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# ON CAKE DIVIDING

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We consider Steinhaus' geometrical game on cake dividing.

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Рассматривается геометрическая задача Штейнгауза о разделе торта.

Hugo Steinhaus in his popular book [1] (see also [2]) considered the following game (Problem 51). Pavel and Havel are dividing a cake as follows. Pavel selects a point $P$ and Havel draws a line $l$ through the point $P$ and gets his piece of the cake. What form of the cake is the most advantageous for Havel and which part of the cake he obtains in this case?

Now we consider a formal interpretation of the game.

Let $\mu$ be the Lebesgue measure on the space $\mathbb{R}^n$. Let $\mathcal{B}^n$ be the family of compact subsets of space $\mathbb{R}^n$ with non zero measure. For every point $a \in \mathbb{R}^n$ put $l_a = \{x \in \mathbb{R}^n : (x, a) \geq 0\}$. Put

$$b_n = \inf_{A \in \mathcal{B}^n} \sup_{x \in A} \inf_{a \in \mathbb{R}^n} \frac{\mu(A \cap (x + l_a))}{\mu(A)}.$$

Steinhaus proved that $1/4 \leq b_2 \leq 1/3$.

**Theorem 1.** *For every $n$ holds $b_n = 1/(n+1)$.*

*Proof.* Let $S \subset \mathbb{R}^n$ be the regular simplex with vertices $a_0, \dots a_n$. Every point $a \in S$ has a unique representation $a = \sum \lambda_i a_i$ such that $\lambda_i \geq 0$ for every $i$ and $\sum \lambda_i = 1$. Put $S_i = a_i - S$ for every $i$. If $i \neq j$ then $S_i \cap S_j = \{0\}$. Indeed, suppose that there exist points $s_i, s_j \in S$ such that $x = a_i - s_i = a_j - s_j$. Let $s_j = \sum \lambda_{ik} a_k$ and $s_j = \sum \lambda_{jk} a_k$ be the representations of the points $s_i$ and $s_j$. The uniqueness of the representation of the point $(a_i + s_j)/2 = (a_j + s_i)/2$ gives that $1 + \lambda_{ji} = \lambda_{ii}$. Thus $\lambda_{ii} = 1$, $s_i = a_i$ and $x = 0$.

Put $A = \bigcup S_i$. Then $\mu(A) = (n+1)\mu(S)$. Let $a \in S$, $a = \sum \lambda_i a_i$ be the representation of the point $a$ and $\lambda_j = \max \lambda_i$. We show that $S_j \subset l_a$. Indeed let $a_j - x = \sum \mu_i a_i$ be the representation of a point $a_j - x$, where $x \in S_j$, and $\alpha = (a_i, a_j)$, where $i \neq j$. Then $(x, a_j) = 1 - \mu_j - \alpha \sum_{i \neq j} \mu_i \geq 1 - \sum \mu_i \geq 0$. Thus, the constructed set $A$ shows that $b_n \leq 1/(n+1)$.

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Show now that $b_n \geq 1/(n+1)$. Consider an arbitrary set $A \in \mathcal{B}^n$. For every $a \in S$ and $0 \leq \varepsilon \leq 1/(n+1)$ put $A(a,\varepsilon) = \{x \in \mathbb{R}^n : \mu(A \cap (x + l_a)) \geq 1/(n+1) - \varepsilon\}$. Then every $A(a,\varepsilon)$ is a convex closed set and $\mu(A(a,\varepsilon) \cap A) = n/(n+1) + \varepsilon$. Now fix an arbitrary number $\varepsilon > 0$. Let $c_0, \dots, c_n \in S$. Since $\mu(A) > \sum \mu(A \setminus A(c_i,\varepsilon))$, there exists a point $c(\varepsilon) \in A \cap \bigcap A(c_i,\varepsilon)$. Since $A$ is compact, the set $\{c(1/m) : m \in \mathbb{N}\}$ has a cluster point $c$ belonging to $A \cap \bigcap A(c_i, 0)$. Then the Helly Theorem implies that $\bigcap\{A(c,0) : c \in \mathbb{R}^n\} \neq \varnothing$, and thus $b_n \geq 1/(n+1)$. □

## REFERENCES

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