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ON RUPPERT'S PREODERING

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For group topologies τ_1, τ_2 on a group G, $\tau_1 \leq_c \tau_2$ means that every Cauchy ultrafilter on (G, τ_2) is a Cauchy ultrafilter on (G, τ_1), and $\tau_1 \leq_b \tau_2$ means that for every neighborhood V of the identity in (G, τ_1) there exists a finite subset K such that KV is a neighborhood of the identity in (G, τ_2). It is proved that the preorders $\leq_c$ and $\leq_b$ coincide.

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Для групповых топологий τ_1, τ_2 на группе G $\tau_1 \leq_c \tau_2$ означает, что каждый ультрафильтр Коши на (G, τ_2) является ультрафильтром Коши на (G, τ_1), а $\tau_1 \leq_b \tau_2$ означает, что для каждой окрестности V единицы в (G, τ_1) существует конечное подмножество K такое, что KV является окрестностью единицы в (G, τ_2). Доказано, что предпорядки $\leq_c$ и $\leq_b$ совпадают.

In [1] Ruppert defined a preodering $\tau_1 \leq_c \tau_2$ on the set of all group topologies on a group G. Namely, $\tau_1 \leq_c \tau_2$ if and only if every Cauchy ultrafilter on (G, τ_2) is a Cauchy ultrafilter on (G, τ_1). By means of this preodering he established a Galois connection between some class of group topologies on an Abelian group G and the set of all idempotents of weak almost periodic compactification of discrete group G.

Another preodering $\tau_1 \leq_b \tau_2$ arose in [2]. Namely, $\tau_1 \leq_b \tau_2$ if and only if, for every neighbourhood V of the identity in (G, τ_1), there exists a finite subset K such that KV is a neighbourhood of the identity in (G, τ_2). By means of this preodering a method of calculation of ultrarank of topological group was developed.

This note is to show that these preoderings coincide. Since they came from mutually remote points of topological algebra, there is a hope that it might be useful in other investigations.

Theorem. *Let G be a group and let τ_1, τ_2 be a group topologies on G. Then $\tau_1 \leq_c \tau_2$ if and only if $\tau_1 \leq_b \tau_2$.*

Proof. We use the following simple observation. An ultrafilter φ on a topological group (G, τ) is a Cauchy ultrafilter if and only if, for every neighborhood U of the identity of (G, τ), there exist $a, b \in G$ such that $aU \in \varphi$ and $Ub \in \varphi$.

$\tau_1 \leq_c \tau_2 \Rightarrow \tau_1 \leq_b \tau_2$. Suppose the contrary and fix a neighborhood V of the identity in (G, τ_1) such that KV is not a neighborhood of the identity in (G, τ_2) for every finite subset

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K of G. Take a converging to the identity ultrafilter φ on (G,τ_2) such that $KV\notin\varphi$ for every finite subset K if G. Since φ is a Cauchy ultrafilter on (G,τ_2) and $\tau_1\leq_c\tau_2$, we see that φ is a Cauchy ultrafilter on (G,τ_1). Take $a\in G$ such that $aV\in\varphi$, a contradiction with the choice of V and φ.

$\tau_1\leq_b\tau_2\Rightarrow\tau_1\leq_c\tau_2$. Suppose the contrary and choose a Cauchy ultrafilter (G,τ_2) which is not a Cauchy ultrafilter on (G,τ_1). Fix a symmetric neighborhood V of the identity in (G,τ_1) such that either $aV\notin\varphi$ or $Va\notin\varphi$ for every element $a\in G$. Since $\tau_1\leq_b\tau_2$, there exists a finite subset K of G such that KV, VK^{-1} are neighborhoods of the identity in (G,τ_2). Since φ is a Cauchy ultrafilter on (G,τ_2), there exist $b_1,b_2\in G$ such that $b_1KV\in\varphi$, $VK^{-1}b_2\in\varphi$. Then $b_1g_1V\in\varphi$, $Vg_2^{-1}b_2\in\varphi$ for some elements $g_1,g_2\in K$, a contradiction. □

Remark. This proof shows that $\tau_1\leq_c\tau_2$ if and only if every converging ultrafilter on (G,τ_2) is a left Cauchy ultrafilter on (G,τ_1).

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