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ON THE LOGARITHMIC DERIVATIVE OF AN ENTIRE FUNCTION

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A conjecture on the existence of an entire function f with a prescribed asymptotics of $f'(r)/f(r)$ as $r \to +\infty$ is formulated.

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Сформулирована гипотеза о существовании целой функции f с заданной асимптотикой для $f'(r)/f(r)$ при $r \to +\infty$.

A. Daniluk [1] showed that for any $0 \le a < b \le +\infty$ there exists an entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \ge 0\,(n \ge 0), \tag{1}$$

such that $\varliminf\limits_{r\to+\infty} f'(r)/f(r) = a$ and $\varlimsup\limits_{r\to+\infty} f'(r)/f(r) = b$. In view of this result the following problem arises: for which positive continuous function γ on $[a, +\infty)$, $a > 0$, there exists an entire function of form (1) such that $f'(r)/f(r) \sim \gamma(r)$, $r \to +\infty$?

The following conjecture is plausible.

Conjecture. *For every positive continuous function γ on $[a, +\infty)$ there exists an entire function of form (1) such that $f'(r)/f(r) \sim \gamma(r)$, $r \to +\infty$.*

I can prove the conjecture in some special case.

Proposition. *For every positive continuously differentiable function γ on $[a, +\infty)$ such that $r\gamma(r) \uparrow +\infty$ as $r \to +\infty$ and $r\gamma'(r)/\gamma(r) \to 0$ as $r \to +\infty$ there exists an entire function of form (1) such that $f'(r)/f(r) \sim \gamma(r)$, $r \to +\infty$.*

Proof. Let $\psi(r) = r\gamma(r)$ and let ω be the inverse function to $\psi(r)$. We suppose that for every $\varepsilon \in (0, 1)$

$$\lim_{x\to+\infty} \sup \frac{\omega(x)}{x\omega'(x)} \ln \frac{\omega(x)}{\omega((1-\varepsilon)x)} < \frac{\varepsilon}{1-\varepsilon}, \tag{2}$$

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and show that there exists an entire function of form (1) such that $f'(r)/f(r) \sim \gamma(r)$, $r \to +\infty$.

Indeed, let Ω be the class of positive on $(-\infty, A)$ functions Φ such that the derivative Φ' is continuous, positive and increasing to $+\infty$ on $(-\infty, +\infty)$. For $\Phi \in \Omega$ let φ be the inverse function to Φ'. Then φ is continuous on $(0, +\infty)$ and increasing to $+\infty$. J. Clunie [2] showed that for every function $\Phi \in \Omega$ there exists an entire function of form (1) such that $\ln f(r) \sim \Phi(\ln r)$, $r \to +\infty$.

Let F be a convex differentiable function on $(-\infty, +\infty)$ and $\Phi \in \Omega(+\infty)$. A. V. Bratishchev [3] showed that if $\alpha(\varepsilon) = \varlimsup\limits_{x\to+\infty} \dfrac{\Phi(x^*) + \Phi'(x^*)(x - x^*)}{\Phi(x)} < 1$, where $\Phi'(x^*) = (1-\varepsilon)\Phi'(x)$, then $\lim\limits_{x\to+\infty} \dfrac{F(x)}{\Phi(x)} = 1 \implies \lim\limits_{x\to+\infty} \dfrac{F'(x)}{\Phi'(x)} = 1$.

Now, we choose $\Phi \in \Omega$ such that $\Phi'(x) = \psi(e^x)$ for $x \ge \ln a$, that is $\Phi'(\ln r) = \psi(r)$ for $r \ge a$. For this function Φ by Clunie theorem we construct an entire function of form (1) such that $\ln f(r) \sim \Phi(\ln r)$, $r \to +\infty$.

In view of (2), we have

$$
\begin{aligned}
\alpha(\varepsilon) &= \varlimsup_{x\to+\infty} \frac{\Phi(\varphi((1-\varepsilon)\Phi'(x))) + (1-\varepsilon)\Phi'(x)(x - \varphi((1-\varepsilon)\Phi'(x))}{\Phi(x)} = \\
&= \varlimsup_{t\to+\infty} \frac{\Phi(\varphi((1-\varepsilon)t)) + (1-\varepsilon)t(\varphi(t) - \varphi((1-\varepsilon)t))}{\Phi(\varphi(t))} \le \\
&\le \varlimsup_{t\to+\infty} \Bigg(\frac{(1-\varepsilon)\Phi'(\varphi((1-\varepsilon)t))\varphi'((1-\varepsilon)t)}{\Phi'(\varphi(t))\varphi'(t)} + \\
&+ \frac{(1-\varepsilon)(\varphi(t) - \varphi((1-\varepsilon)t)) + (1-\varepsilon)t((\varphi'(t) - (1-\varepsilon)\varphi'((1-\varepsilon)t)))}{\Phi'(\varphi(t))\varphi'(t)} \Bigg) = \\
&= \varlimsup_{t\to+\infty} \frac{(1-\varepsilon)(\varphi(t) - \varphi((1-\varepsilon)t) + (1-\varepsilon)t\varphi'(t)}{t\varphi'(t)} = \\
&= (1-\varepsilon)\left(1 + \varlimsup_{t\to+\infty} \frac{(\varphi(t) - \varphi((1-\varepsilon)t)}{t\varphi'(t)}\right) = \\
&= (1-\varepsilon)\left(1 + \varlimsup_{t\to+\infty} \frac{\omega(t)}{t\omega'(t)} \ln \frac{\omega(t)}{\omega((1-\varepsilon)t)}\right) < (1-\varepsilon)\left(1 + \frac{\varepsilon}{1-\varepsilon}\right) = 1.
\end{aligned}
$$

Since $\ln f(e^x) \sim \Phi(x)$, $x \to +\infty$, by the Bratishchev theorem we have $\lim\limits_{x\to+\infty} \dfrac{e^x f'(e^x)}{f(e^x)\Phi'(x)} = 1$, that is $\dfrac{f'(r)}{f(r)} \sim \dfrac{\Phi'(\ln r)}{r} = \gamma(r)$, $r \to +\infty$.

Now, we show that if $r\gamma'(r)/\gamma(r) \to 0$, $r \to +\infty$, then (2) holds. Indeed, if $r\gamma'(r)/\gamma(r) \to 0$, $r \to +\infty$, then $r\psi'(r)/\psi(r) \to 1$, $r \to +\infty$, and $r\omega'(r)/\omega(r) \to 1$, $r \to +\infty$. We have also $\ln \omega(r) - \ln \omega((1-\varepsilon)r) = \dfrac{\omega'(\xi)}{\omega(\xi)}\varepsilon r \le \dfrac{\omega'(\xi)}{\omega(\xi)}\dfrac{\varepsilon\xi}{1-\varepsilon}$, $(1-\varepsilon)r \le \xi \le r$. Hence, (2) holds. Proposition is proved. □

Choosing properly a function γ we can obtain from Proposition the result of A. Daniluk. For example, it is sufficient to choose $\gamma(r) = (b + a + (b - a)\sin(\ln\ln r))/2$ in the case $0 < a < b < +\infty$.

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