УДК 512.54

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EXTREMAL TOPOLOGIES ON GROUPS

I. V. Protasov. *Extremal topologies on groups*, Matematychni Studii, **15** (2001) 9–22.

We present some new constructions of refinements of a left invariant topology on a group. On one hand, these refinements are very close to the original topology of the group. On the other hand, these refinements have a wide spectrum of extremal topological properties.

И. В. Протасов. *Экстремальные топологии на группах* // Математичнi Студiї. – 2001. – Т.15, №1. – C.9–22.

Приводятся новые конструкции усиления левоинвариантной топологии на группе. С одной стороны, эти усиления очень близки к исходной топологии группы. С другой стороны, они охватывают широкий спектр экстремальных топологических свойств.

A topological space without isolated points is called maximal if it has an isolated point in every stronger topology. A topological (left topological) group is called maximal if its underlying space is maximal. Maximality is a very exotic phenomenon in the category of topological groups. Every maximal topological group has a countable open subgroup of period 2. Examples of maximal topological groups were constructed under additional to ZFC set-theoretical assumption [3]. There exist ZFC-models without maximal topological groups [5].

In contrast to topological groups every nondiscrete left topological group has a plenty of left topological refinements that are maximal [4, 6]. But from the topological point of view these refinements could be very far from the original left topological group.

In this paper we introduce some new kinds of refinements of left topological groups. On one hand, these refinements are very close to the original left topological group. On the other hand, these refinements have a wide spectrum of extremal topological properties.

In §1 we give some new characterizations of the left topological filters on a group. This is a remake of the papers [5, 2]. The novelty is in using products of filters instead of products of ultrafilters. Given any filter on a group, we also describe the strongest left invariant topology in which this filter converges to the identity of the group.

In §2 we introduce three types of refinements of a left invariant topology, namely open, dense and open-dense refinements. The maximal open and the maximal open-dense refinements are characterized in terms of open ultrafilters.

In §3 we describe nodec left invariant topologies in terms of open-dense refinements. We also discuss the relationships between nodec, irresolvable, maximal and submaximal left invariant topologies.

2000 *Mathematics Subject Classification:* 22A05, 54A20, 54G05, 54H11.

The main results of the paper are concentrated in §4. We show the way in which every extremely disconnected left invariant topology can be obtained from some o-maximal topology. The key role in this constructions is played by the interior of a filter introduced in §1. It is proved that every regular left topological group has a Hausdorff zero-dimensional extremely disconnected bounded refinement. For every ultrafilter on a group, we prove the extremal disconnectedness of the strongest left invariant topology in which this ultrafilter converges to the identity. The results of this paper witness that the category of left topological groups is much more appropriate environment for the life of extremal objects than the category of topological groups. Moreover, the extremal objects are not exotic in this category because they can be obtained from any left topological group in some very natural ways.

I should like to record my debt to the referee whose 11 remarks improved the previous version of the paper.

§1. Left topological filters

Definition 1.1. Let φ be a filter on a group G, $A \subseteq G$. The subset $\mathrm{cl}(A, \varphi) = \{g \in G : gF \cap A \neq \varnothing$ for every subset $F \in \varphi\}$ is called the *closure of A by the filter φ*.

The subset $\mathrm{int}(A, \varphi) = \{g \in G : gF \subseteq A$ for some subset $F \in \varphi\}$ is called the *interior of A with respect to the filter φ*.

Definition 1.2. A filter φ on a group G with the identity e is called *pointed* if $e \in F$ for every subset $F \in \varphi$. Given any filter ψ on a group G, denote $\psi_e = \{F \cup \{e\} : F \in \psi\}$.

Lemma 1.1. *The following statements hold:*

(a) $\mathrm{cl}(A \cup B, \varphi) = \mathrm{cl}(A, \varphi \cup \mathrm{cl}(B, \varphi)$ *for any subsets $A, B \subseteq G$ and for every filter φ on G,*

(b) $A \subseteq \mathrm{cl}(A, \varphi)$ *for every subset $A \subseteq G$ and for every pointed filter φ on G,*

(c) $\mathrm{cl}(\varnothing, \varphi) = \varnothing$ *for every filter φ on G,*

(d) $\mathrm{cl}(A, \varphi) = \bigcup\{\mathrm{cl}(A, p) : p$ *is an ultrafilter,* $\varphi \subseteq p\}$,

(e) $\mathrm{int}(A, \varphi) \subseteq A$ *for every subset $A \subseteq G$ and for every pointed filter φ on G,*

(f) $\mathrm{int}(A, \varphi) = \bigcap\{\mathrm{int}(A, p) : p$ *is an ultrafilter,* $\varphi \subseteq p\}$,

(g) $\mathrm{int}(A, p) = \mathrm{cl}(A, p)$ *for every subset $A \subseteq G$ and for every ultrafilter p on G.*

Proof. Use Definition 1.1. □

Definition 1.3. Let φ, ψ be filters on a group G. Define the product $\varphi\psi$ by the following rule $A \subseteq G$, $A \in \varphi\psi$ if and only if $\mathrm{int}(A, \psi) \in \varphi$. Note that the family $\{\bigcup_{x \in F} xH_x : F \in \varphi, H_x \in \psi, x \in F\}$ forms a base of the filter $\varphi\psi$. This implies that the product of filters is associative and the product of ultrafilters is an ultrafilter. For semigroups of ultrafilters and their applications see [7] and [1].

Lemma 1.2. *For any filters φ, ψ on a group G and for every subset $A \subseteq G$, the following statements hold: (a)* $\mathrm{int}(\mathrm{int}(A, \psi), \varphi) = \mathrm{int}(A, \varphi\psi)$, *(b)* $\mathrm{cl}(\mathrm{cl}(A, \psi), \varphi) = \mathrm{cl}(A, \varphi\psi)$.

Proof. (a) Let $g \in \mathrm{int}(\mathrm{int}(A, \psi), \varphi)$. Take a subset $F \in \varphi$ with $gF \subseteq \mathrm{int}(A, \psi)$. For every element $x \in F$, pick $H_x \in \psi$ such that $gxH_x \subseteq A$. Put $P = \bigcup_{x \in F} xH_x$. Then $P \in \varphi\psi$, $gP \subseteq A$ so $g \in \mathrm{int}(A, \varphi\psi)$.

Let $g \in \mathrm{int}(A, \varphi\psi)$. Choose $F \in \varphi$, $H_x \in \psi$, $x \in F$ such that $gxH_x \subseteq A$ for every element $x \in F$. Then $gF \in \mathrm{int}(A, \psi)$ so $g \in \mathrm{int}(\mathrm{int}(A, \psi), \varphi)$.

(b) The inclusion $\subseteq$ follows from Definitions 1.1 and 1.3. To prove $\supseteq$ suppose that $g \in \mathrm{cl}(A, \varphi\psi)$ but $g \notin \mathrm{cl}(A\,\mathrm{cl}(A, \psi), \varphi)$. Then there exists a subset $F \in \varphi$ such that $gF \cap \mathrm{cl}(A, \psi) = \varnothing$. For every $x \in F$ take a subset $H_x \in vfi$ with $gxH_x \cap A = \varnothing$. Put $B = \bigcup_{x \in F} xH_x$. Then $B \in \varphi\psi$ and $gB \cap A = \varnothing$. Hence, $g \notin \mathrm{cl}(A, \varphi\psi)$, a contradiction. □

Definition 1.4. A topology τ on a group G is called *left invariant* if, for every element $g \in G$, the left shift $x \mapsto gx$ is continuous. Clearly, every left invariant topology τ is uniquely determined by the filter τ of neighborhoods of the identity. A group G endowed with a left invariant topology τ is called left topological, and it will be denoted by (G, τ).

Definition 1.5. A filter τ on a group G is called *left topological* if τ is a filter of neighborhoods of the identity for some left invariant topology on G.

Definition 1.6. Given any filter φ on a group G, denote by $\mathrm{int}(\varphi)$ the filter with the base $\{\mathrm{int}(A, \varphi) : A \in \varphi\}$.

Definition 1.7. Given any filter φ on a group G, denote by Cl_φ the mapping determined by the rule $\mathrm{Cl}_\varphi(A) = \mathrm{cl}(A, \varphi)$, $A \subseteq G$.

Theorem 1.3. *For every filter φ on a group G the following statements are equivalent:*

(a) φ is left topological,

(b) φ is pointed and $\varphi\varphi = \varphi$,

(c) $\mathrm{int}(\varphi) = \varphi$,

(d) Cl_φ is a closure operator.

Proof. $(a) \Rightarrow (b)$. Since φ is pointed, $\varphi\varphi \subseteq \varphi$. Take any subset $F \in \varphi$ and choose an open subset $U \subseteq (G, \varphi)$ such that $U \in \varphi$, $U \subseteq F$. For every element $x \in U$, pick a subset $V_x \in \varphi$ with $xV_x \subseteq U$. Put $V = \bigcup_{x \in U} xV_x$. Then $V \in \varphi\varphi$ and $V \subseteq F$. Hence, $\varphi \subseteq \varphi\varphi$.

$(b) \Rightarrow (c)$. See Definitions 1.3 and 1.6. $(b) \Rightarrow (d)$. Apply Lemma 1.1 (a,b,c) and Lemma 1.2 (b). $(d) \Rightarrow (a)$. Denote by $\mathcal{T}$ the topology on G determined by the closure operator Cl_φ. If $F \subseteq G$, $g \in G$ and $\mathrm{Cl}_\varphi(F) = F$ then $\mathrm{Cl}_\varphi(gF) = gF$. This shows that $\mathcal{T}$ is left invariant. □

Show that φ is a filter of neighborhoods of the identity e in $\mathcal{T}$. Take any subset $F \subseteq G$ such that $F = \mathrm{cl}(F, \varphi)$, $e \notin F$. Put $U = G \setminus F$. Since $e \in U$ and $\mathrm{int}(U, \varphi) = U$, we get $U \in \varphi$. Hence, every neighborhood of e in $\mathcal{T}$ is an element of the filter φ.

Take any subset $V \in \varphi$. Since $eV \cap (G \setminus V) = \varnothing$, we have $e \notin \mathrm{cl}(G \setminus V, \varphi)$. Hence, V is a neighborhood of e in $\mathcal{T}$.

Definition 1.8. Let G be a group. Endow G with a discrete topology and denote by βG the Stone-Čech compactification of G. Identify βG with the set of all ultrafilters on G. Given any subset $A \subseteq G$, put $\bar{A} = \{p \in \beta G : A \in p\}$. Then the family $\{\bar{A} : A \subseteq G\}$ forms an open base of the space βG. Identify G with the subspace of all principal ultrafilters on G and put $G^* = \beta G \setminus G$. Then G^* is a closed subsemigroup of the semigroup βG. Given any filter φ on G, put $\bar{\varphi} = \bigcap_{A \subseteq \varphi} \bar{A}$, $\varphi^* = \bar{\varphi} \cap G^*$. By Theorem 1.3 (a,b) is a subsemigroup of βG for every left topological filter φ on G.

Now we point out two ways of construction of left topological filters from an arbitrary filter on G. These constructions will be used in §4.

Lemma 1.4. *Let φ be a filter on a group G and let $p \in \beta G$. Then $p \in \overline{\operatorname{int}\varphi}$ if and only if $\varphi \subseteq p\varphi$.*

Proof. Let $p \in \overline{\operatorname{int}\varphi}$. Take any subset $F \in \varphi$. Since $\operatorname{int}(F,\varphi) \in \operatorname{int}(\varphi)$, we get $\operatorname{int}(F,\varphi) \in p$. By Definition 1.3, $F \in p\varphi$ and thus $\varphi \subseteq p\varphi$.

Let $\varphi \subseteq p\varphi$. Take any subset $F \in \varphi$. Since $F \in p\varphi$, we have $\operatorname{int}(F,\varphi) \in p$. Hence, $p \in \overline{\operatorname{int}\varphi}$. □

Theorem 1.5. *For every filter φ on a group G, the filter $\operatorname{int}\varphi$ is left topological.*

Proof. Since $\operatorname{int}\varphi$ is pointed, $\operatorname{int}\varphi \subseteq \operatorname{int}(\operatorname{int}\varphi)$. By Theorem 1.3 (a,c) , it suffices to show that $\operatorname{int}(\operatorname{int}\varphi) \subseteq \operatorname{int}\varphi$. □

Fix any subset $B \in \operatorname{int}\varphi$ and choose a subset $A \in \varphi$ such that $\operatorname{int}(A,\varphi) \subseteq B$. By Lemma 1.1 (f) and Lemma 1.2 (a) we have

$$\operatorname{int}(\operatorname{int}(A,\varphi),\operatorname{int}\varphi) = \bigcap_{p\in\overline{\operatorname{int}\varphi}} \operatorname{int}(\operatorname{int}(A,\varphi),p) = \bigcap_{p\in\overline{\operatorname{int}\varphi}} \operatorname{int}(A,p\varphi).$$

By Lemma 1.4, $\operatorname{int}(A,\varphi) \subseteq \operatorname{int}(A,p\varphi)$ for every ultrafilter $p \in \overline{\operatorname{int}\varphi}$. Hence, $\operatorname{int}(A,\varphi) \subseteq \operatorname{int}(\operatorname{int}(A,\varphi),\operatorname{int}\varphi) \subseteq \operatorname{int}(B,\operatorname{int}\varphi)$ and $\operatorname{int}(B,\operatorname{int}\varphi) \in \operatorname{int}\varphi$.

Definition 1.9. For every filter φ on a group G denote by $\operatorname{hull}\varphi$ the strongest left topological filter on G such that $\operatorname{hull}\varphi \subseteq \varphi$.

Describe the filter $\operatorname{hull}\varphi$. Put $F_0 = \{e\}$ and suppose that the subsets $F_0,\dots,F_n$ of G have been chosen. For every element $x \in F_n$ pick some subset $F(x) \in \varphi$ and put $F_{n+1} = \bigcup_{x\in F_n} xF(x)$. A subset $\bigcup_{n\in\omega} F_n$ is called a standard neighborhood of e determined by φ. A family of all standard neighborhoods forms a base of some filter ψ on G. Show that $\psi = \operatorname{hull}\varphi$ and every standard neighborhood is open in $(G,\operatorname{hull}\varphi)$.

Take any open neighborhood U of the identity in $(G,\operatorname{hull}\varphi)$. Clearly, $F_0 \subseteq U$ where $F_0 = \{e\}$. Since φ converges to $\{e\}$ in $(G,\operatorname{hull}\varphi)$, we conclude that $U \in \varphi$. Put $F_1 = U$. For every element $x \in F_1$ choose $F(x) \in \varphi$ such that $xF(x) \subseteq U$. Put $F_2 = \bigcup_{x\in F_1} xF(x)$ and so on. We see that $\bigcup_{n\in\omega} F_n \subseteq U$. Hence, $\operatorname{hull}\varphi \subseteq \psi$.

Take any standard neighborhood W from ψ. For every point $x \in W$ we can take a standard neighborhood W_x such that $xW_x \subseteq W$. By Theorem 1.3 (a,c), ψ is a left topological filter. Since $\operatorname{hull}\varphi \subseteq \psi$, we get $\psi = \operatorname{hull}\varphi$. After that the inclusions $xW_x \subseteq W$, $x \in W$ show that W is open in $(G,\operatorname{hull}\varphi)$.

REMARK 1.1. Given any ultrafilter p on a group G, denote by C_p the smallest closed subsemigroup of βG that contains p. Since $\overline{\operatorname{hull} p}$ is a closed subsemigroup of βG and $p \in \overline{\operatorname{hull} p}$ we obtain $C_p \subseteq \overline{\operatorname{hull} p}$. If p has a finite order in the semigroup βG then $C_p = \overline{\operatorname{hull} p}$ [5, Example 3.3]. By [2, Theorem 2.27] $C_p \neq \overline{\operatorname{hull} p}$ for some element p of infinite order in βG.

§2. Refinements

Definition 2.1. Let τ, φ be left topological filters on a group G. We say that (G, φ) is an *open refinement of* (G, τ) if $\tau \subseteq \varphi$ and every open subset of (G, φ) contains an open subset of (G, φ). If $\tau \subsetneq \varphi$, then (G, τ) is called a proper open refinement of (G, τ). A left topological group (G, τ) is called o-maximal if it has no proper open refinements.

By Zorn Lemma, for every left topological group, there exists a maximal open refinement. Every maximal open refinement is o-maximal. To describe these refinements we use some definitions from [8].

Definition 2.2. Let (G, τ) be a left topological group. A filter φ on (G, τ) is called an o-*filter* if $\tau \subseteq \varphi$ and φ has a base consisting of open subsets in (G, τ). A filter on (G, φ) that is maximal in the class of all o-filters is called an o-*ultrafilter*. Given any filter φ on (G, τ), denote by $o(\varphi)$ the strongest o-filter ψ such that $\psi \subseteq \varphi$.

Note that $o(\varphi)$ is nothing else but the filter generated by all open subsets belonging to φ.

Definition 2.3. Let (G, φ) be a left topological group. An ultrafilter $p \in \bar{\tau}$ is called an *absorbing ultrafilter* if $\text{int}(F, \tau) \in p$ for every closed subset $F \in p$.

Theorem 2.1. *Let (G, τ) be a left topological group and let φ be a filter on (G, τ) such that $\tau \subseteq \varphi$. The following statements are equivalent:*

(a) φ is an o-ultrafilter,

(b) $\varphi = o(p)$ for every ultrafilter $p \in \bar{\varphi}$,

(c) every ultrafilter $p \in \bar{\varphi}$ is absorbing.

Proof. [8, Lemma 2.6]. □

Lemma 2.2. *Let τ, φ be left topological filters on a group G, $\tau \subseteq \varphi$. The following statements are equivalent:*

(a) (G, φ) is an open refinement of (G, τ),

(b) there exists an o-ultrafilter ψ on (G, τ) such that $\varphi \subseteq \psi$.

Proof. $(a) \Rightarrow (b)$. Since (G, φ) is an open refinement of (G, τ), $\text{int}(F, \tau) \neq \varnothing$ for every subset $F \in \varphi$. Denote by φ' the filter on G with the base $\{\text{int}(F, \tau) : F \in \varphi\}$. By Zorn Lemma there exists an o-ultrafilter ψ on (G, τ) such that $\varphi' \subseteq \psi$. Clearly, $\varphi \subseteq \psi$.

$(b) \Rightarrow (a)$. Evidently. □

Lemma 2.3. *Let (G, τ) be a left topological group and let φ be an o-filter on (G, τ). Then φ_e is a left topological filter.*

Proof. Apply Theorem 1.3 (a,c). □

Theorem 2.4. *Let τ, φ be left topological filters on a group G, $\tau \subseteq \varphi$. Then (G, φ) is a maximal open refinement of (G, φ) if and only if there exists an o-ultrafilter ψ on (G, τ) such that $\varphi = \psi_e$.*

Proof. Apply Lemma 2.2 and Lemma 2.3. □

Theorem 2.5. *Let (G, τ) be a left topological group. The following statements are equivalent:*

(a) (G, τ) is o-maximal,

(b) $\tau = \psi_e$ for some o-ultrafilter ψ on (G, τ),

(c) for every open subset U in (G, τ) with $e \in \mathrm{cl}(U, \tau)$, the subset $\{e\} \cup U$ is a neighborhood of the identity.

Proof. Apply Theorem 2.4. □

Let (G, τ) be a left topological group and let φ be a filter on G. Then φ is an o-filter if and only if $\varphi\tau = \varphi$. Hence, $\bar{\varphi}$ is a closed right ideal of the semigroup $\bar{\tau}$.

Question 2.1. Let (G, τ) be an o-maximal left topological group and let φ be a filter on G such that $\bar{\varphi}$ is a right ideal of the semigroup $\bar{\tau}$. Is it true that $\tau = \varphi_e$?

Definition 2.4. Let τ, φ be left topological filters on a group G, $\tau \subseteq \varphi$. We say that (G, φ) is a *dense refinement of* (G, τ) if $\mathrm{cl}(U, \tau) \in \tau$ for every subset $U \in \varphi$. If $\tau \subsetneq \varphi$ then (G, φ) is called a *proper dense refinement of* (G, τ). A left topological group is called d-*maximal* if it has no proper dense refinement.

By Zorn Lemma every left topological group has a maximal dense refinement and each such refinement is d-maximal. The next lemma shows a way for construction of dense refinements.

Lemma 2.6. *Let (G, τ) be a left topological group and let φ be a filter on G such that $\tau \subseteq \varphi$. Then $(\tau\varphi)_e$ is a left topological filter and $(G, (\tau\varphi)_e)$ is a dense refinement of (G, τ).*

Proof. To check that $(\tau\varphi)_e$ is a left topological filter we may use Theorem 1.3 (a,b). Show that $(G, (\tau\varphi)_e)$ is a dense refinement of (G, τ). Fix any subset $A \in (\tau\varphi)_e$. Choose the subsets $B \in \tau$, $C_x \in \varphi$, $x \in B$ such that $xC_x \subseteq A$ for every element $x \in B$. Then $B \subseteq \mathrm{cl}(A, \varphi)$. Since $\tau \subseteq \varphi$, we see that $B \in \mathrm{cl}(A, \tau)$. Hence, $\mathrm{cl}(A, \tau) \in \tau$. □

Theorem 2.7. *Let (G, τ) be a d-maximal left topological group. Then $(\tau p)_e = \tau$ for every ultrafilter $p \in \bar{\tau}$.*

Proof. Apply Lemma 2.6. □

Let (G, τ) be a d-maximal left topological group and let L be a left ideal of semigroup $\bar{\tau}$. Take any element $p \in L$. Since $\bar{\tau}p \subseteq L$, by Theorem 2.7 $\bar{\tau} = L \cup \{e\}$.

Question 2.2. Let (G, τ) be a left topological group such that $\bar{\tau} = L \cup \{e\}$ for every left ideal L of semigroup $\bar{\tau}$. Is (G, τ) d-maximal?

Definition 2.5. Let τ, φ be left topological filters on a group G, $\tau \subseteq \varphi$. We say that (G, φ) is an *open-dense refinement of* (G, τ) if it is both open and dense refinement. If $\tau \subset \varphi$ then (G, τ) is called a *proper open-dense refinement.* A left topological group is called od-*maximal* if it has no proper open-dense refinements.

We shall show that every left topological group has unique maximal open-dense refinement and describe its structure.

Lemma 2.8. *Let τ, φ be left topological filters on a group G and let (G, φ) be an open-dense refinement of (G, τ). Then $\bar{\varphi}$ contains all absorbing ultrafilters on (G, τ).*

Proof. Fix any absorbing ultrafilter p on (G,τ) and take any subsets $P \in p$, $U \in \varphi$ such that $U = \mathrm{int}(U,\varphi)$. It suffices to show that $P \cap U \neq \varnothing$. Put $F = \mathrm{cl}(P,\tau)$, $V = \mathrm{int}(F,\tau)$. Since p is absorbing we get $e \in \mathrm{cl}(V,\tau)$. Since (G,φ) is a dense refinement of (G,τ), we get $U \cap V \neq \varnothing$. Since $U \cap V$ is open in (G,φ) and (G,φ) is an open refinement of (G,τ), there exists an open subset W in (G,τ) such that $W \subseteq U \cap V$. Clearly, $W \cap P \neq \varnothing$. Hence, $P \cap U \neq \varnothing$. □

Definition 2.6. Given any left topological filter φ on a group G, consider the strongest filter ψ on G containing all o-filters on (G,τ). Call ψ the *open core of* τ.

Clearly, ψ is the strongest filter containing all o-ultrafilters on (G,τ). By Theorem 2.1, $p \in \bar{\psi}$ if and only if p is an absorbing ultrafilter on (G,τ). By Theorem 1.3 (a,b), ψ_e is a left topological filter. For every subset $A \subseteq G$, we have $A \in \psi$ if and only if $\mathrm{int}(A \cap U,\tau) \neq \varnothing$ for every open subset U in (G,τ) such that $e \in \mathrm{cl}(U,\tau)$.

Theorem 2.9. *Let (G,τ) be a left topological group and let ψ be an open core of τ. Then (G,ψ_e) is unique maximal open-dense refinement of (G,τ).*

Proof. Apply Lemma 2.8 and above paragraph. □

Corollary 2.10. *A left topological group (G,τ) is od-maximal if and only if $\tau = \psi_e$ where ψ is an open core of τ.*

Corollary 2.11. *Every o-maximal left topological group is od-maximal.*

Proof. Apply Theorem 2.5 and Corollary 2.10. □

Corollary 2.12. *Every d-maximal left topological group (G,τ) is od-maximal.*

Proof. We may suppose that (G,τ) is nondiscrete. By Theorem 2.7, τ^* is a minimal left ideal of the semigroup $\bar{\tau}$. By [9, Theorem 3.11] every ultrafilter from a minimal left ideal of $\bar{\tau}$ is absorbing. Apply Corollary 2.10. □

Remark 2.1. The following cardinal invariants of left topological groups are stable under any open refinements: density, cellularity, boundedness index. Under the bounded index of a left topological group (G,τ) we understand the smallest cardinal α such that, for every subset $U \in \tau$, there exists a subset $K \subseteq G$ such that $G = KU$ and $|K| < \alpha$. All these invariants could increase under dense refinements.

§3. Nodec and irresolvable topologies

Definition 3.1. A left topological group (G,τ) is called *nodec* if every nowhere dense subset in (G,τ) is closed.

Note that every nowhere dense subset of a nodec group is discrete.

Theorem 3.1. *For every left topological group (G,τ) the following statements are equivalent:*

(a) (G,τ) is nodec,

(b) every ultrafilter $p \in \tau^*$ is absorbing,

(c) (G, τ) is od-maximal.

Proof. $(a) \Rightarrow (b)$. Compare Definition 3.1 and Definition 2.3.
$(b) \Rightarrow (c)$. Apply Corollary 2.10. □

Corollary 3.2. *Every o-maximal left topological group is nodec.*

Proof. Apply Theorem 3.1 and Corollary 2.11. □

Corollary 3.3. *Every d-maximal left topological group is nodec.*

Proof. Apply Theorem 3.1 and Corollary 2.12. □

Definition 3.2. A left topological group (G, τ) is called *irresolvable* if it can not be partitioned into two dense subsets. A left topological group is called submaximal if every dense subset in (G, τ) is open.

Clearly, every submaximal left topological group is irresolvable and nodec. By [10, Theorem 1.3], (G, τ) is irresolvable if and only if every o-ultrafilter on (G, τ) is an ultrafilter. By [10, Theorem 1.5] (G, τ) is submaximal if and only if every ultrafilter from τ^* is an o-ultrafilter.

Theorem 3.4. *A left topological group is submaximal if and only if it is irresolvable and nodec.*

Proof. Apply Theorem 3.1 (a,c) and Corollary 2.10. □

Using the standard topological arguments the referee noted that Theorem 3.4 is valid for every topologically homogeneous space.

Theorem 3.5. *A nondiscrete left topological group is maximal if and only if it is o-maximal and nodec.*

Proof. Apply Theorem 2.5. □

Theorem 3.6. *Let (G, τ) be an irresolvable left topological group. Then the semigroup $\bar{\tau}$ has only one minimal left ideal.*

Proof. Take an ultrafilter $p \in \bar{\tau}$ with base consisting of open subsets. Since $\{p\}$ is a right ideal of and every right ideal intersects every left ideal, then p belongs to every minimal left ideal. □

For applications of Theorem 3.6 and some more general results see [11]. By Theorem 2.7, $\bar{\tau}$ has only one minimal left ideal for every d-maximal left topological group (G, τ).

Question 3.1. Let (G, τ) be a d-maximal left topological group. Is (G, τ) irresolvable?

Example 3.1. We construct a resolvable left topological group (G, τ) such that the semigroup $\bar{\tau}$ has only one minimal left ideal. This example will show that the conversion of Theorem 3.6 is not true. Put $G = \mathbb{Z}$ and denote by φ the filter on $\mathbb{Z}$ with the base $\{2^n\mathbb{Z} : n \in \omega\}$. Take any idempotent p from the minimal ideal of the semigroup $\bar{\varphi}$. Put $\tau = (\varphi p)_e$. By Theorem 3.1, τ is a left topological filter on G. Since $\bar{\tau} = \bar{\varphi}p \cup \{e\}$, the semigroup $\bar{\tau}$ has only one minimal left ideal $\bar{\varphi}p$. By [1, Chapter 7], $\bar{\varphi}p$ contains an infinite free subgroup. By [10] the minimal ideal of every irresolvable left topological group is a left zero semigroup. Hence, (G, τ) is resolvable.

§4. Extremely disconnected topologies

Definition 4.1. A left topological group (G,τ) is called *extremely disconnected* if one of the following equivalent statement holds:

(a) the closure of every open subset of (G,τ) is open,

(b) if U is an open subset and $g\in \mathrm{cl}(U,\tau)$ then $\mathrm{cl}(U,\tau)$ is a neighborhood of g,

(c) there exists only one o-ultrafilter on (G,τ),

(d) if p is an absorbing ultrafilter on (G,τ) then $\mathrm{cl}(U,\tau)\in\tau$ for every subset $U\in p$.

Let (G,τ) be an extremely disconnected left topological group and let φ be an open core of τ. By Definition 4.1 (c) φ is an o-ultrafilter on (G,τ). By Theorem 2.5 (G,φ_e) is unique maximal open refinement of (G,τ) and (G,φ_e) is extremely disconnected. In what follows we show how to construct the filter τ from its open core φ.

Remind that a topological space is called zero-dimensional if every its point has a base consisting of the clopen (=closed and open) subsets. It is worth to mention that each zero-dimensional space is regular and each regular extremally disconnected space is zero-dimensional.

Theorem 4.1. *Let (G,τ) be a left topological group and let φ be an o-ultrafilter on (G,τ). Then the left topological group $(G,\mathrm{int}\,\varphi)$ is zero-dimensional and extremely disconnected.*

Proof. Take any ultrafilter $q\in\overline{\mathrm{int}\,\varphi}$. By Lemma 1.4 $\varphi\subseteq q\varphi$. Since $q\varphi$ is an o-filter then $\varphi=q\varphi$. Hence $\overline{\mathrm{int}\,\varphi}=\{q\in\beta G: q\varphi=\varphi\}$.

Fix any subset $A\in\varphi$ and show that the subset $\mathrm{int}(A,\varphi)$ is clopen in $(G,\mathrm{int}\,\varphi)$ so $(G,\mathrm{int}\,\varphi)$ is zero-dimensional. By Lemma 1.1 (d,g) and Lemma 1.2 (a) we have

$$\mathrm{cl}(\mathrm{int}(A,\varphi),\mathrm{int}\,\varphi)=\bigcup_{q\in\overline{\mathrm{int}\,\varphi}}\mathrm{cl}(\mathrm{int}(A,\varphi),q)=$$

$$=\bigcup_{q\in\overline{\mathrm{int}\,\varphi}}\mathrm{int}(\mathrm{int}(A,\varphi),q)=\bigcup_{q\in\overline{\mathrm{int}\,\varphi}}\mathrm{int}(A,q\varphi)=\mathrm{int}(A,\varphi).$$

By Lemma 1.3 (f) and Lemma 1.2 (a) we have

$$\mathrm{int}(\mathrm{int}(A,\varphi),\mathrm{int}\,\varphi)=\bigcap_{q\in\overline{\mathrm{int}\,\varphi}}\mathrm{int}(\mathrm{int}(A,\varphi),q)=\bigcap_{q\in\overline{\mathrm{int}\,\varphi}}\mathrm{int}(A,q\varphi)=\mathrm{int}(A,\varphi).$$

Let U be an open subset in $(G,\mathrm{int}\,\varphi)$, $e\in\mathrm{cl}(U,\mathrm{int}\,\varphi)$. Show that $\mathrm{cl}(U,\mathrm{int}\,\varphi)\in\mathrm{int}\,\varphi$.

Note that U is open in (G,φ_e) and $q\varphi=\varphi$ for every $q\in\overline{\mathrm{int}\,\varphi}$. This implies $\mathrm{cl}(U,\mathrm{int}\,\varphi)=\mathrm{int}(U,\varphi)$.

Since $e\in\mathrm{cl}(U,\mathrm{int}\,\varphi)$, we have $U\in\varphi$. Hence $\mathrm{int}(U,\varphi)\in\mathrm{int}\,\varphi$ and $\mathrm{cl}(U,\mathrm{int}\,\varphi)\in\mathrm{int}\,\varphi$. □

Remark 4.1. Theorem 4.1 is valid in the following more general form. Let φ be a filter on a group G such that $\mathrm{int}\,\varphi\subseteq\varphi$ and $q\varphi=\varphi$ for every ultrafilter $q\in\overline{\mathrm{int}\,\varphi}$. Then the left topological group $(G,\mathrm{int}\,\varphi)$ is zero-dimensional and extremally disconnected.

Remark 4.2. In notation of Theorem 4.1 let ψ be an open core of $\mathrm{int}\,\varphi$. Clearly, $\psi\subseteq\varphi$ but the equality $\psi=\varphi$ could be wrong even in the case $\tau=\varphi_e$.

Theorem 4.2. *Let (G, τ) be a left topological group and let φ be an o-ultrafilter on (G, τ). Then (G, τ) is extremely disconnected if and only if* $\operatorname{int}\varphi \subseteq \tau$.

Proof. Let (G, τ) be extremely disconnected but $\operatorname{int}\varphi \not\subseteq \tau$. Choose an ultrafilter $q \in \bar{\tau} \setminus \overline{\operatorname{int}\varphi}$. By Lemma 1.4 $\varphi \not\subseteq q\varphi$. Take an open subset $U \in \varphi$ such that $U \notin q\varphi$. Choose a subset $Q \in q$ such that $Q \cap \operatorname{int}(U, \varphi) = \varnothing$. Since U is open and φ is an o-ultrafilter, for every element $x \in Q$, we can take an open subset $V_x \in \varphi$ such that $xV_x \cap U = \varnothing$. Put $V = \bigcup_{x \in Q} xV_x$. Then U, V are open subsets of (G, τ), $U \cap V = \varnothing$ and $e \in \operatorname{cl}(U, \tau)$, $e \in \operatorname{cl}(V, \tau)$, a contradiction with extremal disconnectedness of (G, τ).

Suppose that $\operatorname{int}\varphi \subseteq \tau$ but (G, τ) is not extremely disconnected. Then there exist disjoint open subsets U, V of (G, τ) such that $U \in \varphi$, $e \in \operatorname{cl}(V, \tau)$. Clearly, $V \cap \operatorname{int}(U, \varphi) = \varnothing$ so $\operatorname{int}(U, \varphi) \notin \tau$, a contradiction with $\operatorname{int}(U, \varphi)$ and $\operatorname{int}\varphi \subseteq \tau$. □

Remind that a topological space is called regular if every its point has a base of neighborhoods consisting of the closed subsets.

Theorem 4.3. *Let (G, τ) be a regular left topological group and let φ be an o-ultrafilter on (G, τ). Then (G, τ) is extremely disconnected if and only if $\tau = \operatorname{int}\varphi$.*

Proof. Apply Theorem 4.1 and Theorem 4.2. □

Theorem 4.4. *Every regular left topological group has a zero-dimensional extremely disconnected open refinement.*

Proof. Apply Theorem 4.1. □

Theorem 4.5. *Let G be a group and let p be an idempotent of semigroup βG. Then a left topological group $(G, \operatorname{int} p)$ is Hausdorff zero-dimensional and extremely disconnected.*

Proof. Let τ be a filter on G such that $\bar{\tau} = \{p, e\}$. By Theorem 1.3 τ is left topological. Clearly, p is an o-ultrafilter on (G, τ). By Theorem 4.1 $(G, \operatorname{int} p)$ is zero-dimensional and extremely disconnected. Suppose that an ultrafilter q converges to e and to some element x. Then $q \in \operatorname{int} p$, $x^{-1}q \in \operatorname{int} p$. Hence, $qp = p$ and $x^{-1}qp = p$ so $x^{-1} = e$. This shows that $(G, \operatorname{int} p)$ is Hausdorff. □

Definition 4.2. A regular left topological group (G, φ) is called a *bounded refinement of a regular left topological group* (G, τ) if $\tau \subseteq \varphi$ and, for every subset $U \in \varphi$, there exists a finite subset $K \subseteq G$ such that $KU \subseteq \tau$. If $\tau \subset \varphi$ then (G, φ) is called a proper bounded refinement of (G, τ). We say that (G, τ) is b-maximal if it has no proper bounded refinement.

By Zorn Lemma every regular left topological group has a maximal bounded refinement. Clearly, every maximal bounded refinement is b-maximal. For characterization of b-maximal left topological group we need the following description of ultrafilters from the minimal ideal of an arbitrary closed subsemigroup of βG.

Theorem 4.6. *Let G be a group and let φ be a filter on G such that is a subsemigroup of the semigroup βG. Then an ultrafilter p belongs to the minimal ideal of $\bar{\varphi}$ if and only if, for every subset $A \in \operatorname{int} p$, there exists a finite subset $K \subseteq G$ such that $KA \in \varphi$.*

Proof. [9, Theorem 3.1] or [7, §11]. □

Lemma 4.7. *Let (G,τ) be a regular left topological group and let p be an idempotent from the minimal ideal of semigroup $\bar{\tau}$. Then $(G,\operatorname{int} p)$ is a bounded refinement of (G,τ).*

Proof. Apply Theorem 4.5 and Theorem 4.6. □

Theorem 4.8. *A regular left topological group (G,φ) is a maximal bounded refinement of a regular left topological group (G,τ) if and only if there exists an idempotent p from the minimal ideal of semigroup $\bar{\tau}$ such that $\varphi = \operatorname{int} p$.*

Proof. Suppose that (G,φ) is a maximal refinement of (G,τ). Fix any idempotent p from the minimal ideal of $\bar{\varphi}$. By Lemma 4.7 $\varphi = \operatorname{int} p$. Since (G,φ) is a bounded refinement of (G,τ), by Theorem 4.6, p belongs to the minimal ideal of $\bar{\tau}$.

Assume that $\varphi = \operatorname{int} p$ for some idempotent p from the minimal ideal of $\bar{\tau}$. By Lemma 4.7 (G,φ) is a bounded refinement of (G,τ). Note that $\bar{\varphi} = \{x \in \beta G : xp = p\}$. Take any bounded refinement (G,ψ) of (G,φ). By Lemma 4.7 we may suppose that $\bar{\psi} = \{x \in \beta G : xq = q\}$ for some idempotent q from the minimal ideal of $\bar{\psi}$. Since (G,ψ) is a bounded refinement of (G,φ), by Theorem 4.6, q belongs to the minimal ideal of $\bar{\varphi}$. Since $\{p\}$ is a minimal left ideal of semigroup $\bar{\varphi}$, we have $\{px : x \in \bar{\varphi}\}$ is a minimal ideal of $\bar{\varphi}$. Hence, $pq = q$ and $\bar{\varphi} = \bar{\psi}$. This proves maximality of (G,φ). □

Corollary 4.9. *A regular left topological group (G,τ) is b-maximal if and only if $\tau = \operatorname{int} p$ for some idempotent p from the minimal ideal of semigroup $\bar{\tau}$.*

Corollary 4.10. *Every regular left topological group admits a Hausdorff zero-dimensional extremely disconnected bounded refinement.*

Definition 4.3. A left topological group (G,τ) is called *totally bounded* if, for every subset $U \in \tau$, there exists a finite subset $K \subseteq G$ such that $G = KU$.

Corollary 4.11. *Every group G admits a Hausdorff totally bounded zero-dimensional extremely disconnected left invariant topology.*

Proof. Put $\tau = \{G\}$ and apply Corollary 4.10. □

Remark 4.3. Evidently, every regular open refinement of a regular left topological group is bounded, but there exists a regular o-maximal group that is not b-maximal. The simplest example is (G,τ) where $\tau = \{G\}$.

Remark 4.4. Let (G,τ) be a Hausdorff regular nondiscrete left topological group. By [6] a left topological group (G,φ) is a maximal regular nondiscrete refinement of (G,τ) if and only if there exists a right maximal idempotent p of semigroup τ^* such that $\varphi = \operatorname{int} p$.

Remark 4.5. Definition 4.2 is a left topological version of the following definition from [9]. A topological group (G,φ) is called a TB-refinement of a topological group (G,τ) if, for every subset $U \in \varphi$, there exists a finite subset $K \subseteq G$ such that $KU \in \tau$. For every topological group (G,τ), there exists a unique maximal TB-refinement that is called a TB-modification of (G,τ). For the construction and applications of TB-modifications see [9, §§7,8]. Note that a maximal bounded refinement of a regular left topological group need not to be unique.

Remark 4.6. We can drop the word "regular" in Definition 4.2. By Zorn Lemma there exist the maximal refinements in this sense, but we can not give a characterization of these refinements.

Definition 4.4. A left topological group (G,τ) is called *strongly extremely disconnected* if, for every open nonclosed subset U in (G,τ), there exists an element $g\in \mathrm{cl}(U,\tau)\setminus U$ such that $\{g\}\cup U$ is a neighborhood of g.

Show that every strongly extremely disconnected group (G,τ) is extremely disconnected. Suppose not and choose an open subset U such that $\mathrm{cl}(U,\tau)$ is not a neighborhood of some element $g\in \mathrm{cl}(U,\tau)$. Put $V=G\setminus \mathrm{cl}(U,\tau)$. Then $g\in \mathrm{cl}(V,\tau)\setminus V$ so V is an open nonclosed subset. Since (G,τ) is strongly extremely disconnected there exists $h\in \mathrm{cl}(V,\tau)\setminus V$ such that $\{h\}\cup V$ is a neighborhood of h, a contradiction with $h\in \mathrm{cl}(U,\tau)\setminus U$.

By Theorem 2.5 every o-maximal left topological group is strongly extremely disconnected. Using Definition 1.9 we show another way to produce the strongly extremely disconnected left topological groups.

Theorem 4.12. *For every group G and for every ultrafilter p on G, the left topological group $(G,\mathrm{hull}\,p)$ is strongly extremely disconnected.*

Proof. Let U be an open nonclosed subset in $(G,\mathrm{hull}\,p)$. Fix any point $x\in \mathrm{cl}\,U\setminus U$. Suppose that there exists a subset $P\in p$ such that $xP\subseteq U$. Then there is a standard neighborhood V of e in $(G,\mathrm{hull}\,p)$ such that $xV\subseteq U\cup\{x\}$. Hence $\{x\}\cup U$ is open.

Assume that $xP\cap U=\varnothing$ for some subset $P\in p$. We can take a subset $P_1\in p$ such that $xP_1\subseteq \mathrm{cl}\,U\setminus U$. Otherwise there exists a standard neighborhood of x which does not intersect U. Take any point $y\in xP_1$. If $yP\subseteq U$ for some subset $P\in p$ then $\{y\}\cup U$ is open. Otherwise, for every point $y=xa$, $a\in P_1$, pick a subset $P_a\in p$ with $xaP_a\cap U=\varnothing$, $xaP_a\subseteq \mathrm{cl}\,U\setminus U$. Put $P_2=\bigcup_{a\in P_1} aP_a$. Consider the set $x(P_1\cup P_2)$ and repeat this argument. Since $x\in \mathrm{cl}\,U$, this process will stop on some finite step. □

Definition 4.5. The left topological groups (G_1,τ_1) and (G_2,τ_2) are called *locally isomorphic* if there exist an open neighborhood U_1 of the identity e_1 in (G_1,τ_1), an open neighborhood U_2 of the identity e_2 in (G_2,τ_2) and a homeomorphism $f\colon U_1\to U_2$ such that, for every $x\in U_1$, there exists a neighborhood V of e_1 in (G_1,τ_1) such that $xV\subseteq U_1$ and $f(xy)=f(x)f(y)$ for every element $y\in V$. If (G_1,τ_1) and (G_2,τ_2) are locally isomorphic, then the semigroups $\bar\tau_1$, $\bar\tau_2$ are topologically isomorphic.

Definition 4.6. Let G be a group and let $p,q\in\beta G$. We say that p,q are *locally isomorphic* if the left topological groups $(G,\mathrm{hull}\,p)$, $(G,\mathrm{hull}\,q)$ are locally isomorphic.

The problem of classification of ultrafilters on a group up to the local isomorphism is very difficult. We shall announce a solution of this problem for a class of right cancellable ultrafilters on countable groups.

Definition 4.7. An ultrafilter p on a group G is called *right cancellable* if, for any ultrafilters q,r on G, the equation $qp=rp$ implies $q=r$.

An ultrafilter p on a countable group G is right cancellable if and only if there exists a subset $P\in p$ such that the identity of G is only a limit point of subset P in $(G,\mathrm{hull}\,p)$. For every right cancellable ultrafilter p on a countable group G the left topological group $(G,\mathrm{hull}\,p)$ is Hausdorff and zero-dimensional.

Theorem 4.13. *Let G be a countable group and let p,q be the right cancellable ultrafilters on G. Then p,q are locally isomorphic if and only if there exists a bijection $f\colon G\to G$ such that $f(p)=q$ where $f(p)$ is an ultrafilter with the base $\{f(P): P\in p\}$.*

The following theorem shows that, given any ultrafilter p on an arbitrary group G, we could hardly expect more continuity in $(G, \operatorname{hull} p)$ than the continuity of shifts. We omit a proof referring the reader to [12] for a method.

Theorem 4.14. *Let G be an Abelian group and let $m \in \mathbb{Z}$, $m \neq 0, 1$. Suppose that the subgroup $\{x \in G : m(m-1)x = 0\}$ is finite. Then the mapping $x \to mx$ is not continuous in $(G, \operatorname{hull} p)$ for every ultrafilter $p \in G^*$.*

Remind that every group of period 2 is called a Boolean group. Clearly, every Boolean group is Abelian.

Example 4.1. Let G be an infinite Boolean group and let p be a Ramsey ultrafilter on G. Show that $(G, \operatorname{hull} p)$ is a topological group. For every subset $P \in p$ denote by $\langle P \rangle$ the subgroup of G generated by P. Denote by τ the filter on G with the base $\{\langle P \rangle : P \in p\}$. Clearly, (G, τ) is a topological group and $\tau \subseteq \operatorname{hull} p$. Take any standard neighborhood $\bigcup_{n\in\omega} F_n$ of zero in $(G, \operatorname{hull} p)$. For every $n > 0$ denote by $[G]^n$ the family of all n-subsets of G and define the coloring $\chi_n \colon [G]^n \to \{0, 1\}$ by the rule $\chi_n(g_1, \ldots, g_n) = 1$ if and only if $g_1 + \cdots + g_n \in F_n$. Since p is a Ramsey ultrafilter, there exists a subset $P_n \in p$ with monochrome subset $[P_n]^n$. Clearly, $\chi_n(A) = 1$ for every subset $A \in [P_n]^n$. We may assume that $P_1 \supset P_2 \supset \cdots \supset P_n \supset \ldots$. Since p is selective, there exists $P \in p$ such that $P \subseteq P_1$ and $|P \cap (P_n \setminus P_{n+1})| \leq 1$ for every $n > 0$. Then $\langle P \rangle \subseteq \bigcup_{n\in\omega} F_n$. This shows that $\operatorname{hull} p \subseteq \tau$.

Remark 4.7. The above example is not novelty indeed. The first example of nondiscrete extremely disconnected group topology on a countable Boolean group $G = \bigoplus_\omega \mathbb{Z}_2$ was constructed by Sirota [13]. First, using CH he constructed a Ramsey ultrafilter q on ω. Second, for every $Q \in q$, he put $S(Q) = \{x \in G : \operatorname{supp} x \subseteq Q\}$. Third, he proved that the group topology on G with the family $\{S(Q) : Q \in q\}$ as a base of neighborhoods of zero is extremely disconnected. Example 4.1 shows that Sirota's topology is the strongest left invariant topology on G in which some Ramsey ultrafilter converges to zero.

Remark 4.8. A filter φ on a left topological group (G, τ) is called c-filter if $\tau \subseteq \varphi$ and φ has a base consisting of the subsets closed in the subspace $G \setminus \{e\}$. A filter that is maximal in the class of all c-filters is called c-ultrafilter. Theorem 4.12 is valid with the same proof in the following more general form. Let (G, τ) be a left topological group and let φ be a c-ultrafilter on G. Then the left topological group $(G, \operatorname{hull} \varphi)$ is strongly extremely disconnected.

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Received 7.06.2000