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STABILITY STRUCTURE OF LINEAR GROUP OVER RINGS

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It is shown that from the stability of factor-rings on Jacobson's radical or from the stability of localization of the associative rings on maximal ideals of their centers the stability of the rings follows. It is also proved that matrices with at least one zero element are stable and from the stability of all the elements of the general linear group over the associative ring its stability follows. The most important results of H. Bass, L. Vaserstein, S. H. Khlebutin, A. A. Suslin, I. S. Wilson, I. Z. Golubchik about the stability of the associative rings from the unified position are exposed.

В. М. Петечук. *Стабильная структура линейных групп над кольцами* // Математичні Студії. – 2001. – Т.16, №1. – С.13–24.

Показано, что из стабильности фактор-кольца по радикалу Джекобсона или из стабильности локализаций ассоциативного кольца по всем максимальным идеалам его центра следует стабильность самого кольца. Доказано также, что матрицы с хотя бы одним нулевым элементом — стабильны, а из стабильности всех элементов полной линейной группы над ассоциативным кольцом следует его стабильность. С единых позиций изложены наиболее значимые классические результаты Х. Басса, Л. Васерштейна, С. Г. Хлебутина, О. О. Суслина, И. Уилсона, И. З. Голубчика про стабильность ассоциативных колец.

Let R be an associative ring with 1, R^* the group of invertible elements of ring R, r an arbitrary element R, I an arbitrary ideal R, $\Lambda_I\colon R \to R/I$, X an arbitrary subset R, $\bar{X} = \Lambda_I(X)$, $J(R)$ the Jacobson radical R.

Let $\mathrm{GL}(n,R)$ be the general linear group $n\times n$ matrices over ring R, $g=(g_{ij})$ an element of the group $\mathrm{GL}(n,R)$, $g^{-1}=(G_{ij})$, e_{ij} a standard matrix unit, $t_{ij}(r)=1+re_{ij}$, $i\neq j$, $a_{ij}=t_{ij}(1)t_{ji}(-1)t_{ij}(1)$, $d_i(c)=1+(c-1)e_{ii}$, $c\in R^*$.

We will call the element $t_{ij}(r)$, for $r\neq 0$ a *transvection.*

Let E_X be the subgroup of $\mathrm{GL}(n,R)$ which is generated by the transvection $t_{ij}(r)$, $r\in X$, $1\le i\neq j\le n$, $E(n,R)=E_R$, $E(n,I)$ a normal subgroup of $E(n,R)$ generated by E_I, $C(n,I)$ the inverse image of the center of the group $\mathrm{GL}(n,R/I)$ under the homomorphism $\Lambda_I\colon \mathrm{GL}(n,R)\to \mathrm{GL}(n,R/I)$, N any subgroup of $\mathrm{GL}(n,R)$ which is invariant with respect to $E(n,R)$ and does not contain transvections.

Let S be a subset of the center of the ring R which is closed under multiplication operation and R_S the classical division ring of the ring R by S.

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A natural homomorphism $\Lambda\colon R \to R_S$ defined by the rule $\Lambda : r \mapsto \frac{rs}{s}$ for any $s \in S$, induces the homomorphism $\Lambda\colon \mathrm{GL}(n, R) \to \mathrm{GL}(n, R_S)$.

Let $a^b = bab^{-1}$, $[a, b] = aba^{-1}b^{-1}$. Then commutatorial formulas hold:

$$[ab, c] = [b, c]^a[a, c], \qquad [a, bc] = [a, b][a, c]^b,$$

$$[a^{-1}, b, c]^a[c^{-1}, a, b]^c[b^{-1}, c, a]^b = 1 \text{ (F. Holl identity)},$$

and also the matrix commutatorial formulas for the group $\mathrm{GL}(n, R)$, $n > 2$:

$$[t_{ik}(x), t_{lj}(y)] = \begin{cases} t_{ij}(\delta_{kl}xy), & i \neq j \\ t_{lk}(-\delta_{ij}yx), & l \neq k, \end{cases}$$ where δ_{ij}, δ_{kl} are Kroneker's symbols.

Note, that the commutator of two elements at matrix commutatorial equations commutes with both of these elements.

Let T be a group generated by the elements $t_{ij}(I)^{t_{ji}(R)}$, $1 \leq i \neq j \leq n$. It is clear that $E_I \in T$. It follows from matrix commutatorial equations that for $(s, t) \neq (j, i)$ the inclusions $t_{ij}(I)^{t_{st}(R)} \subset E_I$ hold. Thus the result of conjugation E_I by transvection or by the product of two commutative transvections belongs to the group T. From previous and matrix commutatorial equations we see that T is invariant with respect to transvections. Consequently, it is proved that T coincides with the group $E(n, I)$.

Therefore, $E(n, I^2)$ belongs to the group, generated by the elements $[t_{ik}(I), t_{kj}(I)]^{t_{ji}(R)}$ of the group E_I. Thus $E(n, I^2) \subset E_I$.

An associative ring R with 1 is called *commutatorial*, if

$$[C(n, I), E(n, R)] = E(n, I) \triangleleft \mathrm{GL}(n, R)$$

for all ideals I of ring R.

An associative ring R with 1 is called *partially-normal*, if all N which are invariant with respect to $E(n, R)$ and do not contain transvections belong to groups of central scalar matrices of $\mathrm{GL}(n, R)$.

An associative ring R with 1 is called *normal*, if for any subgroup G of the group $\mathrm{GL}(n, R)$ invariant with respect to $E(n, R)$ there exists an ideal I_0 of the ring R such that

$$E(n, I_0) \subset G \subset C(n, I_0).$$

It is clear that factor-ring of any normal ring is partially-normal.

An associative ring R which is both commutatorial and normal is called *stable*.

An associative ring R with 1 is called weak commutatorial if the commutatorial equations hold:

$$[C(n, I), E(n, R), ..., E(n, R)] = E(n, I),$$

where I is an ideal of the ring R. It is clear, that the commutatorial rings are weak commutatorial.

Lemma 1. *Let $n > 2$. Any weak commutatorial ring which has only partially-normal factor-rings is stable.*

Proof. Let G be the subgroup of $\mathrm{GL}(n,R)$ invariant with respect to $E(n,R)$ and I_0 be a maximal ideal of ring R such that $E(n,I_0) \subset G$. If $\Lambda_{I_0} E(n,J) \subset \Lambda_{I_0}(G)$, where $I_0 \subset J$, then

$$E(n,J) \subset GC(n,I_0) \text{ and } E(n,J) = [E(n,J), E(n,R), ..., E(n,R)] \subset G$$

that contradicts to the assumption. Therefore $\Lambda_{I_0}(G)$ does not contain transvections. Since the ring R/I_0 is partially-normal, $G \subset C(n,I_0)$. This proves that R is a normal ring.

Let H be a subgroup of $\mathrm{GL}(n,R)$ conjugated with $E(n,R)$ and H_0 the normal closure of H with respect to the group $E(n,R)$. Since R is normal, $E(n,R) \subset H_0$. Since the groups H and $[E(n,R), H]$ are contained in the groups $[H_0, H]$, we have $H_0 \subset [H_0, H] \subset ... \subset [H_0, H, ..., H] = H$. Therefore $H_0 = H$, $E(n,R)$ and $E(n,I)$ are normal subgroups of group $\mathrm{GL}(n,R)$. The F. Holl identity shows that R is commutatorial and, consequently, a stable ring. □

From Lemma 1 it follows that all weak commutatorial normal rings are stable.

It should be noted that from commutatority of the ring R and the inclusion $E(n,I) \subset G \subset C(n,I)$ it follows that the group G is invariant with respect to $E(n,R)$ and uniquely determines the ideal I.

Lemma 2. *Let R be an associative ring with 1, $g \in N$, $g_{ij}x = 0$, $x \in R$, $n > 2$. Then $g \in C(n, \mathrm{Ann}\, RxR)$ for $i \neq j$ and $x = 0$ for $i = j$.*

Proof. Let $g_1 = [g, t_{jk}(x)]$, $1 \leq k \neq j \leq n$. Then $g_1 \in N$, and the i-th rows of elements at g_1 and $t_{jk}(-x)$ coincide.

Suppose that $i \neq j$. If $g_1 \neq 1$, we make a transvection which belongs to N by using commutator g_1 with transvections. Since N does not contain transvections, $g_1 = 1$, and $g_{kj}x = xg_{ks}$, for all $1 \leq s \neq k \leq n$. Analogously $[g^{-1}, t_{sk}(x)] = 1$, $g_{ts}x = 0$ for all $1 \leq t \neq s \leq n$. Therefore g is commutative with all elements of the group E_x. This means that $gx = xg$ is scalar (not necessarily central) matrix. Furthermore, g commutes with elements of the group E_{RxR}. This means that $g \in C(n, \mathrm{Ann}\, RxR)$.

Analogously, if $g \in N$, $xg_{ij} = 0$, $i \neq j$, then $g \in C(n, \mathrm{Ann}\, RxR)$.

In the particular case, $g \in N$, $g_{ij} = 0$, $i \neq j$, we can choose $x = 1$. Thus g is a central scalar matrix of the group $\mathrm{GL}(n,R)$.

Since g_1 contains a zero nondiagonal element, so $g_1 = 1$ also in the case $i = j$. This proves that $x = 0$ in the case $i = j$. □

Lemma 3. *If R is an associative ring with 1, $n > 2$, $g \in N$, $g_{i1}x_1 + ... + g_{in}x_n = 0$ and at least one of the elements $x_1, ..., x_n$ is zero. Then $g \in C(n, \mathrm{Ann}\, R\langle x_1, ..., x_n\rangle R)$.*

Proof. Let $x_j = 0$, $g_1 = [g, t_{1j}(x_1)...t_{nj}(x_n)]$. Then $g_1 \in N$, $(g_1)_{ii} = 1$ and g_1 contains a zero nondiagonal element. By Lemma 2 $g_1 = 1$, $g \in C(n, \mathrm{Ann}\, R\langle x_1, ..., x_n\rangle R)$.

Similarly, if $n > 2$, $g \in N$, $x_1 g_{1j} + .. + x_n g_{nj} = 0$ and at least one of the elements $x_1, .., x_n$ is zero, then $g \in C(n, \mathrm{Ann}\, R < x_1, ..., x_n > R)$. □

In a particular case, $n > 2$, $g \in N$ and some element g_{ij} has an inverse, then one of the elements $x_1, .., x_n$ could be chosen 1. Then g is the central scalar matrix of the $\mathrm{GL}(n,R)$ group.

Let $u = (u_1...u_n)^t$ be the column, $v = (v_1...v_n)$ the row. We assume that U is a matrix $n \times n$ with the first column u and the rest zeros, and V the matrix $n \times n$ with the first row v and the rest zeros. If $v_t = 0$, then

$$U = \begin{pmatrix} a & 0 \\ \alpha & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \text{ for } t = 1 \text{ and}$$

$$U^s = \begin{pmatrix} \alpha & 0 \\ a & 0 \end{pmatrix}, \quad V^s = \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} \text{ for } s = \begin{cases} a_{tn}, & 1 < t < n, \\ 1, & t = n. \end{cases}$$

If in this case $VU = 0$, then $\beta\alpha = 0$,

$$1 + UV = \left[\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & a\beta \\ 0 & 1 \end{pmatrix} \text{ for } t = 1 \ ,$$

$$1 + U^s V^s = \left[\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ a\beta & 1 \end{pmatrix} \text{ for } 1 < t \le n.$$

Note that in the above received decomposition of the element $1 + UV$ it follows the decomposition of the element $1 + (Ur)V = 1 + U(rV)$, $r \in R$. For this purpose, in the decomposition of the element $1 + UV$ it is sufficient to multiply for r each element U or V from the right or respectively the left side.

Therefore, if it is possible to decompose $1 + UV$ over the ring R_s, then $1 + UsRV$ can be decomposed over R, where s is the product of denominators of the elements U and V.

Let A be a matrix. We will denote by $\tilde{A}$ a such matrix that all the elements $\tilde{A} - A$ belong to I. We assume that $\tilde{U}$ is a matrix with the first column $\tilde{u}$ and the rest zeros, and $\tilde{V}$ is a matrix with the first column $\tilde{v}$ and the rest zeroes.

Since $t_{ij}(cr) \in t_{ij}(c\tilde{r})E(n, cI)$, where $r \in R$ and c is an element of the centre of R, the following holds.

Lemma 4. *If* $VU = \tilde{V}\tilde{U} = 0$ *and there exists a number* $1 < t \le n$ *such that* $v_t = \tilde{v}_t = 0$, *then* $1 + UcV \in (1 + \tilde{U}c\tilde{V})E(n, cI)$.

Proof. It is obvious that $t = 1$. If $1 < t \le n$, then, as in the previous example, $1 + U^s cV^s \in (1 + \tilde{U}^s c\tilde{V}^s)E(n, cI)$. Conjugation by the element s^{-1} proves Lemma 4. □

Let $g \in \mathrm{GL}(n, R)$, $U = ge_{ii}$, $V = e_{ij}g^{-1}$. It is clear, that $VU = \delta_{ij}$. We assume, that $i \ne j$, $VU = 0$.

Let $V(k)$ be the matrix $n \times n$ with the i-th row $\left(x_{1k}...x_{kk}\left(g^{-1}\right)_{jk}...x_{nk}\right)$ and the rest zeroes, where $x_{1k}, ..., x_{nk}$ are elements of R, $1 \le k \le n$. Put $W(k) = x_{kk}V - V(k)$. It is obvious that $W(k)_k = W(k)_{ik} = 0$.

Lemma 5. *Let* $V(k)U = 0$, $x_{lk} = 0$ *for some* $1 \le l \le n$, c *is an element of the centre of a ring* R. *Then*

1) $t_{ij}(Icx_{kk})^g \subset E(n, cI)$, *for* $g \in \mathrm{GL}(n, R)$;
2) $[C, t_{ij}(Rcx_{kk})] \subset E(n, cI)$, *for* $g = \alpha C$, $C \in C(n, I)$, $\alpha_{ki} = 0$ *or* $\left(\alpha^{-1}\right)_{jk} = 0$ *and* $x_{tk} \in I$ *for all* $1 \le t \ne k \le n$.

Proof. If $x_{kk} = 0$, then Lemma 5 is obvious. Hence, we may assume that $l \neq k$. If we choose $\widetilde{V(k)_l} = 0$, $\widetilde{W(k)_k} = 0$, $\widetilde{V(k)}\widetilde{Ur} = \widetilde{W(k)}\widetilde{Ur} = 0$, then by Lemma 4,

$$t_{ij}(rcx_{kk})^g = 1 + Urcx_{kk}V =$$

$$= (1 + UrcV(k))(1 + UrcW(k)) \in \big(1 + \widetilde{Ur}c(\widetilde{V(k)} + \widetilde{W(k)})\big)E(n, cI).$$

□

Consider consequences of this inclusion:

1) If $r \in I$, then we can choose $\widetilde{Ur} = 0$. Then $t_{ij}(rcx_{kk})^g \in E(n, cI)$.

2) If $r \in R$, then we can choose $\widetilde{Ur} = \alpha C_{ii} r e_{ii}$, $\widetilde{V(k)} = x_{kk}\left(C^{-1}\right)_{jj}\left(\alpha^{-1}\right)_{jk} e_{ik}$, $\widetilde{W(k)} = e_{ij} x_{kk}\left(C^{-1}\right)_{jj}\alpha^{-1} - \widetilde{V(k)}$. Then $t_{ij}(rcx_{kk})^g \in t_{ij}(C_{ii}rcx_{kk}\left(C^{-1}\right)_{jj})^{\alpha} E(n, cI)$.

According to Theorem 4 α normalizes the group $E(n, cI)$. Then $[C, t_{ij}(rcx_{kk})] \subset E(n, cI)$. It is obvious that α can be either a diagonal matrix or transvection.

Lemma 6. *Let $V(k)U = 0$, $x_{lk} = 0$ for a $1 \leq l \leq n$, $g \in N$, $n > 2$. Then $g \in C(n, \operatorname{Ann} Rx_{tk}R)$ for all $1 \leq t \leq n$.*

Proof. According to Lemma 3 we have $g \in C(n, \operatorname{Ann} R\langle x_{1k}, ..., x_{kk}G_{jk}, ..., x_{nk}\rangle R)$. Then $g_1 = [t_{ij}(x_{kk}), g] \in N$, $(g_1)_{jk} = \delta_{jk}$. By Lemma 3 g_1 is a central scalar matrix of the group $\mathrm{GL}(n, R)$, $g \in C(n, \operatorname{Ann} Rx_{kk}R)$. □

We say that an element g of the group $\mathrm{GL}(n, R)$ satisfies the condition of the left (R, i, j)-stability, if exist $V(1), .., V(n)$ such that $R =_R \langle x_{11}, ..., x_{nn}\rangle$.

If an element g of the group $\mathrm{GL}(n, R)$ satisfies the condition of the left (R, i, j)-stability for such $1 \leq i \neq j \leq n$, and $E(n, R) = \langle t_{ij}(R)\rangle$, then we will say that it satisfies the condition of the left $R-$stability. It is clear that if g satisfies one of the above mentioned conditions of stability, then $\Lambda_I(g)$ satisfies the respective condition of stability for any ideal I of the ring R.

We say that elements from $E(n, R)gE(n, R)$ with $g \in \mathrm{GL}(n, R)$, $E(n, I)gE(n, I)$ with $g \in C(n, I)$, $[g, E(n, R)]^{E(n,R)}$ with $g \in N$ are obtained from g up to transvections. Any element obtained from g up to transvections will be denoted by g_e.

It is clear that if g_e satisfies the condition of left R-stability, then by Lemmas 5 and 6, the following inclusions hold:

$$E(n, I)^g \subset E(n, I) \text{ whenever } g \in \mathrm{GL}(n, R), \quad [g, E(n, R)] \subset E(n, I) \text{ whenever } g \in C(n, I),$$

$$g \text{ is the central scalar matrix of the group } \mathrm{GL}(n, R) \text{ whenever } g \in N.$$

Analogous inclusions take place, if g is the product of the corresponding elements that up to transvections, satisfy the conditions of left R-stability.

If all element of the group $\mathrm{GL}(n, R)$, $n > 2$, up to transvections, satisfy the conditions of left R-stability, then R is a commutatorial and partially-normal ring. In this case the factor-rings of R is a partially-normal rings. By Lemma 1, R is a stable ring.

Thus, the following holds.

Theorem 1. *Let R be an associative ring with 1 , $n > 2$. The left, up to transvections, R-stability of the elements of the group $\mathrm{GL}(n, R)$ implies in the stability of the ring R.*

It is clear that the condition of the left R-stability in Theorem 1 can be replaced by the condition of the right $R-$stability.

If $g \in C(n, J(R))$, then we can choose $x_{jj} = 1$, $x_{ij} = -G_{jj}g_{ji}g_{ii}^{-1}$. Then $x_{jj}G_{jj}g_{ji}+x_{ij}g_{ii} = 0$. Therefore, g satisfies the condition of the left (R, i, j)-stability. From Lemmas 5 and 6 and the definition of left R-stability, the inclusions follow:

1) $[C(n, J(R)), E(n, I)] \subset E(n, I)$;
2) $[C(n, I \cap J(R)), E(n, R)] \subset E(n, I)$;
3) $N \cap C(n, J(R))$ is the group of central scalar matrices.

Lemma 7. *Let R be an associative ring with 1, $n > 2$. The subgroup G of the group $C(n, J(R))$ invariant with respect to $E(n, R)$, uniquely determines the ideal $I_0 \subset J(R)$ such that $E(n, I_0) \subset G \subset C(n, I_0)$.*

Proof. Let G be the subgroup of the group $C(n, J(R))$ invariant with respect to $E(n, R)$ and I_0 is the maximal ideal of R for which $E(n, I_0) \subset G$. It is obvious that $I_0 \subset J(R)$. From 2) it can been seen that $[C(n, I_0), E(n, R)] \subset E(n, I_0) \subset G$. Therefore, as in Lemma 1, it is proved that the group $\Lambda_{I_0}(G)$ does not contain transvections.

Since $J(R)/I_0 \subset J(R/I_0)$, we see $\Lambda_{I_0}(G) \subset C(n, J(R/I_0))$. From 3) it can been seen that $\Lambda_{I_0}(G)$ is a group of central scalar matrices. It means that $G \subset C(n, I_0)$. □

Lemma 8. *Let R be an associative ring with 1, $R/J(R)$ a partially-normal ring $n > 2$. Then R is a partially-normal ring.*

Proof. If $\Lambda_{J(R)}N$ contains transvection $\Lambda_{J(R)}t_{ij}(r)$, $r \notin J(R)$, then N contains h which belongs to $t_{ij}(r) \operatorname{Ker} \Lambda_{J(R)}$. Since $h_{11} \in R^*$, by Lemma 3, h is a central scalar matrix. In this case $r \in J(R)$. The obtained contradiction shows that $\Lambda_{J(R)}N$ does not contain transvections. Therefore $N \subset C(n, J(R))$. By Lemma 3 N is a group of central scalar matrices. Hence, R is a partially-normal ring. □

If in Lemma 8 the ring $R/J(R)$ is normal, then all factor-rings $\overline{R}$ of the ring R are partially-normal. Thus, the factor-rings $\overline{R}/J(\overline{R})$ of the normal ring $R/J(R)$ are partially-normal. By Lemma 8, the rings $\overline{R}$ are partially-normal.

Lemma 9. *Let R be an associative ring with 1, $R/J(R)$ is a commutatorial ring, $n > 2$. Then R is a weak commutatorial ring.*

Proof. By the condition

$$[C(n, I), E(n, R)] \subset E(n, I) \operatorname{Ker} \Lambda_{I \cap J(R)} \subset E(n, I)D_I,$$

where D_I is a group of diagonal matrices and $D_I \subset \operatorname{Ker} \Lambda_I$. Then

$$[C(n, I), E(n, R), E(n, R)] \subset [E(n, I)D_I, E(n, R)] \subset E(n, I).$$

It means that R is a weak commutatorial ring. In fact we can prove that R is a commutatorial ring. □

From Lemmas 8 and 9, and also Lemma 1 it follows

Theorem 2. *Let R be an associative ring with 1, $R/J(R)$ a stable ring, $n > 2$. Then R is a stable ring.*

It is obvious that if $G_{jk}g_{ki} = 0$, where $1 \le k \le n$, then g satisfies the condition of the left (R, i, j)-stability.

Theorem 3. *Let R be an associative ring with 1, $g \in \mathrm{GL}(n, R)$, $n > 2$ and at least one element of g is zero. Then up to transvections g satisfies the condition of left R-stability.*

Proof. a) Let $g_{ij} = G_{ji} = 0$. Then $g_{ij}G_{js} = 0 = G_{si}g_{ij}$ and $g_{sj}G_{ji} = 0 = G_{ji}g_{is}$, where $1 \le s \le n$. In this case the element g satisfies the conditions of the left (R, j, s) and (R, s, j)-stability, and g^{-1} satisfies the conditions of the left (R, s, i) and (R, i, s)-stability. Because

$$E(n, R) = \langle t_{is}(R),\ t_{si}(R) |\ 1 \le s \ne i \le n \rangle = \langle t_{sj}(R),\ t_{js}(R) |\ 1 \le s \ne j \le n \rangle,$$

g satisfies the conditions of the left R-stability.

b) Let

$$g_{ij} = 0, \quad g_1 = \prod_l t_{il}(G_{jl})g.$$

Then $(g_1)_{ij} = 1$. Let

$$g_2 = \prod_l t_{li}(-g_{lj}))g_1 \prod_l t_{jl}((G_{ji} - 1)g_{il}).$$

Then $(g_2)_{ik} = \delta_{jk}$, $(g_2)_{kj} = \delta_{ik}$, $(g_2^{-1})_{jk} = \delta_{ik}$, $(g_2^{-1})_{ki} = \delta_{jk}$, where $1 \le k \le n$.

For $g \in C(n, I)$ the inequality $i \ne j$ holds, the element $g_3 = t_{ji}(g_{ii})t_{ij}(-G_{jj})t_{ji}(g_{jj})g_2$ belongs to $C(n, I)$, and a number exists $1 \le t \ne i,\ j \le n$, such that $(g_3)_{tj} = 0 = (g_3^{-1})_{jt}$. By a), the element g, up to transvections, satisfies the conditions of the left R-stability.

For $g \in N$, from Lemma 2 it follows that $i \ne j$ and g is a central scalar matrix. □

From Theorem 3 and Lemma 5 it follows that if g is the product of elements of the groups $C(n, I)$ which have at least one zero element, then $[g, E_{cR}] \subset E(n, cI)$ for all elements c from the centre of R.

Lemma 10. *Let R be an associative ring with 1, $n > 2$. The group ΛN does not contain transvections in the group $\mathrm{GL}(n, R_s)$.*

Proof. It is clear that the group ΛN is invariant with respect to $\Lambda E(n, R)$. If ΛN contains a transvection τ, then a transvection $t \in E_S$ exists such that $\Lambda t_{ij}(r) = [\tau, \Lambda(t)] \in \Lambda N$ for some $r \in R$ and $rS \ne 0$. Therefore, there exist $h \in \mathrm{GL}(n, R)$ and $s \in S$ such that $t_{ij}(r)h \in N$, $\Lambda h = 1$, $hs = s$. Since s annihilates the nondiagonal element of the matrix $t_{ij}(r)h$, by Lemma 2, $t_{ij}(r)hs$ is a scalar matrix. Hence $rs = 0$. The obtained contradiction shows that ΛN does not contain transvections in the group $\mathrm{GL}(n, R_s)$. □

Let R_s be a partially-normal ring. Then ΛN is a group of central scalar matrices and for any element $g \in N$ there exists an element $s \in S$ such that sg is a scalar matrix.

Lemma 11. *Let R be a weak commutatorial ring and R_S be normal rings for all maximal ideals J of a subring K of the centre R, $1 \in K$, $S = K - J$, $n > 2$. Then R is a normal ring.*

Proof. Let G be a subgroup of $\mathrm{GL}(n, R)$, which is invariant with respect to $E(n, R)$ and I_0 the maximal ideal of R such that $E(n, I_0) \subset G$. From the proof of Lemma 1 it follows that $\Lambda_{I_0}(G)$ does not contain transvections.

Let $g \in G$ and $J' = \{s \in K \mid \Lambda_{I_0}(sg)$ are scalar matrices $\}$. It is obvious that J' is an ideal of K and $I_0 \cap K \subset J'$. If $J' \neq K$, then there exists a maximal ideal J of the ring K which contains J'. Let $S = K - J$ and $\overline{R} = \Lambda_{I_0}(R)$. It is easy to see that $\overline{S} = \overline{K} - \overline{J}$ is not an empty multiplicatively closed subset of the centre $\overline{R}$ which does not contain the zero element. By Lemma 10 the group $\Lambda\Lambda_{I_0}(G)$ does not contain transvections in the group $\mathrm{GL}(n, \overline{R}_{\overline{S}})$. The ring $\overline{R}_{\overline{S}}$ is a factor-ring of a normal ring R_S. That is why $\overline{R}_{\overline{S}}$ is a partially-normal ring and, by Lemma 10, there exists an element of the set S which belongs to J'. The obtained contradiction shows that $J' = K$, $1 \in J'$, $\Lambda_{I_0}(G)$ are scalar matrices. Since the group G is invariant with respect to $E(n, R)$, we see that $\Lambda_{I_0}(G)$ are central scalar matrices. It is proved that $G \subset C(n, I_0)$. Consequently, R is a normal ring. □

Lemma 12. *Let R be an associative ring with 1, $g \in \mathrm{GL}(n, R)$, $n > 2$ and $\Lambda(g) \in E(n, I_S)$. Then there exists $s \in S$ such that $[g, t_{ij}(sR)] \subset E(n, I)$ for any pair $1 \leq i \neq j \leq n$.*

Proof. Denote $R' = R[x, y]$ and by I' an ideal of R' which is generated by the ideal I of the ring R, where x commutes with the elements of the centre of the ring R and y belongs to the centre of R'. By Theorem 3, there exists $s_0 \in S$, such that $[g, t_{ij}(s_0 xy)] \subset E(n, yI') \operatorname{Ker} \Lambda$ for any pair $1 \leq i \neq j \leq n$. That is why there exists a polynomial $f(y)$ $(f(0) = 0)$ over the matrix ring $M(n, R[x])$ such that $s_1 f(y) = 0$ for some $s_1 \in S$. Let $s = s_0 s_1$. It is clear that s annihilates the coefficient of the polynomial $f(y)$. Thus, $f(s) = 0$ and, consequently, $[g, t_{ij}(sx)] \subset E(n, I')$. We have proved that $[g, t_{ij}(sR)] \subset E(n, I)$. □

Lemma 13. *Let R_S be commutatorial rings for all maximal ideals J of a subring K of the centre R, $1 \in K$, $S = K - J$, $n > 2$. Then R is a weak commutatorial ring.*

Proof. Consider any elements $g \in C(n, I)$ and $e \in E(n, R)$. Let

$$J' = \{s \in K \mid [g, e, t_{ij}(sR)] \subset E(n, I) \text{ for any pair } 1 \leq i \neq j \leq n\}.$$

It is clear that J' is an ideal of K. If $J' \neq K$, then there exists a maximal ideal J of the ring K which contains J'. Let $S = K - J$. Since I_S is an ideal of commutatorial ring R_S, we see that $\Lambda([g, e]) \subset E(n, I_S)$. According to Lemma 12 there exists an element of the set S which belongs to J'. We have a contradiction which shows that $J' = K$, $1 \in J'$, R is a weak commutatorial ring. □

From Lemmas 11, 13 and 1 it follows

Theorem 4. *Let R_S be stable rings for all maximal ideals J of a subring K of centre R, $1 \in K$, $S = K - J$, $n > 2$. Then R is a stable ring.*

It is known [16] that not all associative rings with 1 are stable. For example, an algebra over the some field with $2n^2$ generators x_{ij}, y_{ij} , $1 \leq i, j \leq n$ and determining relations which are expressed in a matrix form $(x_{ij})(y_{ij}) = 1 = (y_{ij})(x_{ij})$ is not a stable ring. However, the class of stable rings is large enough.

In the next part of the paper we examine the most important results about stable rings. Stability of fields and skew fields is a consequence of the Jordan-Dickson and Dieudonne theorems, stability of local rings follows from Klingenberg results.

1. H. Bass [1]. Let n be a natural number, R^n a free n-dimensional R module. A vector $(r_1, ..., r_n)$ is called *unimodular* in R^n if there exist elements $t_1, ..., t_n$ from R such that $t_1 r_1 + ... + t_n r_n = 1$.

Let $n \geq 2$. We will say that the *ring R satisfies the stable rank n − 1 condition*, if for any unimodular vector $(r_1, ..., r_n) \in R^n$ there exist $s_2, ..., s_n$ is R such that the vector $(r_2 + s_2 r_1, .., r_n + s_n r_1)$ is unimodular in R^{n-1}.

It is known that if R satisfies the stable rank n condition, then the ring R (see [4]) and its factors (see [2]) satisfy the rank m condition, $m \geq n$. Note that the semilocal rings satisfy the stable rank 1 condition (see [2]).

It turns out that the associative rings with 1 which satisfy the stable rank > 1 stability condition are stable.

Indeed, for any $g \in \mathrm{GL}(n, R)$ and $n > 2$ in R there exist elements $k_2, ..., k_n$ such that the vector $(g_{2n} + k_2 g_{1n}, ..., g_{nn} + k_n g_{1n})$ is unimodular. Then in $g_{1n}R$ there exist elements $s_2, ..., s_n$, such that $g_{1n} + s_2(g_{2n} + k_2 g_{1n}) + ... + s_n(g_{nn} + k_n g_{1n}) = 0$. Let $e_1 = t_{21}(k_2) \cdots t_{n1}(k_n)$, $e_2 = t_{12}(s_2) \cdots t_{1n}(s_n)$, $g_1 = e_2 g^{e_1}$, $g_2 = [g^{e_1}, t_{n2}(1)]^{e_2}$. Then $(g_1)_{1n} = 0 = (g_2)_{1n}$, $g_1 \in C(n, I)$ if $g \in C(n, I)$ and $g_2 \in N$ if $g \in N$. From Theorem 3 it follows that the element g up to transvection, satisfies the condition of the left R-stability. By Theorem 1 R is a stable ring.

2. L. N. Vaserstein [9]. Let rings R_S satisfy the stable rank > 2 condition, for all maximal ideals J of the subring K of centre R, $1 \in K$, $S = K - J$. According to 1, rings R_S are stable. By Theorem 4 R is a stable ring.

In a partial case, if R is a finitely generated ring as a module over a subring K with 1 from the centre of the ring R, then R_S is a finitely generated ring as a module over K_S. According to Nakayama lemma $J(K_S) \subset J(R_S)$. Since $R_S/J(R_S)$ is a finitely generated ring as a module over a field $K_S/J(K_S) \cong K/J$, we see that R_S is a semilocal ring, which satisfies stable rank 1 condition. Then R is a stable ring. It is proved that any associative ring with 1 which is finitely generated as a module over subrings of their centers, is stable (see [9] or [6]).

3. A. A. Suslin [6], J. S. Wilson [7], I. Z. Golubchik [8]. Let R be a commutative ring with 1, $g \in \mathrm{GL}(n, R)$, $n > 2$. Choose $V(k)$ such that $x_{kk} = g_{si}$, $x_{sk} = -G_{jk} g_{ki}$, where $1 \leq k \neq s \leq n$, and the rest $x_{tk} = 0$. Changing s and k it is easy to verify that g satisfies the condition of the left (R, i, j)-stability. By Theorem 1 R is a stable ring.

Notice that A. A. Suslin has proved that the commutative ring with 1 is commutatorial. In $n > 3$ Wilson and in case $n > 2$ I. Z. Golubchik have proved that any commutative ring is normal. We notice also the proof of the normality of commutative ring proposed by Z. I. Borevich, N. O. Vavilov [11] and V. M. Petechuk [14].

4. L. N. Vaserstein [17], S. H. Khlebutin [12]. An associative ring R is called a it von Neumann regular ring, if for any $a \in R$ there exists an element a' such that $aa'a = a$. Let $e = aa'$. Then $ea = a$ and $e^2 = e$.

Let R be a ring with 1 regular in the sense of von Neumann, $g \in \mathrm{GL}(n, R)$, $n > 2$, $a = g_{jj}$, $g_1 = g t_{jk}(-a' g_{jk})$. Then $(g_1)_{jk} = g_{jk} - aa' g_{jk} = (1 - e) g_{jk}, (g_1)_{jj} = g_{jj} = e g_{jj}$ Since $e(1 - e) = (1 - e)e = 0$ and $1 = e + 1 - e$, we see that g_1^{-1} satisfies the condition of the left (R, i, j)-stability . For $g \in C(n, I)$, the element $g_1 \in C(n, I)$. Therefore R is a commutatorial ring.

If $g \in N$, then, according to Lemma 2, a is not a zero divisor, $e = 1$, $a \in R^*$. By Lemma 3 g is a central scalar matrix. This means that R is a partially-normal ring.

Since the factor-ring of von Neumann regular rings are von Neumann regular rings, they are partially-normal. By Lemma 1 it follows that R is a stable ring.

Thus, rings with 1, regular in the sense of von Neumann are stable.

5. S. H. Khlebutin [12]. Let R be an associative ring with 1, $g \in \mathrm{GL}(n, R)$, $n > 2$, $a = g_{jj}G_{jj}$ and there exists a natural number m such that $a^m = sa^{m+1}$, where s is an element of ring R which commutes with a. It is clear that $a^m = s^m a^{2m}$. Let

$$e = (as)^m, \quad b = (1 + a + ... + a^{m-1})g_{jj}, \quad g_1 = g \prod_i t_{ij}(-G_{ij}b), \quad g_2 = g_1 t_{jk}(xg_{jk}),$$

where $x = -G_{jj}s^{m+1}$. Then $e^2 = e$, $ea^m = a^m$, $(g_1)_j k = g_{jk}$, $(g_1)_{jj} = a^m g_{jj}$, $(g_2)_{jk} = (1-e)g_{jk}$, $(g_2)_{jj} = ea^m g_{jj}$, where $1 \le k \ne j \le n$. It means that the element g_2^{-1} satisfies the condition of the left (R, i, j)-stability. It is easy to see that $g_2 \in C(n, I)$ in $g \in C(n, I)$. For $g \in N$, the element a is not a zero divisor, $e = 1$, $a \in R^*$, g is a central scalar matrix.

Let R be a ring algebraic over the Artinian subring K of its centre. Then the module $K(a)$ is Artinian and there exists a natural number m such that $a^{m+1}K(a) = a^m K(a)$ and $a^m = sa^{m+1}$, where s is an element of R, which commutes with a. According to the above mentioned, the element g^{-1} up to transvection, satisfies the condition of the left (R, i, j)-stability. Thus R is a stable ring.

Therefore, the associative rings with 1 algebraic over Artinian subrings of their centers are stable.

We should mention that stability of associative rings with 1 algebraic over a subring of their centers, is derived from 5.

Let K be a subring with 1 of the centre of the ring R, J any maximal ideal of K, $1 \in K$, $S = K - J$, $n > 2$. If R is algebraic over K, then R_S is algebraic over K_S. In this case the module $K_S(a)$ is finally generated as a module over K_S for any $a \in R_S$. By Nakayama's lemma, $J(K_S) \subset J(K_S(a))$. Hence $J(K_S) \subset J(R_S)$. Thus the ring $R_S/J(R_S)$ is algebraic over the field $K_S/J(K_S) \cong K/J$. By Lemmas 5 and Theorem 2 R_S are stable rings. By Theorem 4, R is a stable ring.

6. I. Z. Golubchik [18]. Let R be an associative ring with 1. An ideal F of the ring R is called *weak Noetherian*, if for any elements $y, z \in F$, $m \ge 1$ left and right R modules $\sum Rzy^m$ and $\sum y^m zR$ are finally generated as modules over R.

A ring R is called a weak Noetherian (block algebraic), if there exists a series of ideals

$$0 = I_0 \subset I_1 \subset ... \subset I_{q+1} = R,$$

and the ideals I_{i+1}/I_i of the rings R/I_i are weak Noetherian (algebraic above their centers) for i from 0 to q.

Clearly, the block algebraic rings are weak Noetherian. We should mention that PI-rings are block algebraic (see [10], [21], [22]).

Let R be a weak Noetherian ring, $g \in \mathrm{GL}(n, R)$, $n > 2$ and l the maximal number, where $I_l g_{1n} = 0$. At $l < q + 1$ we select $g_1 \in [g, t_{n1}(I_{l+1})]$, $y = (g_1)_{11}$, $z = (g_1)_{1n}$. Then $y - 1$, z belong to $I_{l+1} \cap g_{1n}R$, $I_l(y-1) = 0$ and there exists a natural number m, where

$$z(y-1)^m - \sum_{p=1}^{m-1} s_p z(y-1)^p \in I_l, \qquad zy^{m+1} = \sum_{p=0}^{m} r_p zy^p, \qquad r_p, s_p \in R.$$

If $g \in N$, then $g_1 \in N$. By Lemma 2 y is not a divisor of zero. Since in R there exists an element r such that $ry + r_0 z = 0$, then, according to Lemma 3, $0 = r_0 z = .. = r_m z = z$, g_1 and, as a consequence, all the elements $[g, t_{n1}(I_{l+1})]$ are central scalar matrices. One can conclude from Lemma 3 that $I_{l+1}g_{1n}=0$. The received contradiction shows, that $l=q+1$, $g_{1n}=0$, and R is a partially-normal ring.

Since the factor-rings of weak Noetherian rings are weak Noetherian too, they are partially-normal.

Let F be an ideal R which belongs to some weak Noetherian ideal of the ring R, y and z are elements of F. Then there exists a natural number m such that $zy^{m+1}=\sum r_p zy^p$, $r_p\in R$ and $1\le p\le m$.

Let λ be any element of R ring centre. We multiply this equality by λ^{m+1} and use the equality $\lambda y=\lambda y-1+1$. Then there exists a polynomial $\psi(\lambda)$ such that $\psi(\lambda)z+a(1-\lambda y)=0$, where $\psi(0)=1$, $a\in F$

Let $C_\lambda=[t_{pq}(-\lambda x),C]$, $g_\lambda=t_{pq}(\lambda x)C_\lambda=t_{pq}(\lambda x)^C$, $g=g_1$, where $C\in C(n,F)$, $x\in R$. Then $g_\lambda=\lambda g-\lambda+1$, $g_\lambda^{-1}=g_{-\lambda}$, $C_\lambda\in C(n,F)$. If $y=1-g_{ii}$, $z=(g^{-1})_{jk}g_{ki}$, where i,j,k are different numbers, then $\psi(\lambda)(g_\lambda^{-1})_{jk}(g_\lambda)_{ki}+\lambda^2a(g_\lambda)_{ii}=0$. By Lemma 5 $[c_\lambda,t_{ij}(R\psi(\lambda)]\subset E(n,F)$. From analogous rightsided conclusions and matrix commutatorial formulas one can conclude, that for every pair $1\le i\ne j\le n$ there exists a polynomial $\psi_{ij}(\lambda)$ such that $[c_\lambda,t_{ij}(R\psi_{ij}(\lambda)R]\subset E(n,F)$, where $\psi_{ij}(0)=1$. Let $f(\lambda)=\prod\psi_{ij}(\lambda)$ for all pairs $1\le i\ne j\le n$, $I=Rf(\lambda)R$, $K(\lambda)=I^2f(1-\lambda)I^2$. According to proved above $[c_\lambda,E_I]\subset E(n,F)$ and $[c_{1-\lambda},E_{K(\lambda)}]\subset E(n,F)$. By Lemma 5 $E_{K(\lambda)}^{g_{1-\lambda}}\subset E(n,I^2)\subset E_I$. Thus, for $e\in E_{K(\lambda)}$ there following inclusions hold

$$e^g=(e^{g_{1-\lambda}})^{g_\lambda}\subset(e^{g_{1-\lambda}})^{t_{pq}(\lambda x)}E(n,F)\subset(e^{t_{pq}((1-\lambda)x)})^{t_{pq}(\lambda x)}E(n,F)\subset e^{t_{pq}(x)}E(n,F).$$

This means that $[t_{pq}(-x)g,E_{K(\lambda)}]\subset E(n,F)$. Let $I_1=\sum K(\lambda)$ for all λ from the centre of R. If $I_1\ne R$, then because of equality $f(0)=1$ the polynomial $f(\lambda)^2f(1-\lambda)f(\lambda)^2$ is non-zero and the elements of the centre of R are its roots in the ring R/I_1. Thus, if the centre of the ring R contains an infinite field, then $I_1=R$,

$$[t_{pq}(-x)g,E(n,R)]\subset E(n,F),\quad [C(n,F),E(n,R),E(n,R)]=E(n,F),$$

R is a weak commutatorial ring. By Lemma 1 R is a stable ring. Thus, the weak Noetherian rings which contain infinite fields in their centers are stable. The remark to 5 shows that block algebraic rings are stable (see [19]).

We should note that the statement on the stability of certain rings is more systematically stated in [2,15]. The remark in [15] on page 122 in Section 3 does not correspond to the reality.

The Chevalley groups are wide generalization of linear groups (see e.g. [14] and [20]). The stability of Chevalley groups over a commutative rings is proved in [21-24], and in [25] a generalization of these results for the groups of Lie type groups over PI-rings is received.

The stability of the group has notable applications (see, e.g.[26]).

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