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PROBLEMS IN THE THEORY OF ENTIRE FUNCTIONS OF BOUNDED INDEX AND FUNCTIONS OF SINE TYPE

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A connection between the functions of sine type and entire functions of bounded index is established. New problems in the theory of entire functions of bounded index and functions of sine type are formulated.

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Установлена связь между функциями типа синуса и целыми функциями ограниченного индекса. Сформулированы новые задачи теории целых функций ограниченного индекса и функций типа синуса.

1. Introduction. An entire function f of exponential type $\sigma > 0$ is said to be of sine type [1–3] if there exist numbers $m > 0$, $M > 0$ and $H > 0$ such that

$$m \le |f(x+iy)| \exp\{-\sigma|y|\} \le M$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus [-H, H]$. For simplicity we suppose that $f(0) = 0$. Then [2–3] $f(z) = \lim\limits_{n\to\infty} \prod\limits_{k=-n}^{n} \left(1 - \frac{z}{a_k}\right)$. For $\varrho \in (0, +\infty)$ we denote $G_\varrho(f) = \bigcup_{k\ge1}\{z : |z-a_k| \le \varrho\}$. The following properties of functions of sine type are established in [2–3]:

a) for every $\varrho \in (0, +\infty)$ there exist numbers $m(\varrho) > 0$ and $M(\varrho) > 0$ such that $|f(z)| \ge m(\varrho) > 0$ and $|f'(z)/f(z)| \le M(\varrho)$ for all $z \in \mathbb{C} \setminus G_\varrho(f)$;

b) uniformly by $x \in \mathbb{R}$ the following limits exist

$$\lim_{y\to+\infty} \frac{f'(x+iy)}{f(x+iy)} = -i\sigma, \quad \lim_{y\to-\infty} \frac{f'(x+iy)}{f(x+iy)} = i\sigma.$$

By B_σ we denote a class of entire function of exponential type $\sigma > 0$, that are bounded on the real axis.

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Theorem 1. [3] *Let f be a function of sine type and $\psi = (\psi_k)$ be a bounded sequence of complex numbers. The function $f_\psi(z) = \lim\limits_{n\to\infty} \prod\limits_{k=-n}^{n} \left(1 - \frac{z}{a_k+\psi_k}\right)$ is of sine type if and only if there exists an entire function φ from B_σ such that $\varphi(a_k) = \psi_k f'(a_k)$ if a_k is a simple zero and $\varphi(a_k) = \dots = \varphi^{(q-2)}(a_k) = 0$, $\varphi^{(q-1)}(a_k) = \psi_k f^{(q)}(a_k)/q$ if a_k is a zero of the multiciplity q.*

An entire function f is said to be of bounded index [4; 5, p.5] if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$

$$\frac{|f^{(n)}(z)|}{n!} \le \max\left\{\frac{|f^{(k)}(z)|}{k!} : 0 \le k \le N\right\}.$$

The least such integer N is called the index of f and is denoted by $N(f)$.

An entire function f is said to be of bounded value distribution [6; 5, p.49] if for every $r > 0$ there exists $n^*(r) \in \mathbb{N}$ such that $n\left(r, z_0, \frac{1}{f-w}\right) \le n^*(r)$ for each $z_0 \in \mathbb{C}$ and all $w \in \mathbb{C}$, where $n\left(r, z_0, \frac{1}{f-w}\right)$ is the number of w-points of f in $\{z : |z - z_0| \le r\}$.

The following theorem is announced in [7].

Theorem 2. *Each function of sine type, its derivative and primitive are functions of bounded index, of bounded value distribution and solutions of a differential equation of the form*

$$f^{(n+1)} + g_n f^{(n)} + \dots + g_1 f' + g_0 f = 0, \tag{1}$$

where g_j are entire functions of exponential type.

Proof. G. H. Fricke [8; 5, p.128] proved that an entire function f of exponential type is of bounded index if and only if $|f'(z)/f(z)| \le M(\varrho) < +\infty$ for each $\varrho > 0$ and for all $z \in \mathbb{C} \setminus G_\varrho(f)$. Therefore, by property a) each function of sine type is of bounded index.

Directly from the definition of entire function of bounded index it follows that the primitive of a function of bounded index is also of bounded index. Therefore, the primitive of a function of sine type is of bounded index.

Using property b), it is easy to show that the derivative of a function of sine type is also of sine type. Therefore, the derivative of a function of sine type is of bounded index.

W. K. Hayman [9; 5, p.49] showed that f is of bounded value distribution if and only if f' is of bounded index. Therefore, each function of sine type as well as its derivative and primitive are functions of bounded value distribution.

Finnaly, W. Hennekemper [10; 5, p.132] showed that each function of bounded index is a solution of a differential equation of form (1). □

It is clear (see, for example, [11]) that not every function of bounded index is of sine type, because a function of bounded index may have zeros in each half-plane $\{z : \operatorname{Im} z > H\}$.

2. Conjectures. It seems that the following assertion is true.

Conjecture 1. *Each function f of exponential type $\sigma > 0$ and of sine type is of bounded index $N(f) \le \sigma$.*

By Theorem 2 each function of sine type is a solution of a differential equation of form (1) with exponential type coefficients. However, the class of functions of sine type is narrower than that of functions of bounded index and since $\sin z$ and $\cos z$ are solutions of a differential equation of form (1) with bounded coefficients, the following conjecture seems to be plausible.

Conjecture 2. *Each function of sine type is a solution of a differential equation of form (1), where g_j are entire functions from B_σ.*

Let p be an entire function, q be a function of bounded index and $f(z) = p(z)q(z)$. By multiplication theorem [8; 5, p. 34] f is of bounded index if and only if p is of bounded index.

Suppose that f is of bounded index. Then [9; 5, p. 69] $\ln M_f(r) = O(r)$, $r \to +\infty$, $M_f(r) = \max\{|f(z)| : |z| = r\}$. Therefore, by Hadamard theorem either

$$f(z) = Az^m\pi(z), \quad \pi(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{a_k}\right), \quad \text{if} \quad \int_1^{\infty}\frac{\ln M_f(r)}{r^2}dr < +\infty, \tag{2}$$

or

$$f(z) = Az^m e^{az}\pi(z), \quad \pi(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{a_k}\right)e^{z/a_k}, \quad \text{if} \quad \int_1^{\infty}\frac{\ln M_f(r)}{r^2}dr = +\infty, \tag{3}$$

where $A \in \mathbb{C}$, $m \in \mathbb{Z}_+$, $a \in \mathbb{C}$, $a_k \in \mathbb{C} \setminus \{0\}$ and $|a_k| \nearrow +\infty$.

For a sequence $\psi = (\psi_k)$, $k \geq 0$, we define

$$f_\psi(z) = A(z-\psi_0)^m\pi_\psi(z), \quad \pi_\psi(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{a_k+\psi_k}\right), \quad \text{if} \quad \int_1^{\infty}\frac{\ln M_f(r)}{r^2}dr < +\infty, \tag{4}$$

and

$$f_\psi(z) = A(z-\psi_0)^m e^{az}\pi_\psi(z), \quad \pi_\psi(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{a_k+\psi_k}\right)e^{z/(a_k+\psi_k)},$$
$$\text{if} \quad \int_1^{\infty}\frac{\ln M_f(r)}{r^2}dr = +\infty. \tag{5}$$

We assume that a sequence (ψ_k) is such that functions (4) and (5) are entire. A problem consists in finding a necessary and sufficient condition on (ψ_k) in order that f_ψ be of bounded index.

Since $q(z) = Az^m e^{az}$ is of bounded index for every $a \in \mathbb{C}$, by multiplication theorem π is also of bounded index.

Suppose that $\psi_k \equiv h \in \mathbb{C}$. Then the function $f_\psi(z+h)$ has the same zeros as f and $f_\psi(z+h) = Bz^m e^{bz}\pi(z)$. By multiplication theorem $f_\psi(z+h)$ is of bounded index and, therefore, f_ψ is also of bounded index.

On the other hand, *for every positive continuous nondecreasing to $+\infty$ function ξ on $0, +\infty)$ there exist an entire function f of bounded index and a sequence ψ such that $|\psi_k| \leq \xi(k)$ and f_ψ is of unbounded index.* Indeed, we consider the function $f(z) = \sin \pi z$ of bounded index with zeros $\pm k$. Clearly, we can choose a sequence $\psi_k \in \mathbb{N}$ such that $|\psi_k| \leq \xi(k)$ and an increasing sequence (k_j) of positive integers such that $k_j + \psi_{k_j} = \dots = k_{j+l-1} + \psi_{k_{j+l-1}} = k_{j+l}$, $j+l = j+l_j \to \infty$, i.e. f_ψ has zeros of unbounded multiplicity. Evidently, a function with zeros of unbounded multiplicity is of unbounded index.

Hence, it follows that the following conjecture [12] is plausible.

Conjecture 3. *If f is an entire function of bounded index and (ψ_k) is a bounded sequence then f_ψ is also of bounded index.*

Since f is of bounded value distribution if and only if f' is of bounded index, the following conjecture is also plausible.

Conjecture 4. *If f is an entire function of bounded value distribution and (ψ_k) is a bounded sequence then f_ψ is also of bounded value distribution.*

3. Some results. I can prove Conjecture 3 only in some special cases.

We need the following

Lemma 1. *Let $|a_{k+1}| - |a_k| \geq h > 0$ and $h/2 - |\psi_k| \geq l > 0$ for all $k \geq k_0$. Then for every $\varrho \in (0, h/2)$ and $q \in (0, l)$ there exists $P = P(\varrho, q) > 0$ such that for all $z \in \mathbb{C} \setminus (G_\varrho(\pi) \bigcup G_q(\pi_\psi))$, $|z| \geq R = R(\varrho, q)$,*

$$\left| \frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)} \right| \leq P, \tag{6}$$

where π is the canonical product from (2) or (3).

Proof. At first, let π be the canonical product from (2) and π_ψ be the canonical product from (4).

We denote

$$b_k = a_k + \psi_k, \quad c_k = \min\{|b_k| - q, \, |a_k| - \varrho\}, \quad d_k = \max\{|b_k| + q, \, |a_k| + \varrho\}.$$

Then

$$c_k \geq \min\{|a_k| - h/2 + l - q, \, |a_k| - \varrho\} \geq |a_k| - h/2$$

and

$$d_k \leq \max\{|a_k| + h/2 - l + q, \, |a_k| + \varrho\} \leq |a_k| + h/2$$

for $k \geq k_0$. Hence, $d_k \leq c_{k+1}$ for $k \geq k_0$.

Let

$$A_n = \{z : \, c_n \leq |z| \leq d_n, \, |z - b_n| \geq q, \, |z - a_n| \geq \varrho\}$$

and

$$B_n = \{z : \, d_n \leq |z| \leq c_{n+1}\}, \quad n \geq k_0 + 1.$$

For $z \in A_n$ we have

$$\left| \frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)} \right| = \left| \sum_{k=1}^{\infty} \frac{\psi_k}{(z - a_k)(z - b_k)} \right| \leq \sum_{k=1}^{k_0} \frac{|\psi_k|}{(|z| - |a_k|)(|z| - |b_k|)} +$$

$$+ \sum_{k=k_0+1}^{n-1} \frac{|\psi_k|}{(|z| - |a_k|)(|z| - |b_k|)} + \frac{|\psi_n|}{|z - a_n||z - b_n|} + \sum_{k=n+1}^{\infty} \frac{|\psi_k|}{(|a_k| - |z|)(|b_k| - |z|)} \leq$$

$$\leq \sum_{k=1}^{k_0} \frac{|\psi_k|}{(|a_{n-1}|-|a_k|)(|b_{n-1}|-|b_k|)} + \sum_{k=k_0+1}^{n-1} \frac{|\psi_k|}{(|c_n|-|a_k|)(|c_n|-|b_k|)} +$$

$$+ \frac{|\psi_n|}{q\varrho} + \sum_{k=n+1}^{\infty} \frac{|\psi_k|}{(|a_k|-|d_n|)(|b_k|-|d_n|)} \leq$$

$$\leq K_1 + \sum_{k=k_0+1}^{n-1} \frac{|\psi_k|}{(|a_n|-|a_k|-h/2)(|a_n|-h/2-|a_k|-h/2+l)} +$$

$$+ \frac{h-2l}{2q\varrho} + \sum_{k=n+1}^{\infty} \frac{|\psi_k|}{(|a_k|-|a_n|-h/2)(|a_k|-h/2+l-|a_n|-h/2)} \leq$$

$$\leq K_1 + \sum_{k=k_0+1}^{n-1} \frac{|\psi_k|}{((n-k)h-h/2)((n-k-1)h+l)} +$$

$$+ \frac{h-2l}{2q\varrho} + \sum_{k=n+1}^{\infty} \frac{|\psi_k|}{((k-n)h-h/2)((k-n-1)h+l)} \leq$$

$$\leq K_1 + \sum_{k=1}^{\infty} \frac{h-2l}{h(2k-1)(k-1)h+l)} + \frac{h-2l}{2q\varrho} + \sum_{k=1}^{\infty} \frac{h-2l}{h(2k-1)(k-1)(h+l)} \leq$$

$$\leq K_2 + \frac{h-2l}{2q\varrho}, \quad K_2 = K_2(h,\, l). \tag{7}$$

If $z \in B_n$ then by analogy

$$\left|\frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)}\right| \leq \sum_{k=1}^{k_0} \frac{|\psi_k|}{(|z|-|a_k|)(|z|-|b_k|)} +$$

$$+ \sum_{k=k_0+1}^{n} \frac{|\psi_k|}{(|z|-|a_k|)(|z|-|b_k|)} + \sum_{k=n+1}^{\infty} \frac{|\psi_k|}{(|a_k|-|z|)(|b_k|-|z|)} \leq$$

$$\leq \sum_{k=1}^{k_0} \frac{|\psi_k|}{(|a_n|-|a_k|)(|b_n|-|b_k|)} + \sum_{k=k_0+1}^{n} \frac{|\psi_k|}{(|d_n|-|a_k|)(|d_n|-|b_k|)} +$$

$$+ \sum_{k=n+1}^{\infty} \frac{|\psi_k|}{(|a_k|-|c_{n+1}|)(|b_k|-|c_{n+1}|)} \leq K_3 + \frac{h-2l}{2q\varrho} \tag{8}$$

From (7) and (8) it follows that for all $z \in \bigcup_{n>k_0} (A_n \cup B_n)$ the inequality

$$\left|\frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)}\right| \leq P = \frac{h-2l}{2q\varrho} + K_5(h,\, l). \tag{9}$$

In the case when π is the canonical product from (2) Lemma 1 is proved.

Now, let π be the canonical product from (3) and π_ψ be the canonical product from (5).

Then

$$\left|\frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)}\right| \le \left|\sum_{k=1}^{\infty}\left(\frac{1}{b_k} - \frac{1}{a_k} + \frac{1}{z-b_k} - \frac{1}{z-a_k}\right)\right| \le$$
$$\le \sum_{k=1}^{\infty}\frac{|\psi_k|}{|a_k||b_k|} + \left|\sum_{k=1}^{\infty}\frac{\psi_k}{(z-b_k)(z-b_k)}\right|. \qquad (10)$$

Since $|a_{k+1}| - |a_k| \ge h > 0$ and $h/2 - |\psi_k| \ge l > 0$ for all $k \ge k_0$, we have $\sum\limits_{k=1}^{\infty}\frac{\psi_k}{|a_k||b_k|} = K_6(h,l) < +\infty$. The second summand from the right-hand side of (10) has estimate (9). Therefore, we obtain (6) with $P = \dfrac{h-2l}{2q\varrho} + K_5(h,\, l) + K_5(h,\, l)$. □

Theorem 3. *Suppose that the zeros a_k of entire function f of bounded index satisfy the condition $|a_{k+1}| - |a_k| \ge h > 0$, $k \ge k_0$, and $h/2 - |\psi_k| \ge l > 0$, $k \ge k_0$. Then f_ψ is of bounded index.*

Proof. Since $q(z) = Az^m e^{az}$ is of bounded index for every $a \in \mathbb{C}$, from (2) and (3) by the multiplication theorem it follows that π is also of bounded index.

Let $q \in (0,\, l)$ and $\varrho = q/2$. Then by Lemma 1 and Fricke's criterion

$$\left|\frac{\pi'_\psi(z)}{\pi_\psi(z)}\right| \le \left|\frac{\pi'(z)}{\pi(z)}\right| + P(q/2,\, q) \le M(q/4) + P(q/2,\, q) = M^*(q) \qquad (11)$$

for all $z \in \mathbb{C} \setminus (\bigcup\limits_{k\ge 1} C_k)$, $|z| \ge R = R^*(q)$, where $C_k = C'_k \cup C''_k$ and

$$C'_k = \{z : |z - b_k| \le q\}, \quad C''_k = \{z : |z - a_k| \le q/2\}, \quad b_k = a_k + \psi_k.$$

Clearly, $C_k \cap C_{k+1} = \varnothing$ and there exists a sequence R_k such that

$$C_k \subset D_k = \{z : R_{k-1} < |z| < R_k\}, \quad C_j \not\subset \{z : R_{k-1} < |z| \le R_k\}, (j \ne k,\, k \ge k_0).$$

If $C''_k \subset C'_k$ then (11) holds for $D_k \setminus C'_k$. If $C''_k \cap C'_k = \varnothing$ then (11) holds on $\partial C''_k$ and by the modulus principle (11) holds in C''_k i.e. in $D_k \setminus C'_k$. Finally, if $C''_k \cap C'_k \ne \varnothing$ and $C''_k \not\subset C'_k$ then $C''_k \cap \{z : |z - b_k| < q/2\} = \varnothing$ and in $D_k \setminus C'_k$ (11) holds with $M^*(q/2)$ instead $M^*(q)$. Hence,

$$|\pi'_\psi(z)/\pi_\psi(z)| \le M^{**}(q) \qquad (12)$$

for all $z \in \mathbb{C} \setminus G_q(\pi_\psi)$, $|z| \ge R^{**}(q)$, $0 < q < l$. It is clear, (12) holds for $z \in \mathbb{C} \setminus G_q(\pi_\psi)$, $|z| \le R^{**}(q)$ with another constant $M^{**}(q)$. By Fricke's criterion π_ψ is of bounded index. Hence, by the multiplication theorem f_ψ is also of bounded index. □

Using Lemma 1, we can prove also the following

Theorem 4. *Let f be an entire function of bounded index with zeros lying on a finite system of rays and $\psi_k = O(1)$, $k \to \infty$. Then f_ψ is also of bounded index.*

Proof. By the multiplication theorem it is enough to prove that if π is of bounded index then so is π_ψ.

So, let π be a canonical product from (2) or (3) of bounded index with zeros lying on a finite system of rays and $|\psi_k| \le H < +\infty$ for all $k \ge 1$. We shall prove that for every $\varrho \in (0, 2H)$ and $q \in (0, H)$ there exists $P = P(\varrho, q) > 0$ such that (6) holds for all $z \in \mathbb{C} \setminus (G_\varrho(\pi) \cup G_q(\pi_\psi))$.

At first, we suppose that all zeros of π are positive. Since π is of bounded index, for each $s > 0$ there exists [6; 3, p.27] $n^*(s) \in \mathbb{Z}_+$ such that $n(s, z_0, 1/\pi) \le n^*(s)$ for every $z_0 \in \mathbb{C}$. Therefore, each interval $I_n = (4nH, 4(n+1)H]$ contains at most $n^* = n^*(4H)$ zeros of π.

From each interval I_{2m} we choose one from zeros (if there exists) of π and construct a canonical product π_1^* by such zeros, then we choose second zero (if there exists) of π and construct a canonical product π_2^* by such zeros etc. So we construct $n_1 \le n^*$ canonical products π_j^* with zeros $a_k^{(j)}$ satisfying the condition $a_{k+1}^{(j)} - a_k^{(j)} \ge 4H$. Choosing by analogy zeros of π from each interval I_{2m+1}, we construct $n_2 \le n^*$ canonical products π_j^{**} with zeros satisfying the same condition.

Hence, $\pi(z) = \prod_{j=1}^{n_1+n_2} \pi_j$, where π_j are canonical products with zeros $a_k^{(j)}$ satisfying the condition $a_{k+1}^{(j)} - a_k^{(j)} \ge 4H$. Clearly,

$$\left| \frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)} \right| \le \sum_{j=1}^{n_1+n_2} \left| \frac{\pi'_{j,\psi}(z)}{\pi_{j,\psi}(z)} - \frac{\pi'_j(z)}{\pi_j(z)} \right|.$$

Since $h = 4H$, we have $h/2 - |\psi_k| \ge l = H$ and, applying Lemma 1 to π_j, we obtain

$$\left| \frac{\pi'_{j,\psi}(z)}{\pi_{j,\psi}(z)} - \frac{\pi'_j(z)}{\pi_j(z)} \right| \le P_j(\varrho, q).$$

Therefore,

$$\left| \frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)} \right| \le \sum_{j=1}^{n_1+n_2} P_j(\varrho, q) = P(\varrho, q) < +\infty,$$

i. e. in this case (6) holds.

Now, we suppose that all zeros of π lie on m rays and construct m canonical products π_j^* with zeros lying only on one ray. Then $\pi(z) = \prod_{j=1}^{m} \pi_j^*$ and

$$\left| \frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)} \right| \le \sum_{j=1}^{m} \left| \frac{\pi^{*\prime}_{j,\psi}(z)}{\pi^*_{j,\psi}(z)} - \frac{\pi^{*\prime}_j(z)}{\pi^*_j(z)} \right|.$$

Since for each π_j^* inequality (6) holds, we obtain that for every $\varrho \in (0, 2H)$ and $q \in (0, H)$ there exists $P = P(\varrho, q) > 0$ such that (6) holds for all $z \in \mathbb{C} \setminus (G_\varrho(\pi) \cup G_q(\pi_\psi))$.

Now, $q \in (0, H)$ and $\varrho = q/2$. Then by (6) and Fricke's criterion, as in the proof of Theorem 3, we have (11) for all $z \in \mathbb{C} \setminus \left(\left(\bigcup_{k\ge1} C'_k \right) \cup \left(\bigcup_{k\ge1} C'_k \right) \right)$, where $C'_k = \{z : |z - b_k| \le q\}$, $C''_k = \{z : |z - a_k| \le q/2\}$, $b_k = a_k + \psi_k$.

If $C''_j \subset \bigcup_{k\ge1} C'_k$ then the fulfillment of (11) on $\bigcup_{k\ge1} C'_k$ implies the fulfillment of (11) on C''_j. If $C''_j \cap \left(\bigcup_{k\ge1} C'_k \right) = \varnothing$ then (11) holds on $\partial C''_j$ that is (11) holds on C''_j. Finally, if $C''_j \cap \left(\bigcup_{k\ge1} C'_k \right) \ne \varnothing$ and $C''_j \cap \partial \left(\bigcup_{k\ge1} C'_k \right) \ne \varnothing$ then, as in the proof of Theorem 3, we obtain (11) on C''_j with $M^*(q/2)$ instead of $M^*(q)$. Hence, we have (12) for all $z \in \mathbb{C} \setminus \left(\bigcup_{k\ge1} C'_k \right)$, i.e. for all $z \in \mathbb{C} \setminus G_q(\pi)$. By Fricke's criterion π_ψ is of bounded index. □

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