$$V_r = WV_{2^{-l_k}} \cdots V_{2^{-l_n}} \subset WV_{2^{-l_k}} \cdots V_{2^{-l_n}} V_{2^{-l_n}} \subset WV_{2^{-l_k}} \cdots V_{2^{-l_n+1}} \subset$$
$$\subset WV_{2^{-l_k}} \cdots V_{2^{-l_{n-1}}} V_{2^{-l_{n-1}}} \subset \dots \subset WV_{2^{-m_k}} = V_{2^{-m_1}} \cdots V_{2^{-m_k}} \subset$$
$$\subset V_{2^{-m_1}} \cdots V_{2^{-m_k}} V_{2^{-m_{k+1}}} \cdots V_{2^{-m_p}} = V_s.$$

Remark that for every natural l and $r = 2^{-l_1} + 2^{-l_2} + \dots + 2^{-l_n}$ we have

$$V_r V_{2^{-l}} \subset V_{r+2^{-l+2}}. \qquad (*)$$

Indeed if $r + 2^{-l+2} \geq 1$ then embedding (*) is trivial. Suppose that $r + 2^{-l+2} < 1$. If $l > l_n$ then embedding (*) is obvious. Let $l \leq l_n$. Select a number k such that $l_{k-1} < l \leq l_k$. Let $r_1 = 2^{-l+1} - 2^{-l_k} - 2^{-l_{k+1}} - \dots - 2^{-l_n}$ and $r_2 = r + r_1$. Then $r < r_2 < r + 2^{-l+1}$ and

$$V_r V_{2^{-l}} \subset V_{r_2} V_{2^{-l}} \subset V_{r_2+2^{-l}} \subset V_{r+2^{-l+1}+2^{-l}} \subset V_{r+2^{-l+2}}.$$

Define a function $\varphi\colon G \to \mathbb{R}_+$ putting $\varphi(x) = \inf\{r : x \in V_r\}$ and a function $g\colon G \to \mathbb{R}$ putting $g(x) = \sup\{\varphi(zx) - \varphi(z) : z \in G\}$. It is easy to see that for every elements $x, y \in G$ the following conditions hold (a) $\varphi(x) = 0 \iff x \in H$; (b) $g(x) \geq 0$ and $g(e) = 0$; (c) $g(xy) \leq g(x) + g(y)$.

Now we check that the function g satisfies the conditions of the theorem.

Let k be a natural number and $x \in V_{2^{-k}}$. Let z be an arbitrary element of the group G. If $z \in V_r$ then condition (*) implies that $zx \in V_{r+2^{-k+2}}$ hence $\varphi(zx) - \varphi(z) \leq 2^{-k+2}$. Therefore $g(x) \leq 2^{-k+2}$ and condition 2 is satisfied. Condition 3 is satisfied, because for every $x \in U_k$ we have $g(x) \geq \varphi(x) - \varphi(e) = 2^{-k}$. Condition 1 is an implication of conditions 2 and 3.

Suppose now that all neighborhoods U_k are invariant. Then $\varphi(y^{-1}xy) = \varphi(x)$ for every $x, y \in G$ and thus

$$g(y^{-1}xy) = \sup\{\varphi(zy^{-1}xy) - \varphi(z) : z \in G\} = \sup\{\varphi(yzy^{-1}x) - \varphi(yzy^{-1}) : z \in G\} =$$
$$= \sup\{\varphi(tx) - \varphi(t) : t \in G\} = g(x).$$

□

Proposition 3.1. *A paratopological group is quasimetrizable by a left invariant metric if and only if it is first countable. A paratopological group is quasimetrizable by two-side invariant metric if and only if it is a first countable SIN-group.*

Proof. The necessity is obvious. We prove the sufficiency. Let $\{V_n : n \in \mathbb{N}\}$ be a countable base at the unit of the group G. By induction we can construct a sequence $\{U_k\}$ of open neighborhoods of the unit such that $U_{k+1}^2 \subset U_k \cap V_1 \cap \dots \cap V_k$ and if G is a SIN-group then all neighborhoods U_k are invariant. Then the sequence $\{U_k\}$ satisfies the conditions of the previous proposition and therefore there exists an (invariant) quasinorm g on the group G. Since

$$\{x \in G : g(x) < 2^{-k}\} \subset U_k \subset \{x \in G : g(x) < 2^{-k+2}\},$$

the topology of the group G is generated by the quasinorm g. □

Example 3.1. Let (G,τ) be a p-arrow. Then the topology of G can be generated by the quasinorm

$$g(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n < 0 \\ p^{-s}, & \text{where } s \text{ is the maximal degree of } p \text{ dividing } n, \text{ if } n > 0 \end{cases}$$

but the quasinorm g is not continuous on the group (G,τ).

Question 3.1. Is every first countable Tikhonov paratopological group quasimetrizable by a continuous left invariant quasimetric?

Proposition 3.2. *If a paratopological group G is metrizable by a left invariant metric then G is a topological group.*

Proof. Let d be a norm generating the paratopology of the group G. Then $\mathcal{B} = \{\{x \in G : f(x) < \varepsilon\} : \varepsilon > 0\}$ is a base of the unit of the group G consisting of symmetric neighborhoods, hence G is a topological group. □

Example 3.2. The rational points of the Sorgenfrey arrow is a regular space with countable base, hence it is a metrizable paratopological group which is not a topological group.

Question 3.2. Is every Moore paratopological group metrizable?

4. Minimality

Proposition 1.3 implies that every Hausdorff SIN-paratopology on a group can be weakened to a Hausdorff group SIN-topology. In [4] I. Guran asks: can every Hausdorff paratopology on a group be weakened to a Hausdorff group topology? The following example gives the negative answer to this question.

Let F be a free semigroup over a set X. A word $w = y_1 \cdots y_n \in F$, $y_i \in X$ is *reduced* if there is no pair $y_i y_{i+1}$ such that $y_i^{-1} = y_{i+1}$. A reduced word is *cyclic reduced* if $y_1^{-1} \neq y_n$.

Lemma 4.1. [5, Theorem 5.5] *Let G be a group generated by an alphabet $A = \{t, b, c, \dots\}$ with a relation $r^n = 1$ where r is cyclic reduced and $n > 1$. Let w, v be words over the alphabet A and $w = v$ in the group G. Let the word w be reduced and there exists a letter $a \in A$ which is contained in the word w but which is not contained in the word v. Then there exists a subword s of the word w which also is a subword of the word $r^{\pm n}$ such that $l(s) > (n-1)l(r^n)/n$, where $l(s)$ and $l(r^n)$ denote the lengths of the words s and r^n respectively.*

Let G be a group, $A \subset G$ be a set. Then *the normal closure* of the set A is the smallest normal subgroup of the group G containing the set A.

Corollary 4.1. *Let F^2 be a free group over $\{x, y\}$, N be a normal closure of the element $r^2 = (xy^{-1})^2$. Let $S \subset G$ be a semigroup generated by the elements x an y. Then $SS^{-1} \cap N = \{e\}$.*

Proof. Let $w \in SS^{-1} \cap N$ be a nontrivial reduced word. Then Lemma 4.1 implies that w must contain the subword s of length 3 such that $s \notin SS^{-1}$, which is impossible. □

Example 4.1. Let n be a natural number and F_n^2 be a free group over $\{x_n, y_n\}$. Let G be the direct product of the groups F_n^2. Let $S_n \subset F_n^2$ be the semigroup generated by the elements x_n and y_n. Denote the direct product $\prod_{m \geq n} S_m$ by U_n.

We show that the family $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ satisfies Pontrjagin conditions 1-4. Condition 1 is satisfied because $U_n \cap U_m \supset U_{\max(m,n)}$.Conditions 2 and 3 are satisfied since U_n are semigroups. Let $U_n \in \mathcal{B}$ and $w \in G$. There exists a number m such that $w \in \prod_{i=1}^m F_i^2$. Then $w^{-1}U_{\max(m,n)+1}w = U_{\max(m,n)+1} \subset U_n$. Hence condition 4 is satisfied. Therefore $\mathcal{B}$ is a base at the unit of some (not necessarily Hausdorff) paratopology on the group G.

Let F_n be a factor group of the group F_n^2 by the relation $r_n^2 = (x_n y_n^{-1})^2$, $\varphi_n \colon F_n^2 \to F_n$ be the canonical homomorphism and $N_n = \ker \varphi_n$. Let $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ and $\psi_n \colon F_n^2 \to \mathbb{Z}_3$ be a homomorphism such that $\psi_n(x_n) = 0$ and $\psi_n(y_n) = 1$. Define a map $\psi \colon G \to \mathbb{Z}_3 \times \prod F_n$ as follows. Let $w = w_1 \cdots w_n \in G$ where $w_i \in F_i^2$. Put $\psi(w) = (\sum \psi_i(w_i), \prod \varphi_i(w_i))$. Let $G' = \psi(G)$ and τ' be the quotient topology on the group G'.

We show that τ' is a Hausdorff topology. Let $\mathcal{B}' = \{\psi(U_n) : n \in \mathbb{N}\}$. Therefore $\mathcal{B}'$ is a base at the unit of the paratopology τ'. Verify condition 5 for the family $\mathcal{B}'$. Let $w \in G \backslash \ker \psi$. If $w \in G \backslash \prod N_n$ then there exists a number n such that $w \in \prod_{i=1}^n F_i^2$. Then $\psi(U_{n+1}) \cap \psi(wU_{n+1}) = \varnothing$. Suppose that $w \in \prod N_n$. Let $w = w_1 \cdots w_n$, where $w_i \in F_i^2$. Since $\psi(w) \neq e$, we see that $\sum \psi_i(w_i) \neq 0$. Therefore there exists a number i such that $\psi(w_i) \neq 0$. Since $w_i \in N_i$, Corollary implies that $w_i \notin S_i S_i^{-1}$. Then $\psi(U_1) \cap \psi(wU_1) = \varnothing$. Therefore the topology τ' is Hausdorff.

We show that $\psi(U_n)$ is a clopen subset of the group G' for every n and hence τ' is a zero-dimensional topology. Let $w \in G$ and $\psi(w) \in \overline{\psi(U_n)}$. Let $w = w_1 \cdots w_m$, where $w_i \in F_i^2$. There exist elements $u \in U_{m+1}, v \in U_n$ such that $wuv^{-1} \in \ker \psi$. Let $u = u_{m+1} \cdots u_k, v = v_n \cdots v_k$, where $u_i, v_i \in F_i^2$. Then $u_i v_i^{-1} \in N_i$ for $i \geq m+1$. Since $u_i v_i^{-1} \in S_i S_i^{-1}$ for every i, Corollary implies that $u_i = v_i$ for $i \geq m+1$. Therefore $w \prod_{i=n}^m v_i^{-1} = wuv^{-1} \in \ker \psi$ and $\psi(w) = \psi(v_n \cdots v_m) \in \psi(U_n)$

The topology τ' cannot be weakened to a Hausdorff group topology on the group G', because $(\psi(U_n)\psi(U_n)^{-1})^2 \ni (\psi(x_n)\psi(y_n)^{-1})^2 = (1, e)$ for every natural n.

A topology on a quasifield is called a *ring topology* if the multiplication, the addition, and the subtraction on the quasifield are continuous, that is the additive group of the quasifield is a topological group and the multiplicative group of the quasifield is a paratopological group. If the multiplicative group of the quasifield is a paratopological SIN-group then the topology is called a *SIN-topology*. A topology on a quasifield is called a *quasifield topology* if the additive and multiplicative groups of the quasifield are topological groups. Recall that the following problem is still open: can every Hausdorff ring topology on a quasifield be weakened to a Hausdorff quasifield topology?

Proposition 4.1. *Every Hausdorff ring SIN-topology on a quasifield can be weakened to a Hausdorff quasifield SIN-topology.*

Proof. Let (K, τ) be such a quasifield. Let K^* be the multiplicative group of the quasifield K. We can expose Pontrjagin conditions for the base $\mathcal{B} = \mathcal{B}_\tau$ at the unit 1 of K in the following form.

1. $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B}) : W \subset U \cap V$.
2. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^2 \subset U$.
3. $(\forall U \in \mathcal{B})(\forall x \in U)(\exists V \in \mathcal{B}) : xV \subset U$.
4. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})(\forall x \in K^*) : x^{-1}Vx \subset U$.

5. $\bigcap\{UU^{-1} : U \in \mathcal{B}\} = \{1\}$.
6. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V - V + V \subset U$.
7. $(\forall U \in \mathcal{B})(\forall x \in K)(\exists V \in \mathcal{B}) : x(V - V) \subset U - U$.

These conditions show that we can choose a base $\mathcal{B}$ consisting of invariant neighborhoods, that is such that $xU = Ux$ for every $U \in \mathcal{B}$ and $x \in K$. Put $\mathcal{B}' = \{UU^{-1} : U \in \mathcal{B}\}$. Clearly, the family $\mathcal{B}'$ satisfies conditions 1-5 and that $U' = (U')^{-1}$ for every neighborhood $U' \in \mathcal{B}'$. Therefore we must show only that the family $\mathcal{B}'$ satisfies conditions 6 and 7. Let $U \in \mathcal{B}$. Select a neighborhood $V \in \mathcal{B}$ such that $V^3 - V^3 + V^3 \subset U$. Then $VV^{-1} - VV^{-1} + VV^{-1} \subset (V^3 - V^3 + V^3)V^{-3} \subset UU^{-1}$, hence condition 6 is satisfied. Let $U \in \mathcal{B}$ and $x \in K$. Select a neighborhood $V \in \mathcal{B}$ such that $x(V^2 - V^2)\imath U - U$ and $V^2 \imath U$. Then $x(VV^{-1} - VV^{-1})\imath x(V^2 - V^2)V^{-2}\imath(U - U)U^{-1}\imath UU^{-1} - UU^{-1}$. □

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