УДК 513.6

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ALGEBRAIC CURVES OVER n-DIMENSIONAL GENERAL LOCAL FIELDS

V. I. Andriychuk. *Algebraic curves over n-dimensional general local fields*, Matematychni Studii, **15** (2001) 209–214.

Let k be an n-dimensional local field over a pseudofinite residue field k_0, p be a prime number, $p \neq \operatorname{char} k_0$, and X be a complete, smooth, absolutely irreducible curve over k with successive good reductions. Suppose that $1 \leq n \leq 3$, and that k contains the group μ_p, of p-th roots of unity. It is proved that the etale cohomology groups $H^{n+2}(X, \mu_p)$ and $H^1(X, \mu_p)$ are dual.

В. И. Андрийчук. *Алгебраические кривые над n-мерными общими локальными полями* // Математичні Студії. – 2001. – Т.15, №2. – С.209–214.

Пусть k — n-мерное общее локальное поле с псевдоконечным полем вычетов k_0, p-простое число, $p \neq \operatorname{char} k_0$, X — полная, гладкая, абсолютно неприводимая кривая над k с последовательными невырожденными редукциями. Пусть $1 \leq n \leq 3$, и поле k содержит все корни p-ой степени из единицы. Доказано, что группы этальных когомологий $H^{n+2}(X, \mu_p)$ и $H^1(X, \mu_p)$ двойственны.

By an n-dimensional discretely valued field k we mean a chain of fields $k_0, \cdots, k_n = k$, where k_i is a complete discretely valued field with the residue field $k_{i-1}, 1 \leq i \leq n$. We call k an n-dimensional local field if k_0 is finite, n-dimensional general local field if k_0 is quasifinite [1], and n-dimensional pseudoglobal field if k_0 is pseudofinite [2].

The classical one dimensional local class field theory can be generalized to n-dimensional local fields. This higher dimensional local class field theory was constructed by using algebraic K-theory in the works of A. Parshin, K. Kato, I. Fesenko, S. Vostokov and of other mathematicians. At the same time S. Bloch and S. Saito began to develop the class field theory for curves over local and n-dimensional local fields. For this purpose it is of interest to study the cohomology of curves over n-dimensional local fields. In particular, J.-C. Douai [3] investigated the etale cohomology of curves defined over n-dimensional local fields and proved in this situation a duality theorem and an analogue of Tate-Poitou theorem.

On the other hand, B. Bekker [4] has shown that a part of results of n-dimensional local class field theory can be developed for n-dimensional general local fields if $\operatorname{char} k_0 > 0$. It turns out that the same may be accomplished for the case $\operatorname{char} k_0 = 0$.

The purpose of this paper is to show that a part of J.-C. Donai's [3] results remain true for curves over n-dimensional general local fields with pseudofinite k_0 and $1 \leq n \leq 3$.

2000 *Mathematics Subject Classification:* 12G99, 14H05.

Let X be a complete smooth absolutely irreducible curve over an n-dimensional pseudolocal field. Suppose that X has good reductions under successive reductions $k_i \bmod \pi_i$, $1 \leq i \leq n$ (for the notion of good reduction see [5, p. 240]), where π_i is a uniformizing element of k_i. Since k_0 is pseudofinite, X has a k-rational point. Let p be a prime number, $p \neq \operatorname{char} k_0$. We suppose also that k contains all the p-th roots of unity. Let us denote by μ_p the sheet of p-th roots of unity and by $H^r(X, \mu_p)$, $r \in \mathbb{N}$ the etale cohomology groups. $H^r(k, \mu_p)$ denotes the Galois cohomology, A_m and A/mA stand for the kernel and cokernel of multiplication by m in an abelian group A. Let $|A|$ denote the order of finite set A.

We can now state the analogue for n-dimensional pseudolocal fields of the theorem proved by J.-C. Douai [3] in the case of curves over n-dimensional local fields.

Theorem. *With the notations above, for any $n \leq 3$ there exists a nondegenerate pairing*

$$H^{n+2}(X, \mu_p) \times H^1(X, \mu_p) \to \mu_p,$$

so the groups $H^{n+2}(X, \mu_p)$ and $H^1(X, \mu_p)$ are dual.

Remark. The above theorem holds for any $n \geq 0$, but here we present the proof only for $n \leq 3$.

Before proving this theorem we prove a few auxiliary statements, some of them are of independent interest.

Lemma 1. *Let k be an n-dimensional general local field, p be a prime number, $p \neq \operatorname{char} k_0$. Then the cohomological p-dimension $\operatorname{cd}_p k$ of k is $n+1$, and $H^{n+1}(k, \mu_p) \cong \mathbb{Z}/p\mathbb{Z}$.*

Proof. Since k_0 is a quasifinite field, its absolute Galois group is isomorphic to $\hat{\mathbb{Z}}$, so $\operatorname{cd}_p k_0 = 1$ [5]. The first statement follows now by an easy induction using the equality [5, p. 105] $\operatorname{cd}_p k_i = 1 + cd_p k_{i-1}$.

To prove $H^{n+1}(k, \mu_p) \cong \mathbb{Z}/p\mathbb{Z}$ we consider the exact sequence

$$0 \longrightarrow H^r(k_{i-1}, \mu_p) \longrightarrow H^r(k_i, \mu_p) \longrightarrow H^{r-1}(k_{i-1}, \mu_p) \longrightarrow 0, \tag{1}$$

$1 \leq i \leq n+1$, $r \geq 1$ (see [3, p.183] or [6, p.659]). Since $cd_p k_{n-1} = n$, we have, setting $r = n+1$ in (1), $H^{n+1}(k_{n-1}, \mu_p) = 0$, and $H^{n+1}(k, \mu_p) \cong H^n(k_{n-1}, \mu_p)$. Repeated application of this method enables us to write $H^{n+1}(k, \mu_p) \cong H^n(k_{n-1}, \mu_p) \cong \cdots \cong H^1(k_0, \mu_p) = \operatorname{Hom}(\hat{\mathbb{Z}}, \mu_p) \cong \mathbb{Z}/p\mathbb{Z}$. □

The next Proposition was proved by J.-C. Douai [3] in the case of n-dimensional local field. Taking Lemma 1 into account, it is easy to check that it remains true for n-dimensional general local fields.

Proposition 1. *Let k be an n-dimensional general local field, p be a prime number, $p \neq \operatorname{char} k_0$. Then the Galois cohomology groups $H^r(k, \mu_p)$ are finite, and the cup-product*

$$H^r(k, \mu_p) \times H^{n-r+1}(k, \mu_p) \to H^{n+1}(k, \mu_p)$$

induces a duality of the groups $H^r(k, \mu_p)$ and $H^{n-r+1}(k, \mu_p)$ for all $r \geq 0$.

Proof. Let us use again the exact sequence (1). It is known ([3, p. 183] or [6, p. 659]) that this exact sequence splits, so we get

$$H^r(k_i, \mu_p) \cong H^r(k_{i-1}, \mu_p) \oplus H^{r-1}(k_{i-1}, \mu_p) \tag{2}$$

for all $r > 0$, $1 \le i \le n+1$. The finiteness of $H^r(k_i, \mu_p)$ follows from (2) by induction on i. If $n = 0$, then $H^0(k_0, \mu_p) \cong H^1(k_0, \mu_p) \cong \mathrm{Hom}(\hat{\mathbb{Z}}, \mu_p) \cong \mathbb{Z}/p\mathbb{Z}$. In the case $n = 1$ we get a nondegeneracy pairing $H^r(k, \mu_p) \times H^{2-r}(k, \mu_p) \to \mathbb{Z}/p\mathbb{Z}$, according to [5, Ch. II, Sect. 5.2, Exercise 2, p. 113]. For any n and $r = 0$ the nondegenerescence of the pairing $H^0(k, \mu_p) \times H^{n+1}(k, \mu_p) \to \mathbb{Z}/n\mathbb{Z}$ follows from Lemma 1 and [5, Prop. 17, p. 35]. So let us consider the case $n > 1$ and $0 < r \le n$. By splitting the exact sequence (1) we deduce

$$H^r(k, \mu_p) \cong H^r(k_{n-1}, \mu_p) \oplus H^{r-1}(k_{n-1}, \mu_p),$$

$$H^{n-r+1}(k, \mu_p) \cong H^{n-r+1}(k_{n-1}, \mu_p) \oplus H^{n-r}(k_{n-1}, \mu_p).$$

Proceeding by induction, we see that the group $H^r(k_{n-1}, \mu_p)$ is dual to the group $H^{n-r}(k_{n-1}, \mu_p)$, and $H^{r-1}(k_{n-1}, \mu_p)$ is dual to $H^{n-r}(k_{n-1}, \mu_p)$. Thus the groups $H^r(k, \mu_p)$ and $H^{n-r+1}(k, \mu_p)$ are dual to each other. This completes the proof of Proposition 1. □

Corollary. *The group $H^n(k, \mu_p)$ is isomorphic to the group $(k^*\hat{/}k^{*p})$ dual to k^*/k^{*p}.*

Proof. $H^n(k, \mu_p)$ is dual to $H^1(k, \mu_p)$, and the latter group is isomorphic to k^*/k^{*p}. □

Let $\bar{X} = X \times_k k_{sep}$, k_{sep} being a separable closure of k. There is the spectral sequence [7, p. 134]

$$E_2^{r,s} = H^r(k, H^s(\bar{X}, \mu_p)) \Longrightarrow H^{r+s}(X, \mu_p). \tag{3}$$

Denote by J the jacobian of X, let $\hat{J}$ be its dual.

Lemma 2. *For any n-dimensional general local field k there is the exact sequence*

$$0 \longrightarrow H^{n+1}(k, J_p)/\operatorname{im} d_2^{n-1,2} \longrightarrow H^{n+2}(X, \mu_p) \longrightarrow (k^*\hat{/}k^{*p}) \longrightarrow 0,$$

where $d_2^{n-1,2}$ is the corresponding differential from spectral sequence (3).

Proof. First, it is known [7, Ch. 5], that $H^0(\bar{X}, \mu_p) \cong \mu_p$, $H^1(\bar{X}, \mu_p) \cong J_p$, $H^2(\bar{X}, \mu_p) \cong \mu_p$, and $H^r(\bar{X}, \mu_p) = 0$ for $r > 2$. According to Lemma 1, $\mathrm{cd}_p\, k = n+1$, so in the spectral sequence (3) $E_2^{n+2,0} \cong H^2(k, \mu_p) = 0$, $E_2^{n+1,1} \cong H^{n+1}(k, J_p) \cong \hat{J}(k)_p$, $E_2^{n+2-i,i} = 0$ for $3 \le i \le n+2$. By the above corollary, the group $E_2^{n,2} \cong H^n(k, \mu_p)$ is dual to $H^1(k, \mu_p)$ which is isomorphic to k^*/k^{*p}. One sees easily that $E_2^{n,2} = E_\infty^{n,2}$, $E_\infty^{n+2-i,i} = 0$ if either $i = 0$ or $3 \le i \le n+2$.

Consequently, we have the following filtration $H^{n+2}(X, \mu_p) \cong E_0^{n+2} = \cdots = E_n^{n+2} \supset E_{n+1}^{n+2} \supset 0$ of the group $H^{n+2}(X, \mu_p)$, where $E_n^{n+2}/E_{n+1}^{n+2} = E_2^{n,2}$, $E_{n+1}^{n+2} = E_\infty^{n+1,1}$. Thus we get the exact sequence $0 \longrightarrow E_\infty^{n+1,1} \longrightarrow H^{n+2}(X, \mu_p) \longrightarrow (k^*\hat{/}k^{*p}) \longrightarrow 0$. But $E_\infty^{n+1,1} = E_3^{n+1,1} = E_2^{n+1,1}/\operatorname{im} d_2^{n-1,2} \cong H^{n+1}(k, J_p)/\operatorname{im} d_2^{n-1,2}$ which finishes the proof of Lemma 2. □

Proposition 2. *Let k be an n-dimensional pseudolocal field, A be an abelian variety defined over k. Suppose that A has all good reductions under successive reductions of k_i, $1 \le i \le n$. Then $|A(k)/mA(k)| = |A(k)_m|$ for any m, $(m, \mathrm{char}\, k_0) = 1$.*

Proof. We proceed by induction on n. If $n = 0$, then $H^1(k, A) = 0$, since k is pseudofinite. So, the exact sequence $0 \longrightarrow A_m \longrightarrow A \xrightarrow{m} A \longrightarrow 0$ gives rise to the exact sequence $0 \longrightarrow A(k)/mA(k) \longrightarrow H^1(k, A_m) \longrightarrow 0$.

On the other hand, $|H^1(k, A_m)| = |H^0(k, A_m)| = |A(k)_m|$ by [8, p. 322], thus $|A(k)/mA(k)| = |A(k)_m|$, as desired. If $n > 0$, we have the reduction exact sequence

$$0 \longrightarrow A_1 \longrightarrow A \longrightarrow A' \longrightarrow 0, \tag{4}$$

where the reduction kernel A_1 is uniquely divisible by m [9]. Using the snake lemma to exact sequence (4) and to multiplication by n homomorphisms, we get

$$|A(k_n)_m||A'(k_{n-1})_m| = |A'(k_{n-1})/mA'(k_{n-1})| = |A(k_n)/mA(k_n)|.$$

□

Lemma 3. *Let k be a general local field, M be a finite G_k-module, $(|M|, \operatorname{char} k_0) = 1$. Then the groups $H^i(k, M)$ are finite and $|H^0(k, M)||H^2(k, M)| = |H^1(k, M)|$.*

Proof. This is an "elementary case" of the Euler-Poincare characteristic. For a local ground field k this was proved in [5, p. 115], but that proof cn be carried out the case of finite modules over general local fields (see [5, Exercise, p. 117]. □

Lemma 4. *Let M be a finite G_k-module over a general local field k,$(|M|, \operatorname{char} k_0) = 1$, $\hat{M} = \operatorname{Hom}(M, k_{sep}^*)$. Then for all integer r, $0 \le r \le 2$ cup-product induces a duality of finite groups $H^r(k, M)$ and $H^{2-r}(k, \hat{M})$.*

Proof. In the case of local ground field this was proved in [5, Ch. II, Sec. 3,4, Theorem 2]. According to [5, Exercise 2, p. 113] it remains true for finite modules over general local fields. □

Proposition 3. *Let A be an abelian variety with good reduction, defined over a pseudolocal field k, $\hat{A}$ be its dual variety. Then for any m, $(m, \operatorname{char} k_0) = 1$, the Tate-Shafarevich pairing [10] induces a nondegeneracy pairing $A(k)/mA(k) \times H^1(k, \hat{A})_m \longrightarrow \mathbb{Z}/n\mathbb{Z}$.*

Proof. Let i_m and j_m be the homomorphisms from the exact sequences

$$0 \longrightarrow A(k)/mA(k) \xrightarrow{i_m} H^1(k, A_m) \longrightarrow H^1(k, A)_m \longrightarrow 0, \tag{5}$$

$$0 \longrightarrow \hat{A}(k)/m\hat{A}(k) \longrightarrow H^1(k, \hat{A}_m) \xrightarrow{j_m} H^1(k, \hat{A})_m \longrightarrow 0. \tag{6}$$

By [10, Prop. 9, p. 36] the following diagram commutes

$$\begin{array}{ccccc} H^1(k, A_m) \times & H^1(k, \hat{A}_m) & \xrightarrow{W} & \mathbb{Q}/\mathbb{Z} \\ \uparrow i_m & \downarrow j_m & & \| \\ A(k)/mA(k) \times & H^1(k, \hat{A})_m & \xrightarrow{T} & \mathbb{Q}/\mathbb{Z}, \end{array} \tag{7}$$

where the pairing W is induced by the Weil pairing, and the pairing T is induced by the Tate-Shafarevich pairing. The pairing W is a duality of finite groups. Thus it follows from (7) that the left kernel of T is zero. To prove that T is a duality, it is sufficient to check that $|A(k)/mA(k)| = |H^1(k, \hat{A})_m|$. Using Lemmas 3, 4, Proposition 2, and the exact sequences (5),(6), we get $|A(k)/mA(k)|^2 = |\hat{A}(k)_m|^2 = |H^0(k, \hat{A}_m)||H^0(k, \hat{A}_m)| = |H^0(k, \hat{A}_m)||H^2(k, \hat{A}_m)| = |H^1(k, \hat{A}_m)| = |A(k)/mA(k)||H^1(k, \hat{A})_m|$, so $|A(k)/mA(k)| = |H^1(k, \hat{A})_m|$ which completes the proof of Proposition 3. □

Lemma 5. *In the spectral sequence (3)* $d_2^{n-1,2} = 0$ *for* $1 \le n \le 3$.

Proof. We begin with $n = 1$. Let us consider the terms $E_\infty^{0,2}$, $E_\infty^{1,1}$ and $E_\infty^{0,2}$ in the spectral sequence (3). One deduce easily that $E_\infty^{0,2} \subset E_2^{0,2}$, $E_\infty^{1,1} = E_2^{1,1}$, and $E_\infty^{0,2}$ is isomorphic to some quotient group of $E_2^{0,2}$. We claim that the composition of homomorphisms $H^2(X, \mu_p) \to E_\infty^{0,2} \to E_2^{0,2}$ is surjective. Conversely, suppose that $|E_\infty^{0,2}| < |E_2^{0,2}| = H^2(k, \mu_p)| = p$. Considering the filtration $H^2(X, \mu_p) = E_0^2 \supset E_1^2 \supset E_2^2 \supset 0$ with $E_i^2/E_{i+1}^2 = E_\infty^{i,2-i}$, we conclude that $|H^2(X, \mu_p)| < |E_2^{0,2}||E_\infty^{1,1}||E_\infty^{0,2}|$, because $E_2^{1,1} = E_\infty^{1,1}$, and $|E_\infty^{2,0}| < |E_2^{2,0}|$. Let $d = |J(k)_p|$. Then, using Proposition 3 and the exact sequence (4) with $A = J$, we conclude $|H^2(X, \mu_p)| < |H^1(k, J_p)||H^0(k, \mu_p)||H^2(k, \mu_p)| = d^2p^2$.

On the other hand, let us consider the exact sequence of etale cohomologies

$$0 \longrightarrow \operatorname{Pic} X/p \operatorname{Pic} X \longrightarrow H^2(X, \mu_p) \longrightarrow (\operatorname{Br} X)_p \longrightarrow 0, \tag{8}$$

corresponding to the exact sequence of sheaves

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{\ p\ } \mathbb{G}_m \longrightarrow 0. \tag{9}$$

From (8) we find $|H^2(X, \mu_p)| = |\operatorname{Pic} X/p \operatorname{Pic} X||(\operatorname{Br} X)_p|$. One shows easily that

$$|\operatorname{Pic} X/p \operatorname{Pic} X| = p|\operatorname{Pic}^0 X/p \operatorname{Pic}^0 X| = p|J(k)/pJ(k)| = pd.$$

Also, the existence of k-rational points on X gives us the exact sequence

$$0 \longrightarrow (\operatorname{Br} k)_p \longrightarrow (\operatorname{Br} X)_p \longrightarrow H^1(k, J)_p \longrightarrow 0, \tag{10}$$

which follows from the well known (see, for instance, [11]) exact sequence

$$0 \longrightarrow \operatorname{Pic} X \longrightarrow H^0(k, \operatorname{Pic} \bar{X}) \longrightarrow \operatorname{Br} k \longrightarrow \operatorname{Br} X \longrightarrow$$
$$\longrightarrow H^1(k, \operatorname{Pic} \bar{X}) \longrightarrow H^3(k, k_{sep}^*)$$

by passing to p-torsion.

Using (10) and Proposition 3, we find $|(\operatorname{Br} X)_p| = p|H^1(k, J)_p| = p|J(k)_p| = pd$. So, by (8) $H^2(X, \mu_p)| = p^2d^2$, a contradiction. Consequently, $E_\infty^{0,2} = E_2^{0,2}$, and we have the complex $H^2(X, \mu_p) \xrightarrow{\alpha} E_2^{0,2} \xrightarrow{d_2^{0,2}} E_2^{2,1}$, where α is surjective, so $d_2^{0,2} = 0$.

If $n = 2, 3$, the argument used in [3] to prove $d_2^{n-1,2} = 0$ in the case of finite k_0 works in our situation as well. Briefly, for $n = 2$ the differential $d_2^{1,2}$ can be identified with the composition of two homomorphisms $k^*/k^{*p} \cong H^1(k, \mu_p) \to H^1(X, \mu_p) \to H^3(k, J_q) \cong \hat{J}(k)_q$, so $d_2^{1,2} = 0$ in virtue of exactness of the sequence (11) below, which follows from (9),

$$0 \longrightarrow k^*/k^{*p} \longrightarrow H^1(X, \mu_p) \longrightarrow \hat{J}(k)_p \longrightarrow 0. \tag{11}$$

If $n = 3$, then $d_2^{2,2} = 0$ by Merkur'ev-Souslin theorem (see [3, p. 186] for more detail). □

Now we are ready to prove the main result.

Proof of Theorem. By Lemmas 2 and 5 we have the exact sequence

$$0 \longrightarrow H^{n+1}(k, J_p) \longrightarrow H^{n+2}(X, \mu_p) \longrightarrow (k^* \hat{/} k^{*p}) \longrightarrow 0.$$

Comparing it with (11), and using that $H^{n+1}(k, J_p)$ is dual to $\hat{J}(k)_p$ (since $\operatorname{cd}_p k = n + 1$), we conclude that $H^{n+2}(X, \mu_p)$ and $H^1(X, \mu_p)$ are dual. □

Remark. The statement of Theorem is true for $n = 0$ as well [12].

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Received 26.12.2000