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O. M. Sumyk

ON n-MEMBER ASYMPTOTICS FOR LOGARITHM OF MAXIMAL TERM OF ENTIRE DIRICHLET SERIES

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For an entire Dirichlet series with positive increasing to $+\infty$ exponents, conditions on coefficients under which $\ln\mu(\sigma)=\sum_{j=1}^{n-1}T_j e^{r_j\sigma}+(\tau+o(1))e^{r_n\sigma}$, $(\sigma\to+\infty)$, $n\ge 2$, where $\mu(\sigma)=\max\{|a_n|\exp(\sigma\lambda_n): n\ge 0\}$ is the maximal term of the series and $\rho_n<\dots\ \rho_2<\rho_1<+\infty$, $T_1>0$, $T_j\in\mathbb{R}$, $(j=2,\dots,n-1)$, $\tau\in\mathbb{R}$, $n\ge 2$, $\rho_2<(\rho_1+\rho_n)/2$, are established.

О. М. Сумык. *Об n-членной асимптотике логарифма максимального члена целого ряда Дирихле* // Математичні Студії. – 2001. – Т.15, №2. – С.200–208.

Для целого ряда Дирихле с положительными возрастающими к $+\infty$ показателями указаны условия на коэффициенты, при выполнении которых $\ln\mu(\sigma)=\sum_{j=1}^{n-1}T_j e^{r_j\sigma}+(\tau+o(1))e^{r_n\sigma}$, $(\sigma\to+\infty)$, $n\ge 2$, где $\mu(\sigma)=\max\{|a_n|\exp(\sigma\lambda_n): n\ge 0\}$ — максимальный член ряда, а $\rho_n<\dots\ \rho_2<\rho_1<+\infty$, $T_1>0$, $T_j\in\mathbb{R}$, $(j=2,\dots,n-1)$, $\tau\in\mathbb{R}$, $n\ge 2$, $\rho_2<(\rho_1+\rho_n)/2$.

1⁰. Introduction. Suppose (λ_n) is an increasing to $+\infty$ sequence of nonnegative numbers, $F(s)=\sum_{n=0}^{\infty}a_n\exp(s\lambda_n)$, $s=\sigma+it$, an entire Dirichlet series and $\mu(\sigma,F)=\max\{|a_n|\exp(\sigma\lambda_n): n\ \ge\ 0\}$ its maximal term.

Investigating two-member asymptotics of logarithm of the maximal term of an entire Dirichlet series, M. M. Sheremeta [1] proved the following. In order that $\ln\mu(\sigma,F)=T\exp\{\varrho_1\sigma\}+(\tau+o(1))\exp\{\varrho_2\sigma\}$ $(\sigma\to+\infty)$, where $0<\varrho_2<\varrho_1<+\infty$, $T>0$, $\tau\in\mathbb{R}$, it is necessary and sufficient that for arbitrary $\varepsilon>0$:

1) for all $n\ge n_0(\varepsilon)$: $\ln|a_n|\le -\dfrac{\lambda_n}{\varrho_1}\ln\dfrac{\lambda_n}{e\varrho_1 T}+(\tau+\varepsilon)\left(\dfrac{\lambda_n}{\varrho_1 T}\right)^{\varrho_2/\varrho_1}$;
2) there exists an increasing sequence (n_k) of positive integers such that $\lambda_{n_{k+1}}-\lambda_{n_k}=o(\lambda_{n_k}^{\alpha})$ $(k\to\infty)$, $\alpha=\dfrac{\varrho_1+\varrho_2}{2\varrho_1}$ and $\ln|a_{n_k}|\ge -\dfrac{\lambda_{n_k}}{\varrho_1}\ln\dfrac{\lambda_{n_k}}{e\varrho_1 T}+(\tau-\varepsilon)\left(\dfrac{\lambda_{n_k}}{\varrho_1 T}\right)^{\varrho_2/\varrho_1}$.

Here we extend M. Sheremeta's result to the case of n-member asymptotics. Everywhere further $0<\varrho_n<\dots<\varrho_2<\varrho_1<+\infty$, $T_1>0$, $T_j\in\mathbb{R}$, $(j=2,\dots,n-1)$, $\tau\in\mathbb{R}$, $n\ge 2$, $\varrho_2<\frac{\varrho_1+\varrho_n}{2}$.

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Theorem. *In order that*

$$\ln \mu(\sigma, F) = \sum_{j=1}^{n-1} T_j e^{\varrho_j \sigma} + (\tau + o(1)) e^{\varrho_n \sigma}, \quad \sigma \to \infty, \tag{1}$$

it is necessary and sufficient that for arbitrary $\varepsilon > 0$:

1) there exists $m_o = m_o(\varepsilon)$ such that for all $m \geq m_o$

$$\ln |a_m| \leq -\frac{\lambda_m}{\varrho_1} \ln \frac{\lambda_m}{eT_1\varrho_1} + \sum_{j=2}^{n-1} T_j \left(\frac{\lambda_m}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1} + (\tau + \varepsilon)\left(\frac{\lambda_m}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}; \tag{2}$$

2) there exists an increasing sequence (m_k) of positive integers such that

$$\ln |a_{m_k}| \geq -\frac{\lambda_{m_k}}{\varrho_1} \ln \frac{\lambda_{m_k}}{eT_1\varrho_1} + \sum_{j=2}^{n-1} T_j \left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1} + (\tau - \varepsilon)\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1} \tag{3}$$

and

$$\lambda_{m_{k+1}} - \lambda_{m_k} = o(\lambda_{m_k}^{\alpha}) \quad (k \to \infty), \quad \alpha = \frac{\varrho_1 + \varrho_n}{2\varrho_1}. \tag{4}$$

2^0. Auxiliary statements. Denote by Ω the class of positive unbounded on $(-\infty, +\infty)$ functions Φ such that their derivatives Φ' are continuous, positive and increasing to $+\infty$ on $(-\infty, +\infty)$ functions. As in [2], let $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the associated function with Φ in the sense of Newton. Denote by Ψ^{-1} the inverse function to Ψ and by φ the inverse function to Φ'.

Lemma 1. [2] *Let $\Phi \in \Omega$. In order that $\ln \mu(\sigma, F) \leq \Phi(\sigma)$ for all $\sigma < +\infty$, it is necessary and sufficient that $\ln |a_n| \leq -\lambda_n \Psi(\varphi(\lambda_n))$ for all $n \geq 0$.*

For $\Phi \in \Omega$, $0 < a < b < +\infty$ and $q > 0$ we put

$$G_1(a, b, q, \Phi) = \frac{ab}{b-a} \int_a^b \frac{\Phi(\varphi(qt))}{t^2}\, dt, \quad G_2(a, b, q, \Phi) = \Phi\left(\frac{1}{b-a}\int_a^b \varphi(qt)\, dt\right).$$

Lemma 2. [3] *The inequality $G_1(a, b, q, \Phi) < G_2(a, b, q, \Phi)$ holds.*

The following two lemmas are direct corollaries from Theorems 3.1 and 4.1 from [4].

Lemma 3. *Let $\Phi \in \Omega$ and $\ln |a_{m_k}| > -\lambda_{m_k}\Psi(\varphi(\lambda_{m_k}))$ for some increasing to $+\infty$ sequence (m_k) of positive integers.*

Then for all $k \geq k_0$ and all $\sigma \in [\varphi(\lambda_{m_k}), \varphi(\lambda_{m_{k+1}})]$ the inequality

$$\ln \mu(\sigma, F) \geq \Phi(\sigma) + G_1(\lambda_{m_k}, \lambda_{m_{k+1}}, 1, \Phi) - G_2(\lambda_{m_k}, \lambda_{m_{k+1}}, 1, \Phi)$$

holds.

By Ψ_j and φ_j we denote the functions associated in the sense of Newton with $\Phi_j \in \Omega$ and the functions inverse to Φ_j', respectively.

Lemma 4. *Let $\Phi_j \in \Omega, j = 1, 2$. If $\Phi_1(\sigma) \le \ln \mu(\sigma, F) \le \Phi_2(\sigma)$ for all $\sigma \in (\sigma_0, +\infty)$, then $\ln |a_m| \le -\lambda_m \Psi_2(\varphi_2(\lambda_m))$ for all $m > m_0$ and there exists an increasing to $+\infty$ sequence (m_k) of positive integers such that $\ln |a_{m_k}| \ge -\lambda_{m_k} \Psi_1(\varphi_1(\lambda_{m_k}))$ and*

$$\frac{\lambda_{m_k}\lambda_{m_{k+1}}}{\lambda_{m_{k+1}}-\lambda_{m_k}}\int_{\lambda_{m_k}}^{\lambda_{m_{k+1}}}\frac{\Phi_2(\varphi_2(t))}{t^2}\,dt \ge \Phi_1\left(\frac{1}{\lambda_{m_{k+1}}-\lambda_{m_k}}\int_{\lambda_{m_k}}^{\lambda_{m_{k+1}}}\varphi_2(t)\,dt\right).$$

Finally, the proof of the following lemma is elementary.

Lemma 5. *If a function $\Phi \in \Omega$ is such that $\Phi(\sigma) = \sum\limits_{j=1}^{n-1} T_j e^{\varrho_j\sigma} + \tau e^{\varrho_n\sigma}$ for all σ large enough, then when $x \to +\infty$*

$$\varphi(x) = \frac{1}{\varrho_1}\ln\frac{x}{T_1\varrho_1} - \sum_{j=2}^{n-1}\frac{T_j\varrho_j}{T_1\varrho_1^2}\left(\frac{x}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1} - \frac{\tau\varrho_n}{T_1\varrho_1^2}\left(\frac{x}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}(1+o(1)).$$

3^0. Proof of Theorem. Let us start with sufficiency. We remark that if the function φ is differentiable, then

$$(t\Psi(\varphi(t)))' = (t\varphi(t) - \Phi(\varphi(t)))' = \varphi(t) + t\varphi'(t) - \Phi'(\varphi(t))\varphi'(t) = \varphi(t),$$

that is

$$t\Psi(\varphi(t)) = \int_0^t \varphi(x)\,dx, \tag{5}$$

and

$$\int_a^b \frac{\Phi(\varphi(t))}{t^2}\,dt = -\left.\frac{\Phi(\varphi(t))}{t}\right|_a^b + \int_a^b \frac{\Phi'(\varphi(t))\varphi'(t)}{t}\,dt = $$
$$= \left.\left(-\frac{\Phi(\varphi(t))}{t} + \varphi(t)\right)\right|_a^b = \Psi(\varphi(t))|_a^b. \tag{6}$$

Therefore, if the function $\Phi \in \Omega$ is such that $\Phi(\sigma) = \sum_{j=1}^{n-1} T_j e^{\varrho_j\sigma} + \tau e^{\varrho_n\sigma}$ for $\sigma \ge \sigma_0$, then, using Lemma 5, it is easy to obtain the following relation

$$x\Psi(\varphi(x)) = \int_0^x \varphi(t)\,dt =$$
$$= \frac{x}{\varrho_1}\ln\frac{x}{eT_1\varrho_1} - \sum_{j=2}^{n-1} T_j\left(\frac{x}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1} - \tau\left(\frac{x}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}(1+o(1)), \quad x \to +\infty, \tag{7}$$

and hence

$$\Psi(\varphi(x)) = \frac{1}{\varrho_1}\ln\frac{x}{eT_1\varrho_1} - \sum_{j=2}^{n-1}\frac{T_j}{T_1\varrho_1}\left(\frac{x}{T_1\varrho_1}\right)^{\frac{\varrho_j}{\varrho_1}-1} - \frac{\tau}{T_1\varrho_1}\left(\frac{x}{T_1\varrho_1}\right)^{\frac{\varrho_n}{\varrho_1}-1}(1+o(1)), \quad x \to +\infty, \tag{8}$$

Using formulas (6), (7) and (8), we obtain

$$G_1(\lambda_{m_k},\lambda_{m_{k+1}},1,\Phi)=\frac{\lambda_{m_{k+1}}\lambda_{m_k}}{\lambda_{m_{k+1}}-\lambda_{m_k}}\Psi(\varphi(x))|_{\lambda_{m_k}}^{\lambda_{m_{k+1}}}=$$

$$=\frac{\lambda_{m_{k+1}}\lambda_{m_k}}{\lambda_{m_{k+1}}-\lambda_{m_k}}\left\{\frac{\ln\lambda_{m_{k+1}}-\ln\lambda_{m_k}}{\varrho_1}-\sum_{j=2}^{n-1}\frac{T_j}{T_1\varrho_1}\left(\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1}-\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1}\right)-\right.$$

$$\left.-\frac{\tau}{T_1\varrho_1}\left(\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}-\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}\right)(1+o(1))\right\},\quad k\to\infty, \tag{9}$$

and

$$G_2(\lambda_{m_k},\lambda_{m_{k+1}},1,\Phi)=T_1\exp\left\{\frac{\lambda_{m_{k+1}}\ln\lambda_{m_{k+1}}-\lambda_{m_k}\ln\lambda_{m_k}}{\lambda_{m_{k+1}}-\lambda_{m_k}}-\ln(eT_1\varrho_1)-\right.$$

$$-\frac{\varrho_1}{\lambda_{m_{k+1}}-\lambda_{m_k}}\sum_{j=2}^{n-1}T_j\left(\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1}-\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1}\right)-$$

$$\left.-\frac{\tau\varrho_1}{\lambda_{m_{k+1}}-\lambda_{m_k}}\left(\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}-\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}\right)(1+o(1))\right\}+$$

$$+\sum_{j=2}^{n-1}T_j\exp\left\{\frac{\varrho_j}{\varrho_1}\frac{\lambda_{m_{k+1}}\ln\lambda_{m_{k+1}}-\lambda_{m_k}\ln\lambda_{m_k}}{\lambda_{m_{k+1}}-\lambda_{m_k}}-\frac{\varrho_j}{\varrho_1}\ln(eT_1\varrho_1)-\right.$$

$$-\frac{\varrho_j}{\lambda_{m_{k+1}}-\lambda_{m_k}}\sum_{i=2}^{n-1}T_i\left(\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_i/\varrho_1}-\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_i/\varrho_1}\right)-$$

$$\left.-\frac{\tau\varrho_j}{\lambda_{m_{k+1}}-\lambda_{m_k}}\left(\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}-\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}\right)(1+o(1))\right\}+$$

$$+\tau\exp\left\{\frac{\varrho_n}{\varrho_1}\frac{\lambda_{m_{k+1}}\ln\lambda_{m_{k+1}}-\lambda_{m_k}\ln\lambda_{m_k}}{\lambda_{m_{k+1}}-\lambda_{m_k}}-\frac{\varrho_n}{\varrho_1}\ln(eT_1\varrho_1)-\right.$$

$$-\frac{\varrho_n}{\lambda_{m_{k+1}}-\lambda_{m_k}}\sum_{i=2}^{n-1}T_i\left(\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_i/\varrho_1}-\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_i/\varrho_1}\right)-$$

$$\left.-\frac{\tau\varrho_n}{\lambda_{m_{k+1}}-\lambda_{m_k}}\left(\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}-\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}\right)(1+o(1))\right\},\quad k\to+\infty. \tag{10}$$

Let $\lambda_{m_{k+1}}=(1+\theta_k)\lambda_{m_k}$. Then condition (4) implies $\theta_k\to 0$ $(k\to\infty)$. Therefore, using (9) and (10), one can prove that

$$G_2(\lambda_{m_k},\lambda_{m_{k+1}},1,\Phi)-G_1(\lambda_{m_k},\lambda_{m_{k+1}},1,\Phi)=\frac{\lambda_{m_k}}{\varrho_1}\left(1+\frac{\theta_k}{2}-\frac{\theta_k^2}{24}+O(\theta_k^3)\right)+$$

$$+\sum_{j=2}^{n-1}T_j\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1}\left(\frac{\varrho_1-\varrho_j}{\varrho_1}+\frac{\varrho_j(\varrho_1-\varrho_j)}{2\varrho_1^2}\theta_k-\frac{\varrho_j(5\varrho_1^2-6\varrho_1\varrho_j+4\varrho_j^2)}{24\varrho_1^3}\theta_k^2+O(\theta_k^3)\right)+$$

$$+\sum_{j=2}^{n-1}\frac{T_j\varrho_j^2}{2\varrho_1}\frac{\lambda_{m_k}^{(2\varrho_j-\varrho_1)/\varrho_1}}{(T_1\varrho_1)^{2\varrho_j/\varrho_1}}(1+o(1))-\sum_{j=2}^{n-1}\frac{T_jT_2\varrho_2\varrho_j}{T_1\varrho_1^2}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{(\varrho_2+\varrho_j)/\varrho_1-1}(1+o(1))+$$
$$+\tau\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}\frac{\varrho_1-\varrho_n}{\varrho_1}(1+o(1))-\frac{\lambda_{m_k}}{\varrho_1}\left\{1+\frac{\theta_k}{2}-\frac{\theta_k^2}{6}+O(\theta_k^3)\right\}+$$
$$+\sum_{j=2}^{n-1}T_j\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1}\frac{\varrho_j-\varrho_1}{\varrho_1}\left\{1+\frac{\varrho_j}{2\varrho_1}\theta_k+\frac{\varrho_j(\varrho_j-2\varrho_1)}{6\varrho_1^2}\theta_k^2+O(\theta_k^3)\right\}+$$
$$+\tau\frac{\varrho_n-\varrho_1}{\varrho_1}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}(1+o(1))=\frac{\lambda_{m_k}}{\varrho_1}\left\{\frac{\theta_k^2}{8}+O(\theta_k^3)\right\}+$$
$$+\sum_{j=2}^{n-1}T_j\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1}\left(\frac{\varrho_j(\varrho_1-2\varrho_j)}{8\varrho_1^2}\theta_k^2+O(\theta_k^3)\right)+\sum_{j=2}^{n-1}\frac{T_j\varrho_j^2}{2\varrho_1}\frac{\lambda_{m_k}^{(2\varrho_j-\varrho_1)/\varrho_1}}{(T_1\varrho_1)^{2\varrho_j/\varrho_1}}(1+o(1))-$$
$$-\sum_{j=2}^{n-1}\frac{T_jT_2\varrho_2\varrho_j}{T_1\varrho_1^2}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{(\varrho_2+\varrho_j)/\varrho_1-1}(1+o(1))+o(\lambda_{m_k}^{\varrho_n/\varrho_1}),\quad k\to\infty. \tag{11}$$

From (2) by Lemma 1 and relation (7) we obtain the inequality

$$\ln\mu(\sigma,F)\le\sum_{j=1}^{n-1}T_je^{\varrho_j\sigma}+(\tau+\varepsilon)e^{\varrho_n\sigma}\quad(\sigma\ge\sigma_0(\varepsilon)). \tag{12}$$

Further, from (3), taking into account Lemmas 3, 5 and formula (11), for all σ such that

$$\frac{1}{\varrho_1}\ln\frac{\lambda_{m_k}}{T_1\varrho_1}-\sum_{j=2}^{n-1}\frac{T_j\varrho_j}{T_1\varrho_1^2}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1}-\frac{(\tau-\varepsilon)\varrho_n}{T_1\varrho_1^2}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}\le\sigma\le$$
$$\le\frac{1}{\varrho_1}\ln\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}-\sum_{j=2}^{n-1}\frac{T_j\varrho_j}{T_1\varrho_1^2}\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1}-\frac{(\tau-\varepsilon)\varrho_n}{T_1\varrho_1^2}\left(\frac{\lambda_{m_{k+1}}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}$$

we obtain

$$\ln\mu(\sigma,F)\ge\sum_{j=1}^{n-1}T_je^{\varrho_j\sigma}+(\tau-\varepsilon)e^{\varrho_n\sigma}-e^{\varrho_n\sigma}\left\{\frac{\lambda_{m_k}^{(\varrho_1-\varrho_n)/\varrho_1}}{\varrho_1}\left(\frac{\theta_k^2}{8}+O(\theta_k^3)\right)+\right.$$
$$+\sum_{j=2}^{n-1}T_j\frac{\lambda_{m_k}^{(\varrho_j-\varrho_n)/\varrho_1}}{(T_1\varrho_1)^{\varrho_j/\varrho_1}}\left(\frac{\varrho_j(\varrho_1-2\varrho_j)}{8\varrho_1^2}\theta_k^2+O(\theta_k^3)\right)+$$
$$\left.+\sum_{j=2}^{n-1}\frac{T_j\varrho_j^2}{2\varrho_1}\frac{\lambda_{m_k}^{(2\varrho_j-\varrho_n-\varrho_1)/\varrho_1}}{(T_1\varrho_1)^{2\varrho_j/\varrho_1}}(1+o(1))-\sum_{j=2}^{n-1}\frac{T_jT_2\varrho_2\varrho_j}{T_1\varrho_1^2}\frac{\lambda_{m_k}^{(\varrho_2+\varrho_j-\varrho_n-\varrho_1)/\varrho_1}}{(T_1\varrho_1)^{(\varrho_2+\varrho_j-\varrho_1)/\varrho_1}}(1+o(1))+o(1)\right\}.$$

In view of (4) and condition $\varrho_2<(\varrho_1+\varrho_n)/2$, we have

$$\ln\mu(\sigma,F)\ge\sum_{j=1}^{n-1}T_je^{\varrho_j\sigma}+(\tau-2\varepsilon)e^{\varrho_n\sigma},\quad\sigma\ge\sigma_0(\varepsilon). \tag{13}$$

From (12) and (13) we obtain (1). Sufficiency of conditions 1) and 2) is proved.

Let us prove the necessity of conditions 1) and 2). From (1) for arbitrary $\varepsilon > 0$ and all $\sigma \geq \sigma_0(\varepsilon)$ we have

$$\sum_{j=1}^{n-1} T_j e^{\varrho_j \sigma} + (\tau - \varepsilon) e^{\varrho_n \sigma} = \Phi_1(\sigma) \leq \ln \mu(\sigma, F) \leq \Phi_2(\sigma) = \sum_{j=1}^{n-1} T_j e^{\varrho_j \sigma} + (\tau + \varepsilon) e^{\varrho_n \sigma}.$$

Therefore, by Lemma 4, in view of arbitrariness of ε, for all $m \geq m_0$ inequality (2) is valid. Similarly, by the same lemma there exists an increasing sequence (m_k) of positive integers such that inequalities (3) and

$$G_1(\lambda_{m_k}, \lambda_{m_{k+1}}, 1, \Phi_2) \leq \Phi_1 \left(\frac{1}{\lambda_{m_{k+1}} - \lambda_{m_k}} \int_{\lambda_{m_k}}^{\lambda_{m_{k+1}}} \varphi_2(t)\, dt \right) \tag{14}$$

are valid. Using Lemma 5, as mentioned above (see (10)), we obtain

$$\begin{aligned}
\Phi_1 &\left(\frac{1}{\lambda_{m_{k+1}} - \lambda_{m_k}} \int_{\lambda_{m_k}}^{\lambda_{m_{k+1}}} \varphi_2(t)\, dt \right) = \frac{1}{e\varrho_1} \exp\left\{ \frac{\lambda_{m_{k+1}} \ln \lambda_{m_{k+1}} - \lambda_{m_k} \ln \lambda_{m_k}}{\lambda_{m_{k+1}} - \lambda_{m_k}} - \right.\\
&- \frac{\varrho_1}{\lambda_{m_{k+1}} - \lambda_{m_k}} \sum_{j=2}^{n-1} T_j \left(\left(\frac{\lambda_{m_{k+1}}}{T_1 \varrho_1} \right)^{\varrho_j/\varrho_1} - \left(\frac{\lambda_{m_k}}{T_1 \varrho_1} \right)^{\varrho_j/\varrho_1} \right) - \\
&\left. - \frac{(\tau + \varepsilon)\varrho_1}{\lambda_{m_{k+1}} - \lambda_{m_k}} \left(\left(\frac{\lambda_{m_{k+1}}}{T_1 \varrho_1} \right)^{\varrho_n/\varrho_1} - \left(\frac{\lambda_{m_k}}{T_1 \varrho_1} \right)^{\varrho_n/\varrho_1} \right) (1 + o(1)) \right\} + \\
&+ \sum_{j=2}^{n-1} \frac{T_j}{(eT_1\varrho_1)^{\varrho_j \varrho_1}} \exp\left\{ \frac{\varrho_j}{\varrho_1} \frac{\lambda_{m_{k+1}} \ln \lambda_{m_{k+1}} - \lambda_{m_k} \ln \lambda_{m_k}}{\lambda_{m_{k+1}} - \lambda_{m_k}} - \right.\\
&- \frac{\varrho_j}{\lambda_{m_{k+1}} - \lambda_{m_k}} \sum_{i=2}^{n-1} T_i \left(\left(\frac{\lambda_{m_{k+1}}}{T_1 \varrho_1} \right)^{\varrho_i/\varrho_1} - \left(\frac{\lambda_{m_k}}{T_1 \varrho_1} \right)^{\varrho_i/\varrho_1} \right) - \\
&\left. - \frac{(\tau + \varepsilon)\varrho_j}{\lambda_{m_{k+1}} - \lambda_{m_k}} \left(\left(\frac{\lambda_{m_{k+1}}}{T_1 \varrho_1} \right)^{\varrho_n/\varrho_1} - \left(\frac{\lambda_{m_k}}{T_1 \varrho_1} \right)^{\varrho_n/\varrho_1} \right) (1 + o(1)) \right\} + \\
&+ \frac{\tau - \varepsilon}{(eT_1\varrho_1)^{\varrho_n/\varrho_1}} \exp\left\{ \frac{\varrho_n}{\varrho_1} \frac{\lambda_{m_{k+1}} \ln \lambda_{m_{k+1}} - \lambda_{m_k} \ln \lambda_{m_k}}{\lambda_{m_{k+1}} - \lambda_{m_k}} - \right.\\
&- \frac{\varrho_n}{\lambda_{m_{k+1}} - \lambda_{m_k}} \sum_{i=2}^{n-1} T_i \left(\left(\frac{\lambda_{m_{k+1}}}{T_1 \varrho_1} \right)^{\varrho_i/\varrho_1} - \left(\frac{\lambda_{m_k}}{T_1 \varrho_1} \right)^{\varrho_i/\varrho_1} \right) - \\
&\left. - \frac{(\tau + \varepsilon)\varrho_n}{\lambda_{m_{k+1}} - \lambda_{m_k}} \left(\left(\frac{\lambda_{m_{k+1}}}{T_1 \varrho_1} \right)^{\varrho_n/\varrho_1} - \left(\frac{\lambda_{m_k}}{T_1 \varrho_1} \right)^{\varrho_n/\varrho_1} \right) (1 + o(1)) \right\}, \quad k \to +\infty.
\end{aligned} \tag{15}$$

If we take as above $\lambda_{m_{k+1}} = (1 + \theta_k)\lambda_{m_k}$, then, taking into account (9) and (15), we can rewrite (14) in the form

$$\frac{\lambda_{m_k}(1+\theta_k)}{\theta_k}\left\{\frac{\ln(1+\theta_k)}{\varrho_1}-\sum_{j=2}^{n-1}\frac{T_j}{T_1\varrho_1}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1}\left((1+\theta_k)^{\varrho_j/\varrho_1-1}-1\right)-\right.$$
$$\left.-\frac{\tau+\varepsilon}{T_1\varrho_1}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}\left((1+\theta_k)^{\varrho_n/\varrho_1-1}-1\right)(1+o(1))\right\}\geq$$
$$\geq\frac{\lambda_{m_k}}{e\varrho_1}\exp\left\{\frac{1+\theta_k}{\theta_k}\ln(1+\theta_k)-\frac{\varrho_1}{\lambda_{m_k}\theta_k}\sum_{j=2}^{n-1}T_j\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1}\left\{(1+\theta_k)^{\varrho_j/\varrho_1}-1\right\}-\right.$$
$$\left.-\frac{(\tau+\varepsilon)\varrho_1}{\lambda_{m_k}\theta_k}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}\left\{(1+\theta_k)^{\varrho_n/\varrho_1}-1\right\}(1+o(1))\right\}+$$
$$+\sum_{j=2}^{n-1}T_j\left(\frac{\lambda_{m_k}}{eT_1\varrho_1}\right)^{\varrho_j/\varrho_1}\exp\left\{\frac{\varrho_j}{\varrho_1}\frac{1+\theta_k}{\theta_k}\ln(1+\theta_k)-\right.$$
$$-\frac{\varrho_j}{\lambda_{m_k}\theta_k}\sum_{i=2}^{n-1}T_i\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_i/\varrho_1}\left\{(1+\theta_k)^{\frac{\varrho_i}{\varrho_1}}-1\right\}-$$
$$\left.-\frac{(\tau+\varepsilon)\varrho_j}{\lambda_{m_k}\theta_k}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}\left\{(1+\theta_k)^{\varrho_n/\varrho_1}-1\right\}(1+o(1))\right\}+$$
$$+(\tau-\varepsilon)\left(\frac{\lambda_{m_k}}{eT_1\varrho_1}\right)^{\varrho_n/\varrho_1}\exp\left\{\frac{\varrho_n}{\varrho_1}\frac{1+\theta_k}{\theta_k}\ln(1+\theta_k)-\right.$$
$$-\frac{\varrho_n}{\lambda_{m_k}\theta_k}\sum_{i=2}^{n-1}T_i\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_i/\varrho_1}\left\{(1+\theta_k)^{\varrho_i/\varrho_1}-1\right\}-$$
$$\left.-\frac{(\tau+\varepsilon)\varrho_n}{\lambda_{m_k}\theta_k}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}\left\{(1+\theta_k)^{\varrho_n/\varrho_1}-1\right\}(1+o(1))\right\},$$

hence

$$\frac{1+\theta_k}{\theta_k}\left\{\frac{\ln(1+\theta_k)}{\varrho_1}-\sum_{j=2}^{n-1}\frac{T_j}{T_1\varrho_1}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1}\left((1+\theta_k)^{\varrho_j/\varrho_1-1}-1\right)-\right.$$
$$\left.-\frac{\tau+\varepsilon}{T_1\varrho_1}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}\left((1+\theta_k)^{\varrho_n/\varrho_1-1}-1\right)(1+o(1))\right\}\geq$$
$$\geq\frac{1}{e\varrho_1}\exp\left\{\frac{1+\theta_k}{\theta_k}\ln(1+\theta_k)-\frac{\varrho_1}{\lambda_{m_k}\theta_k}\sum_{j=2}^{n-1}T_j\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1}\left\{(1+\theta_k)^{\varrho_j/\varrho_1}-1\right\}-\right.$$
$$\left.-\frac{(\tau+\varepsilon)\varrho_1}{\lambda_{m_k}\theta_k}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1}\left\{(1+\theta_k)^{\varrho_n/\varrho_1}-1\right\}(1+o(1))\right\}+$$
$$+\sum_{j=2}^{n-1}T_j\frac{\lambda_{m_k}^{\varrho_j/\varrho_1-1}}{(eT_1\varrho_1)^{\varrho_j/\varrho_1}}\exp\left\{\frac{\varrho_j}{\varrho_1}\frac{1+\theta_k}{\theta_k}\ln(1+\theta_k)-\right.$$

$$
\begin{aligned}
&- \frac{\varrho_j}{\lambda_{m_k}\theta_k}\sum_{i=2}^{n-1} T_i \left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_i/\varrho_1} \left\{(1+\theta_k)^{\varrho_i/\varrho_1}-1\right\} - \\
&- \frac{(\tau+\varepsilon)\varrho_j}{\lambda_{m_k}\theta_k}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1} \left\{(1+\theta_k)^{\varrho_n/\varrho_1}-1\right\}(1+o(1))\Bigg\} + \\
&+ (\tau-\varepsilon)\frac{\lambda_{m_k}^{\varrho_n/\varrho_1-1}}{(eT_1\varrho_1)^{\varrho_n/\varrho_1}}\exp\left\{\frac{\varrho_n}{\varrho_1}\frac{1+\theta_k}{\theta_k}\ln(1+\theta_k) - \right. \\
&- \frac{\varrho_n}{\lambda_{m_k}\theta_k}\sum_{i=2}^{n-1} T_i \left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_i/\varrho_1} \left\{(1+\theta_k)^{\varrho_i/\varrho_1}-1\right\} - \\
&- \frac{(\tau+\varepsilon)\varrho_n}{\lambda_{m_k}\theta_k}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1} \left\{(1+\theta_k)^{\varrho_n/\varrho_1}-1\right\}(1+o(1))\Bigg\}. \qquad (16)
\end{aligned}
$$

Let us show that $\theta_k = O(1)$, $k\to\infty$. Actually, if we assume that $\theta_{k_j}\to+\infty$, then

$$
\frac{\ln(1+\theta_{k_j})}{\varrho_1} \ge \frac{1}{e\varrho_1}(1+\theta_{k_j}) + \sum_{j=2}^{n-1} T_j \frac{x_{k_j}^{\varrho_j/\varrho_1-1}}{(eT_1\varrho_1)^{\varrho_j/\varrho_1}}(1+\theta_{k_j})^{\varrho_j/\varrho_1} + \frac{(\tau-\varepsilon)x_{k_j}^{\varrho_n/\varrho_1-1}}{(eT_1\varrho_1)^{\varrho_n/\varrho_1}}(1+\theta_{k_j})^{\varrho_n/\varrho_1},
$$

and $\ln(1+\theta_{k_j}) \ge \dfrac{1+\varepsilon}{e}(1+\theta_{k_j})$ which is impossible. Thus, $\overline{\lim\limits_{k\to\infty}}\,\theta_k = \theta < +\infty$. From (16) and Lemma 2 we can see that $\theta\ge 0$ satisfies the equation

$$
\frac{1+\theta}{\theta}\ln(1+\theta) = \frac{1}{e}\exp\left\{\frac{1+\theta}{\theta}\ln(1+\theta)\right\} \qquad (17)
$$

Having done the substitution $x = \dfrac{1+\theta}{\theta}\ln(1+\theta)$, equation (17) reduces to the form $x = e^{x-1}$. Since the line $y = x$ is tangent to the curve $y = e^{x-1}$ at the point $x = 1$, this equation has the unique solution $x = 1$, and therefore equation (17) has the only solution $\theta = 0$. So, (16) can be rewritten in the form

$$
\begin{aligned}
&\frac{1}{\varrho_1}\left\{1+\frac{\theta_k}{2}-\frac{\theta_k^2}{6}+O(\theta_k^3)\right\} + \\
&+ \sum_{j=2}^{n-1} T_j \frac{\varrho_1-\varrho_j}{T_1\varrho_1^2}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1} \left\{1+\frac{\varrho_j}{2\varrho_1}\theta_k + \frac{\varrho_j(\varrho_j-2\varrho_1)}{6\varrho_1^2}\theta_k^2 + O(\theta_k^3)\right\} + \\
&+ (\tau+\varepsilon)\frac{\varrho_1-\varrho_n}{T_1\varrho_1^2}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1} \ge \frac{1}{\varrho_1}\left(1+\frac{\theta_k}{2}-\frac{\theta_k^2}{24}+O(\theta_k^3)\right) + \\
&+ \sum_{j=2}^{n-1}\frac{T_j}{T_1\varrho_1}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1}\left\{\frac{\varrho_1-\varrho_j}{\varrho_1} + \frac{\varrho_j(\varrho_1-\varrho_j)}{2\varrho_1^2}\theta_k - \frac{\varrho_j(5\varrho_1^2-6\varrho_1\varrho_j+4\varrho_j^2)}{24\varrho_1^3}\theta_k^2 + O(\theta_k^3)\right\} + \\
&+ \sum_{j=2}^{n-1}\frac{T_j\varrho_j^2}{2\varrho_1}\frac{\lambda_{m_k}^{2(\varrho_j-\varrho_1)/\varrho_1}}{(T_1\varrho_1)^{2\varrho_j/\varrho_1}}(1+o(1)) - \sum_{j=2}^{n-1}\frac{T_jT_2\varrho_2\varrho_j}{T_1\varrho_1^2}\frac{\lambda_{m_k}^{(\varrho_2+\varrho_j)/\varrho_1-2}}{(T_1\varrho_1)^{(\varrho_2+\varrho_j)/\varrho_1-1}}(1+o(1)) + \\
&+ \tau\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}\frac{\varrho_1-\varrho_n}{T_1\varrho_1^2} - \varepsilon\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}\frac{\varrho_1+\varrho_n}{T_1\varrho_1^2}, \quad k\to\infty,
\end{aligned}
$$

and hence

$$\frac{1}{\varrho_1}\left(\frac{\theta_k^2}{8}+O(\theta_k^3)\right)+\sum_{j=2}^{n-1}\frac{T_j}{T_1\varrho_1}\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_j/\varrho_1-1}\left\{\frac{\varrho_j(\varrho_1-2\varrho_j)}{8\varrho_1^2}\theta_k^2+O(\theta_k^3)\right\}+$$
$$+\sum_{j=2}^{n-1}\frac{T_j\varrho_j^2}{2\varrho_1}\frac{\lambda_{m_k}^{2(\varrho_j-\varrho_1)/\varrho_1}}{(T_1\varrho_1)^{2\varrho_j/\varrho_1}}(1+o(1))-\sum_{j=2}^{n-1}\frac{T_jT_2\varrho_2\varrho_j}{T_1\varrho_1^2}\frac{\lambda_{m_k}^{(\varrho_2+\varrho_j)/\varrho_1-2}}{(T_1\varrho_1)^{(\varrho_2+\varrho_j)/\varrho_1-1}}(1+o(1))\le$$
$$\le\varepsilon\left(\frac{\lambda_{m_k}}{T_1\varrho_1}\right)^{\varrho_n/\varrho_1-1}\frac{2}{T_1\varrho_1},\quad k\to\infty.$$

In view of arbitrariness of ε the latter inequality is valid if and only if $\theta_k^2\lambda_{m_k}^{1-\varrho_n/\varrho_1}\to 0$, $k\to\infty$, that is $\lambda_{m_{k+1}}-\lambda_{m_k}=o\Big(\lambda_{m_k}^{\frac{\varrho_n+\varrho_1}{2\varrho_1}}\Big)$, $k\to\infty$, that means (4). The proof of theorem is completed.

Remark. The condition $\varrho_2<\frac{\varrho_1+\varrho_n}{2}$ of Theorem is essential. For instance, if $n=3$, $\varrho_2=\dfrac{\varrho_1+\varrho_3}{2}$ and conditions (2), (3), (4) hold, then

$$\ln\mu(\sigma,F)=T_1e^{\varrho_1\sigma}+T_2e^{\varrho_2\sigma}+(\tau+O(1))e^{\varrho_3\sigma},\quad \sigma\to+\infty.$$

REFERENCES

1. Шеремета М. Н. *Двучленная асимптотика целых рядов Дирихле,* Теория функций, функц. анализ и их прилож. (1990), Вып. 54, 16–25.

2. Шеремета М. Н., Федыняк С. И. *О производной ряда Дирихле,* Сиб. мат. журн. **39** (1998), №1, 206–223.

3. Заболоцький М. В., Шеремета М. М. *Узагальнення теореми Лiндельофа,* Укр. мат. журн. **50** (1998), №9, 1177–1192.

4. Шеремета М. М., Сумик О. М. *Зв'язок мiж зростанням спряжених за Юнгом функцiй,* Мат. студiї, **11** (1999), №1, 41–47.

Faculty of Mechanics and Mathematics, Lviv National University

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