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ON S-TORSION THEORIES IN R-MOD

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We give a characterization of rings over which every hereditary torsion theory is an S-torsion theory.

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Охарактеризованы кольца, над которыми каждая наследственная теория кручений является теорией S-кручений.

Throughout the whole text, all rings are considered to be associative with unit $1 \neq 0$ and all ring homomorphisms are supposed to preserve the unit. All modules are unitary left modules.

Let R be a ring. The category of left R-modules is denoted by R-Mod.

A radical filter of the ring R is a collection $\mathcal{E}$ of left ideals of R possessing the following properties:

G1. If $I \in \mathcal{E}$ and $I \subseteq J$, where J is a left ideal of R, then $J \in \mathcal{E}$.

G2. If $I \in \mathcal{E}$, then for any $\lambda \in R$ the left ideal $(I:\lambda) = \{x \mid x \in R, x\lambda \in I\}$ is also in $\mathcal{E}$.

G3. If $I \subseteq J$, where I is a left ideal of R, $J \in \mathcal{E}$ and the left ideal $(I:\xi)$ is in $\mathcal{E}$ for any $\xi \in J$, then $I \in \mathcal{E}$.

A torsion theory for R-Mod is a pair $(\mathcal{T}, \mathcal{F})$ of classes of left R-modules such that

(i) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}, F \in \mathcal{F}$.

(ii) If $\mathrm{Hom}_R(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$.

(iii) If $\mathrm{Hom}_R(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

$\mathcal{T}$ is called a torsion class and its modules are torsion modules, while $\mathcal{F}$ is a torsion-free class consisting of torsion-free modules.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is called hereditary if $\mathcal{T}$ is closed under submodules.

Any given class G of left R-modules generates a torsion theory in the following way:

$$\mathcal{F} = \{F \mid F \in \text{R-Mod}, \mathrm{Hom}_R(C, F) = 0 \text{ for all } C \in G\},$$

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$$\mathcal{T} = \{T \mid T \in \text{R-Mod}, \operatorname{Hom}_R(T, F) = 0 \text{ for all } F \in \mathcal{F}\}.$$

This pair $(\mathcal{T}, \mathcal{F})$ is a torsion theory for R-Mod, and $\mathcal{T}$ is the smallest torsion class containing G.

Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory. Then a submodule N of a left R-module M is said to be $\mathcal{T}$-dense in M if M/N is a torsion module.

Set

$$\mathcal{E}(\mathcal{T}, \mathcal{F}) = \{I \mid I \text{ is a left ideal of } R, I \text{ is } \mathcal{T}\text{-dense in } R\}.$$

It can be proved [1] that $\mathcal{E}(\mathcal{T}, \mathcal{F})$ is a radical filter of R.

Let $\mathcal{E}$ be a radical filter of R. Put

$$\mathcal{T}_{\mathcal{E}} = \{M \mid M \in \text{R-Mod}, M = \{m \mid m \in M, \{x \mid x \in R, xm = 0\} \in \mathcal{E}\}\}.$$

This class $\mathcal{T}_{\mathcal{E}}$ is a torsion class for some hereditary torsion theory $(\mathcal{T}_{\mathcal{E}}, \mathcal{F}_{\mathcal{E}})$ [1].

Proposition 1 [1]. *There is a bijective correspondence between the hereditary torsion theories and the radical filters:*

$$(\mathcal{T}, \mathcal{F}) \mapsto \mathcal{E}(\mathcal{T}, \mathcal{F}),$$
$$\mathcal{E} \mapsto (\mathcal{T}_{\mathcal{E}}, \mathcal{F}_{\mathcal{E}}).$$

A torsion theory $(\mathcal{T}, \mathcal{F})$ is semisimple if some class of simple left R-modules generates it. In this case, the torsion theory is a hereditary torsion theory [1].

Definition (O. Horbachuk). A hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ is an *S-torsion theory* if there exists a left ideal H of R satisfying the condition that any left ideal I of R is $\mathcal{T}$-dense in R if and only if $H + I = R$.

A ring R is said to be left semiartinian in case $\operatorname{soc}(M) \neq 0$ for any non-zero left R-module M.

The Jacobson radical of R will be denoted by $J(R)$.

Theorem 1. *Let R be a ring. If every semisimple torsion theory is an S-torsion theory, then R is left semiartinian.*

Proof. Suppose that every semisimple torsion theory is an S-torsion theory.

Let G be the set of all simple left R-modules and $(\mathcal{T}, \mathcal{F})$ a torsion theory generated by G. It is clear that $\mathfrak{M} \in \mathcal{E}(\mathcal{T}, \mathcal{F})$ for any maximal left ideal $\mathfrak{M}$ of R. Since $(\mathcal{T}, \mathcal{F})$ is semisimple, there exists a left ideal H of R such that

$$\mathcal{E}(\mathcal{T}, \mathcal{F}) = \{I \mid I \text{ is a left ideal of } R,\ I + H = R\}.$$

Therefore $H + \mathfrak{M} = R$ for any maximal left ideal $\mathfrak{M}$ of R. Hence $H = R$. Thus, we see that $\mathcal{T} = \text{R-Mod}$, $\mathcal{F} = \{0\}$. By Proposition 8.15.2 [2], R is left semiartinian. □

Corollary 1. *Let R be a ring. If every hereditary torsion theory is an S-torsion theory, then R is left semiartinian.*

A subset I of a ring R is left T-nilpotent whenever for every sequence $a_1, a_2, \ldots$ in I there is an n such that $a_n \ldots a_2 a_1 = 0$.

Recall [1] that if R is left semiartinian, then J(R) is left T-nilpotent.

Corollary 2. *Let R be a ring. If every semisimple torsion theory is an S-torsion theory, then $J(R)$ is left T-nilpotent.*

Recall that a ring R is semisimple if $\operatorname{soc}(M) = M$ for every left R-module M. A ring R is semisimple if and only if $\operatorname{soc}({}_RR) = {}_RR$.

A ring R is right perfect in case $R/J(R)$ is semisimple and $J(R)$ is left T-nilpotent.

Let Ω_l denote the set of isomorphism classes of simple left R-modules. For each subset Ω_1 of Ω_l we let $(\mathcal{T}(\Omega_1), \mathcal{F}(\Omega_1))$ be the torsion theory generated by simple left R-modules with isomorphism class belonging to Ω_1.

Put

$$J_l(\Omega_1) = \bigcap \{\mathfrak{M} \,|\, \mathfrak{M} \text{ is a maximal left ideal of } R, \exists \omega \in \Omega_1 : R/\mathfrak{M} \in \omega\}.$$

A left R-module M is said to be semiartinian whenever $M \in \mathcal{T}(\Omega_l)$.

Theorem 2. *Let R be a ring. If every semisimple torsion theory is an S-torsion theory, then R is right perfect.*

Proof. Suppose that every semisimple torsion theory is an S-torsion theory. By Corollary 2, $J(R)$ is left T-nilpotent. Then, to complete the proof, it is sufficent to show that $R/J(R)$ is semisimple.

Let $\operatorname{soc}({}_R(R/J(R)) = U/J(R)$. It is clear that U is a two-sided ideal such that $U \supseteq J(R)$. Suppose that $U \neq R$. Then there exists a maximal left ideal W containing U. Let $\omega \in \Omega_l$ such that $R/W \in \omega$. It is clear that $U \subseteq J_l(\{\omega\})$. It is easy to see that

$$J(R) = J_l(\Omega_l \setminus \{\omega\}) \cap J_l(\omega) \supseteq J_l(\Omega_l \setminus \{\omega\}) \cap U \supseteq J(R).$$

Therefore $J(R) = J_l(\Omega_l \setminus \{\omega\}) \cap U$. By Theorem 1, R is left semiartinian. Since $\operatorname{soc}({}_RM)$ is an essential submodule of ${}_RM$ for every semiartinian module ${}_RM, U/J(R)$ is an essential submodule of $R/J(R)$. But $J(R) = J_l(\Omega_l \setminus \{\omega\}) \cap U$. Thus, we see that $U/J(R) \cap J_l(\Omega_l \setminus \{\omega\})/J(R) = J(R)/J(R)$.

Therefore $J_l(\Omega_l \setminus \{\omega\}) = J(R)$.

Let $\mathfrak{M}$ be an arbitrary maximal left ideal of R such that $R/\mathfrak{M} \notin \omega$. It is clear that $\operatorname{Hom}_R(R/\mathfrak{M}, R/W) = 0$. Since ω generates $(\mathcal{T}(\{\omega\}), \mathcal{F}(\{\omega\}))$, $R/\mathfrak{M} \in \mathcal{F}(\{\omega\})$. But $\mathcal{T}(\{\omega\}) \cap \mathcal{F}(\{\omega\}) = \{0\}$ and $R/\mathfrak{M} \neq 0$. Hence $R/\mathfrak{M} \notin \mathcal{T}(\{\omega\})$. Therefore $\mathfrak{M} \notin \mathcal{E}(\mathcal{T}(\{\omega\}), \mathcal{F}(\{\omega\}))$. Since $(\mathcal{T}(\{\omega\}), \mathcal{F}(\{\omega\}))$ is a semisimple torsion theory, we see that there exists a left ideal H of R such that

$$\mathcal{E}(\mathcal{T}(\{\omega\}), \mathcal{F}(\{\omega\})) = \{I | I \text{ is a left ideal of } R, I + H = R.\}$$

Since $\mathfrak{M} \notin \mathcal{E}(\mathcal{T}(\{\omega\}), \mathcal{F}(\{\omega\}))$, $\mathfrak{M} + H \neq R$. Then, by maximality, $H \subseteq \mathfrak{M}$.

Thus, we see that $H \subseteq \mathfrak{M}$ for every maximal left ideal $\mathfrak{M}$ of R such that $R/\mathfrak{M} \notin \omega$. Hence $H \subseteq J_l(\Omega_l \setminus \{\omega\})$. But $J_l(\Omega_l \setminus \{\omega\}) = J(R)$. Therefore $H \subseteq J(R)$. Since $J(R)$ is a superfluous submodule of ${}_RR$ and $H \subseteq J(R), H$ is a superfluous submodule of ${}_RR$. Hence $\mathcal{E}(\mathcal{T}(\{\omega\}), \mathcal{F}(\{\omega\})) = \{R\}$. Since $R/W \in \omega, W \in \mathcal{E}(\mathcal{T}(\{\omega\}), \mathcal{F}(\{\omega\})) = \{R\}$. Therefore $W = R$, which is a contradiction. So R is a right perfect ring. □

Corollary 3. *Let R be a ring. If every hereditary torsion theory is an S-torsion theory, then R is right perfect.*

A ring R is said to be a left duo-ring in case every left ideal of R is two-sided.

It is easy to see that every right perfect left duo-ring is a ring direct sum of some local right perfect rings. This follows from 27.6 (a-b), 28.11 [3].

Theorem 3. *Let R be a left duo-ring. Then the following statements are equivalent:*

(a) R is right perfect;

(b) R is a ring direct sum of some local right perfect rings;

(c) Every semisimple torsion theory in R-Mod is an S-torsion theory.

Proof. $(a) \Leftrightarrow (b)$. This is clear.

$(b) \Rightarrow (c)$. Suppose that $R = R_1 \dot{+} \cdots \dot{+} R_n$ for some local right perfect rings. Let I be an idempotent ideal of R. Then we have that $I = R_1 I \oplus \cdots \oplus R_n I$. Next suppose that $R_i I \neq R_i$. Since R_i is local, $R_i I \subseteq J(R_i)$. But $R_i I$ is idempotent and $J(R_i)$ is left T-nilpotent. Thus, by Lemma 28.3 [3], $R_i I = 0$. Therefore $I = R_{k_1} \oplus \cdots \oplus R_{k_t}$ where $\{k_1, \ldots, k_t\} \subseteq \{1, \ldots, n\}$. Set $A = \{1, \ldots, n\} \setminus \{k_1, \ldots, k_t\}$. Hence $\{\mathcal{D} \,|\, \mathcal{D}$ is a left ideal of $R, \mathcal{D} \supseteq I\} = \{\mathcal{D} | \mathcal{D}$ is a left ideal of $R, \mathcal{D} + \oplus_A R_\alpha = R\}$. Now apply Corollary 6.3 [1, p. 192].

$(c) \Rightarrow (a)$ This is immediate from Theorem 2. □

Corollary 4. *Let R be a left duo-ring. Then the following statements are equivalent:*

(a) R is right perfect;

(b) R is a ring direct sum of some local right perfect rings;

(c) Every semisimple torsion theory in R-Mod is an S-torsion theory;

(d) Every hereditary torsion theory in R-Mod is an S-torsion theory [4].

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. This is immediate from Theorem 3.

$(b) \Rightarrow (d)$. Assume (b). Since R is right perfect, every hereditary torsion theory in R-Mod is semisimple. But $(b) \Rightarrow (c)$.

$(d) \Rightarrow (b)$. Assume (d). Then every semisimple torsion theory in R-Mod is an S-torsion theory. But $(c) \Rightarrow (b)$. □

A ring R is semiprimitive in case $J(R) = 0$.

Theorem 4. *Let R be a semiprimitive ring. Then the following statements are equivalent:*

(a) R is semisimple;

(b) Every semisimple torsion theory in R-Mod is an S-torsion theory;

(c) Every hereditary torsion theory in R-Mod is an S-torsion theory.

Proof. $(a) \Rightarrow (c)$. Assume (a). Let I be an idempotent ideal (ideal) of R. Then there exists an ideal S of R such that $R = I \dot{+} S$. It is easy to see that $\mathcal{D} \supseteq I \Leftrightarrow \mathcal{D} + S = R$ for any left ideal $\mathcal{D}$ of R. Now apply Corollary 6.3 [1, p. 192]

$(c) \Rightarrow (b)$. This is clear.

$(b) \Rightarrow (a)$. Assume (b). By Theorem 2, R is right perfect. Since R is right perfect semiprimitive, R is semisimple. □

Corollary 5. *Let R be a regular ring. If every semisimple torsion theory is an S-torsion theory, then R is a semisimple ring.*

Example 1. Let F be a field. Set $R = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$. It is clear that

$$_RR = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

is a composition series of lenght 3 for $_RR$. Thus, R is left artinian. Let $I = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}$. It is easy to see that I is an idempotent ideal of R. Put $\mathcal{E} = \{\mathcal{D} \mid \mathcal{D} \text{ is a left ideal of } R,\ \mathcal{D} \supseteq I\}$. Then $\mathcal{E}$ is a radical filter of R [1].

Suppose that there exists a left ideal S of R such that

$$\mathcal{E} = \{\mathcal{D} \mid \mathcal{D} \text{ is a left ideal of } R,\ \mathcal{D} + S = R\}.$$

Then we have that $I + S = R$. Hence $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S$. Therefore

$$\begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix} = R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \subseteq S.$$

But

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} + S \supseteq \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} + \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix} = R.$$

Hence $\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \in \mathcal{E}$. Whence we see that $\begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix} \subseteq \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, which is a contradiction.

Thus, there exists a hereditary torsion theory for R-Mod which is not an S-torsion theory. And now we have a contradiction to Proposition 2.5 [6].

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