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THE CONCENTRATION INDEX OF SUBHARMONIC FUNCTIONS OF INFINITE ORDER

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The purpose of this paper is to introduce an analogue of the concentration index for the class of subharmonic functions of infinite order. Such an index in the case of finite order is used in the interpolation theory.

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Вводится индекс концентрации для субгармонических функций бесконечного порядка. Такой индекс для случая конечного порядка применяется в теории интерполяции.

We use the standard notation of the potential theory and the value distribution theory [1, 2], nevertheless we recall some of them. Denote by μ_u the Riesz measure of a subharmonic function u. Let $C(z,t) = \{w : |w - z| \le t\}$, $n(z,t) = \mu_u(C(z,t))$, $n(r) = n(0,r)$, and $B(r,u)$ be the maximum of u on the disk $C(0,r)$. Without loss of generality we may assume $u(0) = 0$ and $n(1) = 0$. The set of all subsets of $[1,\infty)$ with finite logarithmic measure is denoted by FLM: if $S \in FLM$, then $\int_1^\infty \chi_S(t)\,d\log(t) < \infty$, where χ_S is the characteristic function of S; M will stand for a positive constant.

The concentration index of an entire function of finite order was introduced into consideration implicitly by Levin [3] and explicitly by Krasichkov [4], who studied its properties. The specific case of zero order was considered in [5, 6].

Define the concentration index $I(z,u)$ of a subharmonic function of infinite order by the formula

$$I(z,u) = -\int\limits_0^{|z|/\log^{\varkappa} n(|z|)} n(z,t)/t\,dt,$$

where $\varkappa$ is a positive number.

We prove

Theorem. *Let u be a subharmonic function of infinite order, $r = |z|$. Then*

$$u(z) = I(z,u) + \exp\left(o(N(r))\right) + O(B(r,u)), \quad z \to \infty, r \notin S \in FLM. \tag{1}$$

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and

$$I(z,u) = o\left(\exp(o(N(r)))\right), \quad z \to \infty, z \notin E, \tag{2}$$

where E is such that for every $r \notin S \in FLM$ there exists an at most countable set of disks $C(z_j, r_j)$, having the following properties:

$$\bigcup_j C(z_j, r_j) \supset E \cap \{w : r < |w| < r + \Delta\}, \tag{3}$$

and

$$\sum_{|z_j| \in [r, r+\Delta]} r_j = o(\Delta), \quad r \to \infty, r \notin S \in FLM,$$

where $\Delta = r/\log^{\text{[illegible]}} n(r)$.

Proof. We start with the construction of a subharmonic function v such that the Riesz measure $\mu_v = \mu_u$ and the growth of the function v is minimal in some sense.

Let real numbers $\varkappa$ and η satisfy the inequalities $0 < \varkappa < \eta$. Following [7], we put

$$v(z) = \int_{\text{[illegible]}} \log |E(z/\xi, [\log^{1+\eta} n(|\xi|)])| \, d\mu_u(\xi), \tag{4}$$

where $E(z,p)$ is the Weierstrass primary factor of genus p. The integral in the right-hand side of (4) converges uniformly on every compact subset of $\mathbb{C}$. This known statement will be proved below too.

We represent $v(z)$ as the sum

$$v(z) = v_1(z) + v_2(z) + v_3(z) + v_4(z) + v_5(z), \tag{5}$$

where ($R = r + \Delta$)

$$v_1(z) = \int_{C(z,\Delta)} \log |1 - z/\xi| \, d\mu_u(\xi),$$

$$v_2(z) = \int_{C(z,\Delta)} \mathrm{Re} \sum_{j=1}^{j=[\log^{1+\eta} n(r)]} j^{-1} (z/\xi)^j \, d\mu_u(\xi),$$

$$v_3(z) = \int_{C(0,r) \setminus C(z,\Delta)} \log |E(z/\xi, [\log^{1+\eta} n(|\xi|)])| \, d\mu_u(\xi),$$

$$v_4(z) = \int_{C(0,R) \setminus (C(0,r) \setminus C(z,\Delta))} \log |E(z/\xi, [\log^{1+\eta} n(|\xi|)])| \, d\mu_u(\xi),$$

$$v_5(z) = \int_{\text{[illegible]} \setminus C(0,R)} \log |E(z/\xi, [\log^{1+\eta} n(|\xi|)])| \, d\mu_u(\xi).$$

We first prove two estimates we will need later on. Applying the Borel-Nevanlinna Theorem [2, p. 120] with

$$u(r) = \log \log n(\exp(r)), \quad \varphi(u) = \exp(-\varkappa u + \log M)$$

we obtain

$$n\left(r\left(1+\frac{M}{\log^{\varkappa} n(r)}\right)\right) \le n(r)^e, \quad r \notin S \in FLM. \tag{6}$$

By Lemma 1.1 [2, p. 433] with $\varepsilon = 1$, $\varphi(t) = N(\exp(t))$, we have

$$n(r) \le N(r)^2, \quad r \notin S \in FLM. \tag{7}$$

Let us estimate $v_1(z)$. We denote by $\nu(z,t)$ the value $\mu_u(C(0,t) \cap C(z,t))$. Assuming $u(z) > -\infty$, we get

$$v_1(z) = \int_0^{\Delta} \log t \, dn(z,t) + \int_0^R \log \frac{1}{t} \, d\nu(z,t) =$$

$$= I(z,t) + \log \Delta \, n(z,t) + \int_0^R \frac{\nu(z,t)}{t} \, dt + \log \frac{1}{R} \, \nu(z,R). \tag{8}$$

Above we represented $v_1(z)$ as the sum of the Stieltjes integrals and integrated by parts. Applying (6), (7), we obtain

$$|\log \Delta n(z,\Delta)| \le n(R)(\log r + \varkappa \log\log n(r)) =$$

$$= O(n(r)^{e+1} \log r) = O(N(r)^{2e+3}), \quad r \to \infty, r \notin S \in FLM. \tag{9}$$

Likewise,

$$\left|\log \frac{1}{R} \nu(z,R)\right| \le n(R) \log R = O(n(r)^e \log r) = O(N(r)^{2e+1}),$$

$$r \to \infty, \; r \notin S \in FLM. \tag{10}$$

Next,

$$\int_0^R \frac{\nu(z,t)}{t} \, dt \le n(R) \log R = O(N(r)^{2e+1}), \quad r \to \infty, r \notin S \in FLM. \tag{11}$$

Combining (8)-(11), we obtain

$$v_1(z) = I(z,u) + O(N(r)^{2e+3}), \quad r \to \infty, \; r \notin S \in FLM. \tag{12}$$

We will need the elementary inequality

$$\sum_{j=1}^{j=p} j^{-1} |w|^j \le a^p (2 + \log p), \tag{13}$$

which holds under the assumptions $|w| < a$ and $a > 1$.

Applying (13) to the estimate of v_2, we get

$$|v_2(z)| \leq 2\left(\frac{R}{r-\Delta}\right)^{\log^{1+\eta} n(R)} \log(\log n(R))\, n(R) \leq$$
$$\leq 2\left(1+3\log^{-\varepsilon} n(r)\right)^{\log^{1+\eta} n(R)} \log(\log n(R))\, n(R) =$$
$$= O(n(r)^{e+1} \exp(O(1)\log^{1+\eta-\varepsilon} n(r))) =$$
$$= O(\exp(o((N(r))), \quad r\to\infty,\, r\notin S\in FLM. \tag{14}$$

Now consider v_3. In view of the inequality

$$|\log|1-z/\xi|| \leq \max\left(\left|\log\frac{\Delta}{r}\right|, \log(1+r)\right) \leq 2(\log\log n(r)+\log r), \tag{15}$$

which holds on the set $C(0,r)\setminus(C(z,\Delta)\cup C(0,1))$, and (13), we obtain

$$|v_3(z)| \leq n(r)(2\log\log n(r)+2\log r) + (2\log\log n(r)+2)\left(\frac{r}{r_0}\right)^{(\log n(r_0))^{1+\eta}},$$

where

$$\left(\frac{r}{r_0}\right)^{(\log n(r_0)^{1+\eta})} = \max_{1\leq|\xi|\leq r}\left(\frac{r}{|\xi|}\right)^{\log^{1+\eta} n(|\xi|)}, \quad 1\leq r_0\leq r,$$

and r_0 is the greatest such a number. It exists, because the function $n(r)$ being nondecreasing and right-continuousis is upper semicontinuous on $[1,r]$. We easily see that $r_0\to\infty$ as $r\to\infty$. Taking into account the inequality

$$N(r) \geq \int_{r_0}^{r}\frac{n(t)}{t}\,dt \geq n(r_0)\log\frac{r}{r_0}$$

and (7), we have

$$|v_3(z)| \leq 4n(r)(\log\log n(r)+\log r)\exp\left(\frac{\log^{1+\eta} n(r_0)}{n(r_0)}N(r)\right) =$$
$$= \exp(o(N(r))), \quad r\to\infty, r\notin S\in FLM. \tag{16}$$

The next term $v_4(z)$ is estimated somewhat in another way. If $\xi\in C(0,R)\setminus(C(0,r)\cup C(z,\Delta))$, then

$$\left|\log\left|1-\frac{z}{\xi}\right|\right| \leq \left|\log\frac{\Delta}{R}\right| \leq \log\log n(r). \tag{17}$$

From (6), (13), and (17) we conclude (compare with(14))

$$|v_4(z)| \leq n(R)\left(\log\log n(r) + \left(\frac{R}{r}\right)\right)^{\log^{1+\eta} n(R)} (\log\log n(R)+2)) \leq$$
$$\leq n(r)^e\left(\log\log n(r) + \left(1+\frac{1}{\log^{\varepsilon} n(r)}\right)^{(e\log n(r))^{1+\eta}}\right)(\log\log n(r)+3)) =$$
$$= O(\exp(o(N(r)))), \quad r\to\infty, r\notin S\in FLM. \tag{18}$$

Finally, we estimate $v_5(z)$. Applying the inequality [3, p.21] $|\log|E(w,p)|| \le |w|^{p+1}$, when $|w| \le \frac{p}{p+1}$, we obtain

$$|v_5(z)| \le \int\limits_{\mathbb{C}\setminus C(0,R)} \left(\frac{r}{|\xi|}\right)^{\log^{1+\eta} n(|\xi|)+1} d\mu_u(\xi) \le$$

$$\le \int\limits_{\mathbb{C}\setminus C(0,R)} \left(\frac{r}{R}\right)^{\log^{1+\eta} n(|\xi|)+1} d\mu_u(\xi) \le \int\limits_{\mathbb{C}\setminus C(0,R)} \left(\frac{r}{R}\right)^{\log^{\varkappa} n(r) \log^{1+\eta-\varkappa} n(|\xi|)} d\mu_u(\xi) \le$$

$$\le \int\limits_{\mathbb{C}\setminus C(0,R)} 2^{-\log^{1+\eta-\varkappa} n(|\xi|)} d\mu_u(\xi) = \int\limits_R^\infty 2^{-(\log n(t))^{1+\eta-\varkappa}} dn(t) = O(1), \quad r\to\infty. \tag{19}$$

Combining (12), (14), (16), (18), and (19), we have

$$v(z) = I(z,u) + \exp(o(N(r))), \quad r\to\infty, r\notin S\in FLM, \tag{20}$$

i.e. the modulus of the difference $v(z) - I(z,u)$ is bounded by a nondecreasing function V, such that

$$V(r) = \exp(o(N(r))), \quad r\to\infty,\ r\notin S\in FLM$$

and

$$N(r)^{2e} = o(V(r)), \quad r\to\infty$$

(We can replace $V(r)$ by a greater function if necessary).

The next step consists in the proof of claims (1)-(4). We will use a method by Hayman [8]. A point z is said to be (β,s)-light with respect to a measure μ if for every $t\in(0,s)$ the inequality $n(z,t) < \beta t$ holds. We denote by $LP(\beta,s,\mu)$ the set of such points. We put $s(z) = s(|z|) = r/\log^{\square} n(r)$, $\mu=\mu_u$. Choose $\beta(z)=\beta(|z|)$ to satisfy

$$\beta(z)s(z) = o(V(r)), \quad r\to\infty,\ r\notin S\in FLM, \tag{21}$$

$$6N(r)^{2e} = o(\beta(r)s(r)), \quad r\to\infty,\ r\notin S\in FLM, \tag{22}$$

For instance, we can put $\beta(r)s(r) = (V(r)N(r)^{2e})^{1/2}$. If $z\in LP(\beta,s,\mu)$, then, applying (21), we have

$$|I(z,\mu)| = \int\limits_0^\Delta \frac{n(z,t)}{t}\,dt \le \int\limits_0^\Delta \beta\,dt = \beta(z)s(z) = o(\exp(o(N(r)))), \quad r\to\infty,\ r\notin S\in FLM.$$

If a point z is heavy (i. e. $z\in HP(\beta,s,\mu) = \mathbb{C}\setminus LP(\beta,s,\mu)$), then there exists a real number $r_z\in(0,s)$, such that $n(z,r_z)\ge\beta r_z$. We obtain a cover $\{C(z,r_z)\}$ of the set $HP(\beta,s,\mu)$. Applying the Besicovitch-Landkof Theorem [9, p.246], we can choose an at most countable subcover $\{C(z_j,r_j)\}$ of multiplicity not exceeding 6.

We note that if $t\in[r,R]$, then

$$\frac{r}{\log^{\square} n(R)} \le s(t) \le \frac{R}{\log^{\square} n(r)},$$

and thus $s(t)\sim\Delta$, $r\to\infty$, $r\notin S\in FLM$.

This implies

$$\sum_{|z_j|\in[r,R]} n(z_j,r_j) \le 6n(R+2\Delta) = 6n\left(r+\frac{3r}{\log^{\varepsilon} n(r)}\right) \le$$
$$\le 6n(r)^{e} \le 6N(r)^{2e}, \quad r\notin S\in FLM. \tag{23}$$

On the other hand,

$$\sum_{|z_j|\in[r,R]} n(z_j,r_j) \ge \sum_{|z_j|\in[r,R]} \frac{\beta(|z_j|)\varphi(|z_j|)}{\varphi(|z_j|)} r_j \ge \beta(r)\sum_{|z_j|\in[r,R]} \frac{\varphi(r)}{\varphi(|z_j|)} r_j \ge \frac{1}{2}\beta(r)\sum_{|z_j|\in[r,R]} r_j. \tag{24}$$

Comparing (23), (24) and using (22), we obtain

$$\sum_{|z_j|\in[r,R]} r_j \le 12N(r)^{2e}\beta(r)^{-1} = o(\Delta), \quad r\to\infty,\, r\notin S\in FLM.$$

To complete the proof of the theorem, we show

$$|u(z)-v(z)| \le M\left(B(r,u)+\exp(o(N(r)))\right) \quad r\to\infty,\, r\notin S\in FLM. \tag{25}$$

According to a result of Goldberg [7] (who considered only the case of entire functions, but his result and naturally changed proof are true for subharmonic functions too.),

$$B(r,v)\le \exp(o(N(r))),\ r\to\infty,\ r\notin S\in FLM.$$

Combining this and Theorem 4.4 [10], we obtain

$$u(z)-v(z)\le M\,T(r,u-v)\le M\left(T(r,u)+T(r,-v)\right)=$$
$$=M(T(r,u)+T(r,v)\le M(T(r,u)+\exp(o(N(r)))),\ r\to\infty,\ r\notin S\in FLM.$$

Above we used the First Main Theorem of the value distribution theory. It should be noted that we do not have any exceptional set of disks, because the function $u-v$ is harmonic. We can apply the same arguments to $v-u$ too, thus we prove (25). □

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