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TOPOLOGIES ON GROUPS DETERMINED BY SEQUENCES: ANSWERS TO SEVERAL QUESTIONS OF I. PROTASOV AND E. ZELENYUK

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We answer on several problems of I. Protasov and E. Zelenyuk concerning topologies on groups determined by sequences.

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Дан ответ на несколько проблем И. Протасова и Е. Зеленюка, касающихся групповых топологий, определяемых последовательностями.

In this note we give answers to several problems posed by I. Protasov and E. Zelenyuk in [3] and [4]. Following [4] we define a sequence $(a_n)_{n\in\omega}$ of elements of a group G to be a *T-sequence* if $(a_n)_{n\in\omega}$ converges to zero in some non-discrete Hausdorff group topology on G. Given a T-sequence $(a_n)_{n\in\omega}$ in G we denote by $(G|(a_n))$ the group G endowed with the strongest topology in which the sequence (a_n) converges to zero. We say that a topological group G is *determined by a T-sequence* if $G=(G|(a_n))$ for some T-sequence $(a_n)_{n\in\omega}$ in G.

1. There is no restriction on the growth of T-sequences in $\mathbb{Z}$

All T-sequences of integers constructed in [3] have exponential growth. This led I. Protasov and E. Zelenyuk to the following question (see [4] and [3, Question 2.2.3]): *is there a monotone T-sequence of integers having polynomial growth*? First our result answers this question affirmatively. We recall that a group topology τ on a group G is called *totally bounded* if for every neighborhood $U\in\tau$ of zero in G there exists a finite subset $F\subset G$ with $G=F\cdot U$.

Theorem 1. *(1) If $(a_n)_{n=1}^{\infty}\subset\mathbb{Z}$ is an increasing T-sequence, then $\lim_{n\to\infty}(a_{n+1}-a_n)=\infty$.*

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(2) *Suppose* $f : \mathbb{N} \to \mathbb{N}$ *and* $\varepsilon : \mathbb{N} \to [0, \infty)$ *are functions such that* $\lim_{n\to\infty} \varepsilon(n) = \infty$ *and* $\lim_{n\to\infty} f(n+1) - f(n) = \infty$. *For every metrizable totally bounded group topology* τ *on* $\mathbb{Z}$ *there exists a converging to zero sequence* $(a_n)_{n\in\omega} \subset (\mathbb{Z}, \tau)$ *such that* $\lim_{n\to\infty} \frac{a_n}{f(n)} = 1$ *and* $|a_n - f(n)| \le \varepsilon(n)$ *for every* $n \in \omega$.

Proof. 1. Suppose $(a_n)_{n\in\omega} \subset \mathbb{Z}$ is an increasing T-sequence with $\lim_{n\to\infty} a_{n+1} - a_n \neq \infty$. This means that for some $C \in \mathbb{N}$ and every $n \in \mathbb{N}$ we can find $m \ge n$ with $a_{m+1} - a_m \le C$. Let τ be a non-discrete Hausdorff group topology on $\mathbb{Z}$ such that $(a_n)_{n=1}^{\infty}$ converges to zero in τ. Pick a τ-open neighborhood $U \subset \mathbb{Z}$ of zero such that $U \cap (i + U) = \varnothing$ for every $1 \le i \le C$ and find $n_0 \in \mathbb{N}$ such that $a_n \in U$ for every $n \ge n_0$. By the choice of the constant C, there exists $m \ge n_0$ with $a_{m+1} - a_m \le C$. Then letting $i = a_{m+1} - a_m$, we get $a_{m+1} = a_m + i \in (i + U) \cap U = \varnothing$, a contradiction.

2. Suppose functions f and ε satisfy the hypotheses of the theorem. Without loss of generality, $\varepsilon(1) = 0$ and $\varepsilon(n) \le \frac{1}{2} \min\{\sqrt{f(n)}, f(n+1) - f(n), f(n) - f(n-1)\}$ for $n > 1$.

Let τ be any metrizable totally bounded group topology on $\mathbb{Z}$ and $\mathbb{Z} = U_0 \supset U_1 \supset U_2 \supset \dots$ be a countable base of neighborhoods of zero in $(\mathbb{Z}, \tau)$. For every $n \in \omega$ let $k(n) = \max\{i \in \omega : U_i \cap [f(n) - \varepsilon(n), f(n) + \varepsilon(n)] \neq \varnothing\}$ and a_n be any point in $U_{k(n)} \cap [f(n) - \varepsilon(n), f(n) + \varepsilon(n)]$ (the number $k(n)$ is finite since the topology τ is Hausdorff). Evidently, $|f(n) - a_n| \le \varepsilon(n)$ for every $n \in \omega$ and

$$0 \le \lim_{n\to\infty} \left|\frac{a_n}{f(n)} - 1\right| \le \lim_{n\to\infty} \frac{\varepsilon(n)}{f(n)} \le \lim_{n\to\infty} \frac{1}{\sqrt{f(n)}} = 0.$$

It remains to verify the convergence of the constructed sequence $(a_n)_{n\in\omega}$ to zero in the topology τ. This will follow as soon as we prove that $\lim_{n\to\infty} k(n) = \infty$. Fix any number $m \in \mathbb{N}$. We have to find $n_0 \in \mathbb{N}$ such that $k(n) \ge m$ for every $n \ge n_0$. Using the total boundedness of the topology τ, find $l \in \mathbb{N}$ such that $\bigcup_{|i|<l}(i + U_m) = \mathbb{Z}$. Since $\lim_{n\to\infty} \varepsilon(n) = \infty$, there exists $n_0 \in \mathbb{N}$ such that $\varepsilon(n) > l$ for all $n \ge n_0$. It follows that for every $n \ge n_0$ there exists $i \in \mathbb{Z}$ such that $|i| < l < \varepsilon(n)$ and $i + U_m \ni f(n)$. Consequently, $U_m \cap [f(n) - \varepsilon(n), f(n) + \varepsilon(n)] \neq \varnothing$ and hence $k(n) \ge m$. □

Remark 1. The requirement of the metrizability of the topology τ in Theorem 1 is essential: according to [3, §5.1], there exists a totally bounded group topology τ on $\mathbb{Z}$ such that the space $(\mathbb{Z}, \tau)$ contains no nontrivial convergent sequence.

Remark 2. Theorem 1 gives a short proof of Theorem 2.2.6 from [3] which states that for every real number $r \ge 1$ there exists a T-sequence $(a_n)_{n\in\omega} \subset \mathbb{Z}$ with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$ (apply Theorem 1 with $f(n) = r^n + n^2$ and $\varepsilon(n) = n$).

2. T-sequences in the ring $\mathbb{Z}$

According to Theorem 2.2.3 [3], if $(a_n)_{n\in\omega} \subset \mathbb{Z}$ is a sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ is a transcendental real number, then $(a_n)_{n\in\omega}$ is a T-sequence in the group $\mathbb{Z}$. In [4] (see also [3, Question 3.4.1]) I. Protasov and E. Zelenyuk asked whether such a sequence $(a_n)_{n\in\omega}$ needs to be a T-sequence in the ring $\mathbb{Z}$, i.e., whether $(a_n)_{n\in\omega}$ converges to zero for some Hausdorff ring topology τ on $\mathbb{Z}$. The following theorem answers this question in negative.

Theorem 2. *For every real number* $r > 1$ *there exists a sequence* $(a_n)_{n\in\omega} \subset \mathbb{Z}$ *such that* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$ *but* $(a_n)_{n\in\omega}$ *is not a* T*-sequence in the ring* $\mathbb{Z}$.

Proof. Given a real number $r > 1$, consider the sequence $(a_n)_{n\in\omega} \subset \mathbb{Z}$ defined by

$$a_n = \begin{cases} [r^{n/2}]^2 + 1, & \text{if } n = 2 \cdot 3^k \text{ for some } k \in \mathbb{N}; \\ [r^n], & \text{otherwise,} \end{cases}$$

where as usual $[x] = \max\{k \in \mathbb{Z} : k \le x\}$ for a real number x. It can be easily shown that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$. Nonetheless, the sequence $(a_n)_{n\in\omega}$ cannot converge to zero in a ring topology τ on $\mathbb{Z}$ because $a_{2\cdot 3^k} - a_{3^k}^2 = 1$ for every $k \in \omega$. □

3. On closed subgroups of topological groups determined by T-sequences

In this section we give an example of a countable Abelian topological group G determined by a T-sequence and a closed subgroup H of G which is not determined by a T-sequence, thus answering Question 2.3.1 of [3]. The group G is a Graev free topological Abelian group $A(S_0)$ over a convergent sequence S_0 under which we understand any countable compactum S_0 with a unique nonisolated point considered as the distinguished point of S_0. We recall that the *Graev free topological Abelian group* $A(X)$ over a pointed Tychonov space $(X, *)$ is uniquely determined by the following three requirements: (1) there is an embedding $X \subset A(X)$ such that the fixed point $*$ of X coincides with the neutral element of the group $A(X)$, (2) $A(X)$ coincides with the group hull of X in $A(X)$, and (3) every continuous map $f : X \to G$ into a topological Abelian group G such that $f(*) = 0$ uniquely extends to a continuous group homomorphism $\bar{f} : A(X) \to G$, see [2].

It will be more convenient to work with the following concrete realization of a free group $A(S_0)$. In the group $\mathbb{Z}^\omega$ consider the sequence $(e_n)_{n\in\omega} \subset \mathbb{Z}^\omega$ of characteristic functions $e_n = \chi_{\{n\}} : \omega \to \{0,1\} \subset \mathbb{Z}$ of one-point subsets $\{n\} \subset \omega$. Clearly, the sequence $(e_n)_{n\in\omega}$ converges to zero in the product topology of $\mathbb{Z}^\omega$. Denote by $\mathbb{Z}^\omega_f$ the group hull of the set $\{e_n : n \in \omega\}$ in $\mathbb{Z}^\omega$. Algebraically, $\mathbb{Z}^\omega_f$ is the direct sum of countably many of cyclic groups $\mathbb{Z}$ and consists of all eventually zero sequences of integers. It can be easily shown that a Graev free topological Abelian group $A(S_0)$ is topologically isomorphic to the group $(\mathbb{Z}^\omega_f|(e_n))$ determined by the T-sequence $(e_n)_{n\in\omega}$.

Theorem 3. *The topological group $A(S_0) = (\mathbb{Z}^\omega_f|(e_n))$ contains a closed subgroup H which is not determined by a T-sequence.*

Proof. Fix any function $f : \omega \to \mathbb{N}$ such that the set $f^{-1}(n)$ is infinite for every $n \in \mathbb{N}$ and consider the closed subgroup

$$H = \big\{(x_i)_{i\in\omega} \in \mathbb{Z}^\omega_f : x_i \in f(i) \cdot \mathbb{Z} \ \text{ for every } \ i \in \omega\big\}$$

in $(\mathbb{Z}^\omega_f|(e_n))$. We claim that the topology of H is not determined by a T-sequence. Suppose the contrary: $H = (H|(b_n))$ for some sequence $(b_n)_{n\in\omega}$ convergent to zero in the topology τ of the group $(\mathbb{Z}^\omega_f|(e_n))$. By the standard arguments (see Chapter 4 of [3]) it can be shown that the group $(\mathbb{Z}^\omega_f|(e_n))$ carries the strongest topology inducing the original (product) topology on each (compact) set

$$\mathbb{Z}^\omega_n = \Big\{(x_i)_{i\in\omega} \in \mathbb{Z}^\omega_f : \sum_{i\in\omega} |x_i| \le n\Big\}, \quad n \in \mathbb{N}.$$

This fact and the convergence of $(b_n)_{n\in\omega}$ in $(\mathbb{Z}^\omega_f|(e_n))$ imply $\{b_n : n\in\omega\}\subset \mathbb{Z}^\omega_{n_0}\cap H$ for some $n_0\in\mathbb{N}$. Observe that $H=H'\oplus H''$, where H' and H'' are the group hulls of the sets $\{f(n)e_n : f(n)\le n_0\}$ and $\{f(n)e_n : f(n)>n_0\}$, respectively. It follows from the inclusion $\{b_n\}_{n\in\omega}\subset \mathbb{Z}^\omega_{n_0}\cap H$ that $\{b_n\}_{n\in\omega}\subset H'$. Since $H=H'\oplus H''$ carries the strongest group topology in which the sequence $(b_n)_{n\in\omega}$ converges to zero, we conclude that the topology of the group H'' is discrete. But this is not so, because H'' contains an infinite compact subset $\{0\}\cup\{(n_0+1)e_i : f(i)=n_0+1\}$. □

Remark 3. Using the Zelenyuk topological classification of countable k_ω-group ([5] or [3, §4.3]) it can be shown that the subgroup $H\subset A(S_0)$ constructed in Theorem 3 is homeomorphic to $A(S_0)$. This shows that the property of a topological group to be determined by a T-sequence is not a topological invariant.

Remark 4. The pathology described in Theorem 3 can not occur in the group $\mathbb{Z}$: for every Hausdorff group topology τ on $\mathbb{Z}$ every closed non-trivial subgroup H of $(\mathbb{Z},\tau)$ has finite index in $\mathbb{Z}$ and thus is open. If $(\mathbb{Z},\tau)$ is determined by a T-sequence, then so does the open subgroup H.

Question. Suppose G is a finitely-generated Abelian topological group determined by a T-sequence. Is every closed subgroup of G determined by a T-sequence?

4. On supremum of group topologies determined by T-sequences

In [4] I. Protasov and E. Zelenyuk posed the following problem: *Suppose τ_1,τ_2 are two group topologies on $\mathbb{Z}$ determined by T-sequences. Is the topology $\tau_1\vee\tau_2$ determined by a T-sequence?* We recall that $\tau_1\vee\tau_2$ is the weakest topology τ on $\mathbb{Z}$ such that the identity maps $(\mathbb{Z},\tau)\to(\mathbb{Z},\tau_i)$, $i=1,2$, are continuous. Clearly, the group $(\mathbb{Z},\tau_1\vee\tau_2)$ may be identified with the diagonal of the product $(\mathbb{Z}\times\mathbb{Z},\tau_1\times\tau_2)$.

It turns out that the supremum $\tau_1\vee\tau_2$ of two topologies determined by T-sequences on a countable Abelian group G may be very wild: it needs not be a k_ω-topology as well as may be a k_ω-topology but not determined by a T-sequence, etc.

We remind that a topological group G is called a *k_ω-group* if G admits a cover $\mathcal{K}$ by compact subspaces such that a subset $U\subset G$ is open in G if and only if $U\cap K$ is open in K for every $K\in\mathcal{K}$. According to [3, Corollary 4.1.5] every countable group G determined by a T-sequence is a k_ω-group.

Given an Abelian group G and a subgroup H of G let $G\oplus_H G$ denote the quotient group of $G\oplus G$ by the subgroup $\Gamma=\{(h,-h) : h\in H\}\subset G\oplus G$. The following result was suggested by I. Protasov.

Theorem 4. *For every subgroup H of a topological group G determined by a T-sequence there exist group topologies τ_1, τ_2 on $G\oplus_H G$ determined by T-sequences such that the topological group $(G\oplus_H G,\tau_1\vee\tau_2)$ contains an open subgroup topologically isomorphic to the group H.*

Proof. Let $(a_n)_{n\in\omega}\subset G$ be a T-sequence determining the topology of the group G. Denote by $\pi: G\oplus G\to G\oplus_H G$ the quotient homomorphism and by $e_1,e_2: G\to G\oplus_H G$ the injective group homomorphisms defined by $e_1(g)=\pi(g,0)=(g,0)+\Gamma$ and $e_2(g)=\pi(0,g)=(0,g)+\Gamma$ for $g\in G$. Observe that $e_1(h)=(h,0)+\Gamma=(0,h)+\Gamma=e_2(h)$ for any $h\in H$ which allows us to define the injective homomorphism $e: H\to G\oplus_H G$ by $e=e_1|H=e_2|H$. It is easy to see that $e(H)=e_1(G)\cap e_2(G)$. Using Theorem 2.1.4 of [3] (or the complementability of the groups $e_i(G)$ in $G \oplus_H G$) one may show that for $i = 1, 2$ the sequence $(e_i(a_n))_{n\in\omega}$ is a T-sequence in $G \oplus_H G$ determining the non-discrete topology τ_i on $G \oplus_H G$. It follows that the map $e_i : G \to (G \oplus_H G, \tau_i)$ is an open embedding. Then the map $e : H \to (G \oplus_H G, \tau_1 \vee \tau_2)$ is a topological embedding and $e(H) = e_1(G) \cap e_2(G)$ is an open subgroup of $(G \oplus_H G, \tau_1 \vee \tau_2)$ isomorphic to H. □

Theorem 4 shows that the supremum $\tau_1 \vee \tau_2$ of two group topologies determined by T-sequences may be as bad as are subgroups of topological groups determined by T-sequences. In particular, Theorems 3 and 4 imply

Corollary 1. *There exists a countable Abelian group G and two topologies τ_1, τ_2 on G determined by T-sequences such that $(G, \tau_1 \vee \tau_2)$ is a k_ω-group not determined by a T-sequence.*

Theorem 4 also yields the existence of a countable Abelian group G and two topologies τ_1, τ_2 on G determined by T-sequences such that $(G, \tau_1 \vee \tau_2)$ is not complete and thus is not a k_ω-group. We shall show that this fact is valid even for the group $G = \mathbb{Z}$ thus answering Question 3 of [4] (we do not know if Corollary 1 is true for the group $G = \mathbb{Z}$).

It follows from Theorems 2.2.3 and 2.2.1 of [3] (see also Exercise 2.2.5 in [3]) that a sequence $(a_n)_{n\in\omega} \subset \mathbb{Z}$ is a T-sequence in $\mathbb{Z}$ provided $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ either is infinite or is a transcendental real number. This implies that for any such a T-sequence $(a_n)_{n\in\omega}$ and any non-zero integer c the sequence $(a_n + c)_{n\in\omega}$ is a T-sequence too. Denote by τ_1, τ_2 the group topologies on $\mathbb{Z}$ determined by the T-sequences $(a_n)_{n\in\omega}$, $(a_n + c)_{n\in\omega}$. The following theorem implies that the topology $\tau_1 \vee \tau_2$ on $\mathbb{Z}$ is not complete and thus is not determined by a T-sequence.

Theorem 5. *Let τ_1 and τ_2 be the group topologies on a countable group G determined by T-sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$, respectively. If for some non-zero element $g \in G$ and every $n_0 \in \omega$ there exist $n, m \geq n_0$ with $g = a_n^{-1} b_m$, then the topological group $(G, \tau_1 \vee \tau_2)$ is not complete and thus is not determined by a T-sequence.*

Proof. To prove the theorem it suffices to verify that the diagonal of the square $G \times G$ (which is identified with $(G, \tau_1 \vee \tau_2)$) is not closed in the product topology $\tau_1 \times \tau_2$. Suppose the contrary: G is closed in $G \times G$. Then for the point $(g, 0) \in G \times G$ beyond the diagonal we may find two neighborhoods $U_i \in \tau_i$, $i = 1, 2$, of zero in G such that $((U_1 g) \times U_2) \cap G = \varnothing$. Since the sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ converge to zero in the topologies τ_1, τ_2, we may find numbers $n, m \in \omega$ such that $a_n \in U_1$, $b_m \in U_2$, and $g = a_n^{-1} b_m$. Then $(b_m, b_m) = (a_n g, b_m) \in (U_1 g \times U_2) \cap G$, a contradiction. □

Remark 5. Using Theorem 4 of [1], one may show that every non-closed subgroup of a countable k_ω-group is not sequential. In particular, the group $(G, \tau_1 \vee \tau_2)$ constructed in Theorem 5 is not sequential. Nonetheless, this group contains a non-trivial convergent sequence (this can be easily shown using the sequentiality of any countable k_ω-group, see Exercise 4.3.1 of [3]).

Finally, we prove the following theorem answering Question 2.5.5 of [3].

Theorem 6. *There exists a countable Abelian group G admitting a group topology τ determined by a T-sequence and a metrizable group topology τ' such that the group $(G, \tau \vee \tau')$ is not discrete but contains no non-trivial convergent sequence.*

Proof. Let $(G,\tau) = (\mathbb{Z}^\omega_f|(e_n))$ be the Graev free topological Abelian group from Theorem 3. As we said the topology τ is inductive with respect to the collection $\{\mathbb{Z}^\omega_n\}_{n\in\omega}$, where $\mathbb{Z}^\omega_n = \{(x_i)_{i\in\omega} : \sum_{i\in\omega}|x_i| \le n\}$ for $n \in \omega$.

Consider the metrizable topology τ' on $\mathbb{Z}^\omega_f$ generated by the base $(U_n)_{n\in\omega}$, where

$$U_n = \{(x_i)_{i\in\omega} \in \mathbb{Z}^\omega_f : x_i \in 2^n \cdot \mathbb{Z} \text{ for all } i \in \omega\}, \quad n \in \omega.$$

Let us show that the topology $\tau \vee \tau'$ is not discrete. To see this, notice that for every $n \in \omega$ and every open neighborhood $U \in \tau$ of zero in $\mathbb{Z}^\omega_f$ the intersection $U_n \cap U$ is infinite (it contains the sequence $(2^n e_i)_{i\ge n_0}$ for some n_0). Next, assume that $(b_n)_{n\in\omega}$ is a sequence convergent to zero in the topology $\tau \vee \tau'$. Since $(b_n)_{n\in\omega}$ is convergent in $(\mathbb{Z}^\omega_f, \tau)$ we get $\{b_n : n \in \omega\} \subset \mathbb{Z}^\omega_{n_0}$ for some n_0. On the other hand, using the convergence of $(b_n)_{n\in\omega}$ to zero in $(\mathbb{Z}^\omega_f, \tau')$ we may find $m_0 \in \mathbb{N}$ such that $\{b_n : n \ge m_0\} \subset U_{n_0}$. Since $U_{n_0} \cap Z^\omega_{n_0} = \{0\}$, we conclude that $b_n = 0$ for all $n \ge m_0$, i.e., the sequence $(b_n)_{n\in\omega}$ is trivial. □

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REFERENCES

1. Banakh T., *On topological groups containing a Fréchet-Urysohn fan,* Matem. Studii **9** (1998), №2, 149–154.
2. Graev M. I. *Free topological groups,* Izvestiya Akad. Nauk SSSR. Ser. Mat. **12** (1948), 279–324. (in Russian); English transl.: Topology and Topological Algebra, Translations Series 1. American Mathematical Society **8** (1962), 305–364.
3. Protasov I., Zelenyuk E. *Topologies on groups determined by sequences,* Monograph Series, Math. Studies. VNTL, Lviv, 1999.
4. Protasov I. V., Zelenyuk E. G. *Topologies on $\mathbb{Z}$ determined by sequences: seven open problems,* Matem. Studii **12** (1999), №1, 111.
5. Зеленюк Е. *Топологии на группах, определяемые компактами,* Мат. Студії (1995), Вып. 5, 5–16.

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