УДК 512.534

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GREEN'S RELATIONS ON $\mathcal{FP}^+(S_n)$

V. S. Mazorchuk. *Green's relations on* $\mathcal{FP}^+(S_n)$, Matematychni Studii, **15** (2001) 151–155.

We describe the Green's relations and study the signature ideals on the semigroup $\mathcal{FP}^+(S_n)$.

В. С. Мазорчук. *Отношения Грина на* $\mathcal{FP}^+(S_n)$ // Математичні Студії. – 2001. – Т.15, №2. – С.151–155.

Описаны отношения Грина и изучены сигнатурные идеалы на полугруппе $\mathcal{FP}^+(S_n)$.

1. Introduction

Description of Green's relations on the semigroup $\mathcal{B}(X)$ of all binary relations on a finite set X forms a classical part of the semigroup theory. This description was obtained and studied by several authors (see for example [5], [6] and references therein). Recently, an approximation subsemigroup $\mathcal{FP}^+(S_n)$ in $\mathcal{B}(X)$ for $X = \{1, 2, \dots, n\}$ was introduced in [1] in a natural way. It was shown that this subsemigroup has a lot of nice properties ([2], [3], [4]), for example it inherits the property of $\mathcal{B}(X)$ to have only inner automorphisms. In this paper we study Green's relations on $\mathcal{FP}^+(S_n)$.

The paper is organized as follows: In Section 2 we collect all necessary preliminaries. In Section 3 we prove several technical lemmas necessary for the proof of our main result presented in Section 4. Finally, in Section 5 we describe a lattice of a natural family of ideals in $\mathcal{FP}^+(S_n)$.

2. Preliminaries

For a fixed positive integer n let X denote the n-element set $\{1, 2, \dots, n\}$. Let S_n be the symmetric group on X. Consider the Boolean $\mathfrak{B}_n$ of S_n as a semigroup under natural operation induced from S_n, and define an equivalence relation, $\sim$, on $\mathfrak{B}_n$ as follows: for A_1 and A_2 from $\mathfrak{B}_n$ we set $A_1 \sim A_2$ if and only if for any $x \in X$ the sets $\{\sigma(x) : \sigma \in A_1\}$ and $\{\sigma(x) : \sigma \in A_2\}$ coincide. It is straightforward that $\sim$ is a well-defined congruence on $\mathfrak{B}_n$. The corresponding quotient semigroup $\mathfrak{B}_n/_\sim$ is called the factorpower of S_n and denoted by $\mathcal{FP}(S_n)$. The last semigroup has an empty set class as an external zero element. Throwing this element away one obtains the semigroup $\mathcal{FP}^+(S_n)$ which we will also call the factorpower of S_n. In what follows we will consider the semigroup $\mathcal{FP}^+(S_n)$ only.

2000 *Mathematics Subject Classification:* 20M20.

Let $\mathcal{B}(X)$ denote the semigroup of all binary relations on X. $\mathcal{FP}^+(S_n)$ can be identified with a subsemigroup of $\mathcal{B}(X)$ in a natural way. To each $A \in \mathcal{FP}^+(S_n)$ we associate a binary relation on X which consists of all pairs $(x, \sigma(x))$ where x runs through X and σ runs through A. One can show that this is in fact a monomorphism of semigroups. Thus, the elements of $\mathcal{FP}^+(S_n)$ can be written down as usual permutations:

$$A = \begin{pmatrix} 1 & 2 & \dots & n \\ A_1 & A_2 & \dots & A_n \end{pmatrix},$$

where $A \in \mathcal{FP}^+(S_n)$ and $A_x = \{\sigma(x) \,|\, \sigma \in A\}$ for all $x \in X$. Using this notation, the elements of $\mathcal{FP}^+(S_n)$ can be multiplied also as permutations. Namely, for A and B in $\mathcal{FP}^+(S_n)$ one has

$$\begin{pmatrix} 1 & 2 & \dots & n \\ A_1 & A_2 & \dots & A_n \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & \dots & n \\ B_1 & B_2 & \dots & B_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \dots & n \\ \bigcup\limits_{x \in B_1} A_x & \bigcup\limits_{x \in B_2} A_x & \dots & \bigcup\limits_{x \in B_n} A_x \end{pmatrix}.$$

Consider a set, $\mathfrak{D}$, consisting of all vectors $(l_1, l_2, \dots, l_n)$ with positive integer coefficients. For an element, $l \in \mathfrak{D}$, and for $1 \leq i \leq n$ by l_i (or $(l)_i$) we will denote the i-th coefficient of l. There are two natural partial preorders on $\mathfrak{D}$. For l and m from $\mathfrak{D}$ we will write $l < m$ if $l_i \leq m_i$ for all i and we will write $l \prec m$ if there exist a permutation $\sigma \in S_n$ such that $l_i \leq m_{\sigma(i)}$ for all i. Clearly, $\prec$ is a partial preorder on $\mathfrak{D}$ and $<$ is a (reflexive) partial order on $\mathfrak{D}$. One also has that $l < m$ implies $l \prec m$.

For an element $A \in \mathcal{FP}^+(S_n)$ by its signature, $\mathrm{Sgn}(A)$, we will mean the element $(|A_1|, |A_2|, \dots, |A_n|) \in \mathfrak{D}$.

3. Some technical lemmas

Lemma 1. *Let $A \in \mathcal{FP}^+(S_n)$ and $Y \subset X$ be a non-empty subset. Then*

$$\left| \bigcup_{y \in Y} A_y \right| \geq |Y|.$$

Proof. Let Z denote the union of all A_y, where y runs through Y. By the definition of $\mathcal{FP}^+(S_n)$, A is an equivalence class in $\mathfrak{B}$, which differs from empty set. Take any representative in A and choose a permutation $\sigma \in S_n$ in it. Clearly, Z contains, as a subset, the set $Z(\sigma) = \{\sigma(y) \,:\, y \in Y\}$. Since σ is a permutation, one has $|Z(\sigma)| = |Y|$. Hence $|Z| \geq |Y|$ as required. □

Lemma 2. *Suppose that $A, B, C \in \mathcal{FP}^+(S_n)$ such that $AB = C$. Then* $\mathrm{Sgn}(B) < \mathrm{Sgn}(C)$.

Proof. Fix $x \in X$. By definition of the multiplication in $\mathcal{FP}^+(S_n)$ we have

$$C_x = \bigcup_{y \in B_x} A_y.$$

From Lemma 1 it follows that $|C_x| \geq |B_x|$ and thus $\mathrm{Sgn}(B) < \mathrm{Sgn}(C)$ as required. □

Lemma 3. *Suppose that $A, B, C \in \mathcal{FP}^+(S_n)$ such that $AB = C$. Then* $\mathrm{Sgn}(A) \prec \mathrm{Sgn}(C)$.

Proof. Fix a representative of B in $\mathfrak{B}_n$ and choose a permutation, σ, in it. Fix $x \in X$. Since

$$C_x = \bigcup_{y \in B_x} A_y$$

and $\sigma(x) \in B_x$, it follows immediately that $|C_x| \geq |A_{\sigma(x)}|$. Hence $\mathrm{Sgn}(A) \prec \mathrm{Sgn}(C)$ as required. □

Corollary 1. *Suppose that* $A, B, C \in \mathcal{FP}^+(S_n)$ *such that* $AB = C$. *Then* $\mathrm{Sgn}(A) \prec \mathrm{Sgn}(C)$ *and* $\mathrm{Sgn}(B) \prec \mathrm{Sgn}(C)$.

4. Green's relations on $\mathcal{FP}^+(S_n)$

Theorem 1. *Let* $A, B \in \mathcal{FP}^+(S_n)$. *Then*

1. $A\mathcal{L}B$ *if and only if* $A = \sigma B$ *for some* $\sigma \in S_n$;
2. $A\mathcal{R}B$ *if and only if* $A = B\sigma$ *for some* $\sigma \in S_n$;
3. $A\mathcal{H}B$ *if and only if* $A = \sigma B = B\tau$ *for some* $\sigma, \tau \in S_n$;
4. $A\mathcal{D}B$ *if and only if* $A = \sigma B \tau$ *for some* $\sigma, \tau \in S_n$;
5. $\mathcal{J} = \mathcal{D}$.

Proof. Clearly, it is sufficient to prove only two first statements.

Suppose that $A = \sigma B$ (resp. $A = B\sigma$) for some $\sigma \in S_n$. Clearly, this implies $A\mathcal{L}B$ (resp. $A\mathcal{R}B$). So, now we can suppose that $A\mathcal{L}B$. This means that $A = XB$ and $B = YA$ for some $X, Y \in \mathcal{FP}^+(S_n)$. In particular, this implies $B \prec A$ and $A \prec B$, or, in other words, for any $1 \leq i \leq n$ we have

$$|\{x : |A_x| = i\}| = |\{x : |B_x| = i\}|.$$

Our goal is to show that one can choose $X \in S_n$.

We have $A = XB$ for some $X \in \mathcal{FP}^+(S_n)$. Choose a representative for X, say $\{\sigma_1, \dots, \sigma_k\} \subset S_n$, and consider the elements $A(j) = \{\sigma_1, \dots, \sigma_j\} \cdot B$ for $1 \leq j \leq k$. One has $A(k) = A$. From Corollary 1 we have $\mathrm{Sgn}(B) \prec \mathrm{Sgn}(A(j))$ for all j. Since

$$A(j+1)_t = A(j)_t \cup \bigcup_{f \in B_t} \{\sigma_{j+1}(f)\},$$

it follows that $\mathrm{Sgn}(A(j)) < \mathrm{Sgn}(A(j+1))$ for all j. But we recall that $\mathrm{Sgn}(A) \prec \mathrm{Sgn}(B)$, which implies that, in fact, $\mathrm{Sgn}(A(j)) = \mathrm{Sgn}(A(j+1))$ for all j, hence $A(j) = A(j+1)$ for all j and thus $A = A(1) = \sigma_1 \cdot B$. This completes the proof for the $\mathcal{L}$ relation.

Now suppose that $A\mathcal{R}B$. This means that $A = BX$ and $B = AY$ for some $X, Y \in \mathcal{FP}^+(S_n)$. In particular, this implies $B \prec A$ and $A \prec B$. Again we will show that one can choose $X \in S_n$.

We have $A = BX$ for some $X \in \mathcal{FP}^+(S_n)$. Choose a representative for X, say $\{\sigma_1, \dots, \sigma_k\} \subset S_n$ and consider the elements $A(j) = B \cdot \{\sigma_1, \dots, \sigma_j\}$ for $1 \leq j \leq k$. One has $A(k) = A$. From Corollary 1 we have $\mathrm{Sgn}(B) \prec \mathrm{Sgn}(A(j))$ for all j. Since

$$A(j+1)_t = A(j)_t \cup B_{\sigma_{j+1}(t)},$$

it follows that $\mathrm{Sgn}(A(j)) < \mathrm{Sgn}(A(j+1))$ for all j. At the same way as above we conclude that $\mathrm{Sgn}(A(j)) = \mathrm{Sgn}(A(j+1))$ for all j, hence $A(j) = A(j+1)$ for all j and thus $A = A(1) = B \cdot \sigma_1$. This completes the proof of the theorem. □

5. Signature ideals in $\mathcal{FP}^+(S_n)$

The problem to describe the ideal structure of $\mathcal{FP}^+(S_n)$ is still open. Nevertheless, technical lemmas presented in Section 3. enable one to describe a natural family of ideals defined by using the notion of signature.

Set $\mathfrak{a} = (1, 1, \dots, 1) \in \mathfrak{D}$ and $\mathfrak{b} = (n, n, \dots, n) \in \mathfrak{D}$. Consider an interval $\mathfrak{D}\{\mathfrak{a}, \mathfrak{b}\} = [\mathfrak{a}, \mathfrak{b}]$ with respect to the preorder $\prec$. Let $\mathfrak{D}(\mathfrak{a}, \mathfrak{b})$ denote a subset of $\mathfrak{D}\{\mathfrak{a}, \mathfrak{b}\}$, which consists of all those $(l_1, l_2, \dots, l_n)$ such that

$$\max_{1\leq i\leq n} l_i \leq n - |\{x \,:\, l_i = 1\}|.$$

Let $\tilde{\mathfrak{D}}(\mathfrak{a}, \mathfrak{b})$ denote the poset associated with $\mathfrak{D}(\mathfrak{a}, \mathfrak{b})$ in which the induced relation $\prec$ becomes a partial order. Let $\tilde{\mathfrak{D}}$ be a subset of $\mathfrak{D}$ consisting of all those vectors, whose coefficients do not decrease. Then $\tilde{\mathfrak{D}}$ is a poset with respect to $<$. One can easily show that the interval $[\mathfrak{a}, \mathfrak{b}]$ in $\tilde{\mathfrak{D}}$ is isomorphic to $\tilde{\mathfrak{D}}(\mathfrak{a}, \mathfrak{b})$.

For $l \in \mathfrak{D}$ let $I(l)$ denote the set of all elements $A \in \mathcal{FP}^+(S_n)$ such that $l \prec \mathrm{Sgn}(A)$. Clearly, $I(l)$ is not empty if and only if $l < \mathfrak{b}$ ($l \prec \mathfrak{b}$). We will call $I(l)$ the signature ideal corresponding to l. To proceed we need the following lemma:

Lemma 4. *For $l \in \mathfrak{D}$ there exists an element $A \in \mathcal{FP}^+(S_n)$ such that $\mathrm{Sgn}(A) = l$ if and only if $l \in \mathfrak{D}(\mathfrak{a}, \mathfrak{b})$.*

Proof. First we prove that $\mathrm{Sgn}(A) \in \mathfrak{D}(\mathfrak{a}, \mathfrak{b})$ for all $A \in \mathcal{FP}^+(S_n)$. Fix $A \in \mathcal{FP}^+(S_n)$. Clearly, $\mathrm{Sgn}(A) \in \mathfrak{D}\{\mathfrak{a}, \mathfrak{b}\}$. Let $1 \leq x_1 < x_2 < \dots < x_k \leq n$ be all indexes such that $|A_{x_j}| = 1$. It follows immediately that

$$\bigcup_{y\in X\setminus\{x_1,x_2,\dots,x_k\}} A_y = X \setminus \Big(\bigcup_{1\leq j\leq k} A_{x_j} \Big).$$

Clearly, $A_{x_i} \neq A_{x_j}$ if $i \neq j$. Hence $|A_y| \leq n-k$ for any $y \in X \setminus \{x_1, x_2, \dots, x_k\}$. This implies that $\mathrm{Sgn}(A) \in \mathfrak{D}(\mathfrak{a}, \mathfrak{b})$.

Now we prove that for any $l \in \mathfrak{D}(\mathfrak{a}, \mathfrak{b})$ there exists an element $A \in \mathcal{FP}^+(S_n)$ such that $\mathrm{Sgn}(A) = l$. Clearly, we can assume that $\min_{1\leq i\leq n} l_i \geq 2$, otherwise one can reduce the statement to the case of smaller n. Set

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ \{1,2\} & \{2,3\} & \{3,4\} & \dots & \{n-1,n\} & \{n,1\} \end{pmatrix}.$$

Clearly, $\tilde{A} \in \mathcal{FP}^+(S_n)$. Now it is enough to prove that for any

$$B = \begin{pmatrix} 1 & 2 & \dots & n \\ B_1 & B_2 & \dots & B_n \end{pmatrix}$$

such that $A_i \subset B_i$ for $1 \leq i \leq n$ the element B belongs to $\mathcal{FP}^+(S_n)$. Fix $x \in X$ and $y \neq x, x+1$ (here we set $n+1 = 1$). Clearly, it is enough to show that there exists a permutation $\sigma \in S_n$ such that $\sigma(x) = y$ and $\sigma(i) \in A_i = \{i, i+1\}$ for $i \neq x$. We have $\sigma(x) = y$. Set $\sigma(y) = y+1$, $\sigma(y+1) = y+2$ and so on till $\sigma(x-1) = x$. Also set $\sigma(y-1) = y-1$, $\sigma(y-2) = y-2$ and so on till $\sigma(x+1) = x+1$. Obviously, σ is a permutation. This completes the proof. □

Theorem 2. *1. $I(l)$ is a two-sided ideal of $\mathcal{FP}^+(S_n)$.*

2. Signature ideals form a lattice which is isomorphic to $\tilde{\mathfrak{D}}(\mathfrak{a},\mathfrak{b})$.

Proof. The first statement follows immediately form Corollary 1.

By virtue of Lemma 4, to prove the rest we first note that $\tilde{\mathfrak{D}}(\mathfrak{a},\mathfrak{b})$ is in fact a lattice. Indeed, let $l, m \in \tilde{\mathfrak{D}}(\mathfrak{a},\mathfrak{b})$. Using the isomorphism mentioned above we can assume that the coefficients of l and m do not decrease. Obviously, in this case

$$\min(l,m) = (\min(l_1,m_1), \min(l_2,m_2), \dots, \min(l_n,m_n))$$

and

$$\max(l,m) = (\max(l_1,m_1), \max(l_2,m_2), \dots, \max(l_n,m_n)).$$

This observation and Corollary 1 imply the second statement of our theorem. □

6. Acknowledgments

The paper was written during the authors visit to Bielefeld University as an Alexander von Humboldt fellow. The financial support of the AvH foundation and perfect conditions for work provided by Bielefeld University are gratefully acknowledged. I am also indebt to Prof. Boris M. Schein and Prof. Olexandr G. Ganyushkin for their valuable notes.

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Received 14.01.2000