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ON SINGLE-VALUED SOLUTIONS OF EULER-POISSON'S EQUATIONS

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In this paper necessary conditions of the existence of single-valued (on $\mathbb{C}$) solutions to Euler-Poisson's equations are obtained.

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В статье получены необходимые условия существования однозначных (в $\mathbb{C}$) решений уравнений Эйлера-Пуассона.

§ **0. Introduction.** As is known, all solutions of the problem on the moving of solids are single-valued in cases found by Euler, Lagrange and Kovalevskaya (see, for example, [1]). Moreover, the last case was found as new one with single-valued solutions. After Kovalevskaya there were found series of new partial cases of integrability by other methods different from the method of Kovalevskaya (see [2] and references therein).

Thus it is natural to find partial cases of integrability as the cases which allow the existence of some single-valued solutions. In the papers [3, 4] an approach developing the method of Kovalevskaya was proposed. It allowed to find the asymptotics of singular points of the solutions to Euler-Poisson's equations [4]. Proceeding from these asymptotics we obtain in this paper the conditions on the parameters of the solid such that Euler-Poisson's equations admit single-valued solutions. These conditions are given as a countable set of polynomial relations. Moreover, it is shown that a single-valued solution, if any, can be given as a linear combination of ζ-functions and $\wp$-functions of Weierstrass [5].

Let us represent Euler-Poisson's equations in the following form :

$$\begin{cases} A\,\dot{p} &= Ap \times p + \gamma \times r, \\ \dot{\gamma} &= \gamma \times p, \end{cases} \tag{0.1}$$

here $p = (p_1, p_2, p_3) \in \mathbb{C}^3$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3$, $Ap = (A_1p_1, A_2p_2, A_3p_3)$, $r = (r_1, r_2, r_3) \in \mathbb{R}^3$, $A_i > 0$.

We also use the notation $z(t) = (p(t), \gamma(t))$.

Define $\mathbb{C}$-scalar product in $\mathbb{C}^3$: $\langle x, y\rangle = \sum_{i=1}^3 x_i y_i$.

We use the cyclic permutation of indices $\sigma = (1, 2, 3)$ for writing the product or sums (for example, $\sum_\sigma A_1A_2 = A_1A_2 + A_2A_3 + A_3A_1$, $\prod_\sigma A_1 = A_1A_2A_3$) and expressions which

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differ one from another only in the cyclic permutation of indices ($\dot{\gamma} = \gamma \times p$, can be written as $\gamma_1 = p_3\gamma_2 - p_2\gamma_3, \sigma$).

Introduce also the notations $B_{ij} = A_i - A_j$, $C_{ij} = 2A_i - A_j$.

§ **1. The singular points of the solutions of Euler-Poisson's equations.** The proofs of the propositions and theorems of this section see in [4, 6].

Let $t_* \in \mathbb{C}$ be a singular point of the solutions $z(t)$ of the system (0.1) (i.e. t_* is a singular point of one of the coordinate functions of $z(t)$). Get rid of the branching at t_*, if any, by the representation $z(t) = \hat{z}\,(\mathrm{Ln}(t - t_*))$, where $\hat{z}\,(\tau)$ is a single-valued function as $\mathrm{Re}\,\tau \to -\infty$.

The system (0.1) is transformed into:

$$\begin{cases} A\,\dot{\hat{p}} & = e^{\tau}(A\hat{p} \times \hat{p} + \hat{\gamma} \times r), \\ \dot{\hat{\gamma}} & = e^{\gamma}(\hat{\gamma} \times \hat{p}), \end{cases}$$

where the derivative is taken by τ.

In order to make the right part of the equation independent of τ we make replacement of variable again, setting $\tilde{p}\,(\tau) = e^{\tau}\,\hat{p}\,(\tau)$, $\tilde{\gamma}\,(\tau) = e^{2\tau}\,\hat{\gamma}\,(\tau)$ and then we have :

$$\begin{cases} A\,\dot{\tilde{p}} & = A\,\tilde{p} \times \tilde{p} + \tilde{\gamma} \times r + A\,\tilde{p}, \\ \dot{\tilde{\gamma}} & = \tilde{\gamma} \times \tilde{p} + 2\,\tilde{\gamma}\,. \end{cases} \tag{1.1}$$

In this case the dependence on solutions of (0.1) and (1.1) is expressed by the relations

$$p(t) = \frac{1}{t - t_*}\,\tilde{p}\,(\mathrm{Ln}(t - t_*)), \quad \gamma(t) = \frac{1}{(t - t_*)^2}\,\tilde{\gamma}\,(\mathrm{Ln}(t - t_*)) \tag{1.2}$$

Proposition 1. *The solutions $p(t)$, $\gamma(t)$ of system (0.1) have not the singularity at the point t_* if and only if the corresponding solution (1.1) by (1.2) have the asymptotic behaviour $\tilde{p}\,(\tau) \sim \tilde{p}_0\, e^{\tau}$, $\tilde{\gamma}\,(\tau) \sim \tilde{\gamma}_0\, e^{2\tau}$, when $Re\ \tau \to -\infty$.*

Remark 1. The solutions $\tilde{z}(\tau)$ which have not the asymptotic behaviour $(\tilde{p}_0 e^{\tau}, \tilde{\gamma}_0 e^{2\tau})$, $\mathrm{Re}\,\tau \to -\infty$, are first, the constant solutions and, second, have trajectories entering singular points.

This fact is fundamental: we can investigate completely the singular points of the differential equation but at the same time apriori we cannot say that all singular points of solutions of (0.1) can be obtained in such a way.

Definition 1. We call the system (the solution of which are singular points of (1.1))

$$\begin{cases} A\tilde{p}^0 \times \tilde{p}^0 + \tilde{\gamma}^0 \times r + A\tilde{p}^0 = 0, \\ \tilde{\gamma}^0 \times \tilde{p}^0 + 2\tilde{\gamma}^0 = 0. \end{cases} \tag{1.3}$$

characteristic (for Euler-Poisson's equations).

Now for convenience we write (p, γ) instead of $(\tilde{p}^0, \tilde{\gamma}^0)$ in this section.

Remark 1.2. By condition $r_3 = 0$ characteristic system (1.3) has symmetry

$$S_3 : (p_1, p_2, p_3, \gamma_1, \gamma_2, \gamma_3) \longleftrightarrow (-p_1, -p_2, p_3, \gamma_1, \gamma_2, -\gamma_3).$$

Theorem 1. *Characteristic system (1.3) has the following solutions*
0) $(p,\gamma)=0$,
1) by conditions $\prod_\sigma B_{12}\neq 0$,

$$\gamma_1=0,\ p_1=\sqrt{\frac{A_2A_3}{B_{12}B_{31}}},\sigma \tag{1.4}$$

here, if (p_1,p_2,p_3) *is a solution of (1.4) then the other solutions are*

$$(-p_1,-p_2,p_3),\quad (-p_1,p_2,-p_3),\quad (p_1,-p_2,-p_3),$$

moreover,

(a) by condition $\prod_\sigma r_1\neq 0$, *8 solutions (with account of the multiplicities of the roots) which can be obtained as follows: if* ξ *is a root of the polynomial*

$$(\xi^2+1)(r_1B_{23}(C_{21}B_{31}+C_{31}B_{12}+C_{31}B_{12}\xi^2)-r_3B_{12}\xi(C_{13}B_{23}+C_{13}B_{23}\xi^2+ \\ +C_{23}B_{31}\xi^2))^2+(r_2B_{31}\xi(C_{32}B_{12}\xi^2-C_{12}B_{23}))^2=0,$$

then

$$p_1=\frac{2B_{23}\sqrt{-\xi^2-1}}{B_{12}\xi^2-B_{23}},\ p_2=\frac{2B_{31}\xi}{B_{12}\xi^2-B_{23}},\ p_3=\frac{2B_{12}\xi\sqrt{-1-\xi^2}}{B_{12}\xi^2-B_{23}},$$

$$\gamma_1=\lambda\xi,\ \gamma_2=\lambda\sqrt{-1-\xi^2},\ \gamma_3=\lambda,\lambda=-\frac{\langle Ap,p\rangle}{2\left(r_1\xi+r_2\sqrt{-1-\xi^2}+r_3\right)}; \tag{1.5}$$

(b) by condition $r_3=0$, r_1, $r_2\neq 0$, *2 solutions lying on the axes of symmetry* S_3

$$p_1=p_2=\gamma_3=0,\ p_3=\pm 2i,\ \gamma_1=\frac{2A_3}{r_1\pm ir_2},\ \gamma_2=\pm\frac{2A_3i}{r_1\pm ir_2},$$

(b1) by condition $r_1^2\ B_{23}=r_2^2\ B_{31}$, *the pair of the* S_3*-symmetric solutions*

$$p_1=\mp\frac{2A_2i}{(A_1+A_2)\,B_{12}}\sqrt{A_1^2\frac{B_{23}}{B_{31}}+A_2^2},\ p_2=\mp\frac{2A_1i}{(A_1+A_2)\,B_{12}}\sqrt{A_1^2+A_2^2\frac{B_{31}}{B_{23}}},$$

$$p_3=\frac{2A_1A_2}{(A_1+A_2)\sqrt{B_{23}B_{31}}},\ \gamma_1=\frac{2A_1^2A_2}{(A_1+A_2)^2\ r_1},\ \gamma_2=\frac{2A_1A_2^2}{(A_1+A_2)^2\ r_2},$$

$$\gamma_3=\mp\frac{2A_1A_2i}{(A_1+A_2)^2\ r_1}\sqrt{A_1+A_2^2\frac{B_{31}}{B_{23}}},$$

(b2) by condition $r_1^2\ B_{23}\neq r_2^2\ B_{31}$, *3 pairs of the* S_3*-symmetric solutions which can be obtained as follows: we find the roots* $\zeta=\gamma_1/\gamma_2$ *of the polynomial of the 3rd power*

$$r_2B_{31}\zeta\left(A_2B_{31}\zeta-C_{12}B_{23}\right)+r_1B_{23}\left(C_{21}B_{31}\zeta^2-A_1B_{23}\right)=0,$$

and then use the relations $\xi=\frac{\zeta}{\sqrt{-1-\zeta^2}}$ *and (1.5),* *(c) by the condition $r_2 = r_3 = 0, r_1 \neq 0$, 4 solutions lying on the axes of S_2, S_3-symmetries*

$$p_1 = p_2 = \gamma_3 = 0,\ p_3 = \pm 2i,\ \gamma_1 = \frac{2A_3}{r_1},\ \gamma_2 = \pm\frac{2A_3 i}{r_1},$$
$$p_1 = p_3 = \gamma_2 = 0,\ p_2 = \pm 2i,\ \gamma_1 = \frac{2A_2}{r_1},\ \gamma_3 = \mp\frac{2A_2 i}{r_1},$$

and the 4 S_2, S_3-symmetric solutions

$$p_1 = -\frac{\sqrt{C_{21}B_{31}}\sqrt{C_{31}B_{12}}}{B_{12}B_{31}},\ p_2 = -\frac{\sqrt{A_1B_{23}}\sqrt{C_{31}B_{12}}}{B_{12}B_{23}},\ p_3 = -\frac{\sqrt{C_{21}B_{31}}\sqrt{A_1B_{23}}}{B_{23}B_{31}},$$

$$\gamma_1 = \frac{A_1}{r_1},\ \gamma_2 = \frac{A_1\sqrt{C_{21}B_{31}}}{r_1\sqrt{A_1B_{23}}},\ \gamma_3 = \frac{A_1\sqrt{C_{31}B_{12}}}{r_1\sqrt{A_1B_{23}}},$$

here, the signs of $\sqrt{C_{21}B_{31}}, \sqrt{C_{31}B_{12}}, \sqrt{A_1B_{23}}$ can be taken arbitrarily but simultaneously in all the formulas,

(d) by condition $r = 0$, other solutions are absent,

2) by condition $A_1 = A_2 \neq A_3$, $r_2 = 0$,

(a) by condition $r_1 \neq 0$,

$$p_1 = p_3 = \gamma_2 = 0,\ p_2 = \pm 2i,\ \gamma_1 = \frac{\pm 2A_1 i}{r_3 \pm ir_1},\ \gamma_3 = \pm\frac{2A_1}{r_3 \pm ir_1},$$

(a1) by condition $C_{31} \neq 0$,

$$p_1 = \mp\frac{2A_3r_3i}{C_{31}r_1},\ p_1 = \frac{2A_3r_3}{C_{31}r_1},\ p_3 = \pm 2i,\ \gamma_1 = \frac{2A_3}{r_1},\ \gamma_2 = \pm\frac{2A_3i}{r_1},\ \gamma_3 = 0,$$

(a2) by condition $C_{31} = 0$,

(a21) by condition $r_3 \neq 0$, other solutions are absent,

(a22) by condition $r_3 = 0$, a one-parameter set of the points

$$\gamma_1 = \frac{A_1}{r_1},\ \gamma_2 = \pm\frac{A_1 i}{r_1},\ \gamma_3 = 0,\ p_3 = \pm 2i,\ p_2 = \pm p_1 i,$$

(b) by condition $r_1 = 0$, $r_3 \neq 0$, a one-parameter set of the points

$$\gamma_3 = \frac{2A_1}{r_3},\ p_3 = 0, p_2 = \frac{\gamma_1 r_3}{A_1},\ p_1 = -\frac{\gamma_2 r_3}{A_1},\ \gamma_1^2 + \gamma_2^2 = -4\frac{A_1^2}{r_3^2},$$

(c) by condition $r = 0$, any solutions are absent.

Definition 2. Characteristic system (1.3) has two types of the solutions: $(\tilde{p}^0, 0)$ and $(\tilde{p}^0, \tilde{\gamma}^0)$, $\tilde{\gamma}^0 \neq 0$. Respectively, define two types of the singularities of $z(t)$: α-singular and β-singular points or α-points and β-points.

Theorem 2. *Suppose in problem (0.1) on a motion of a rigid body all inertia moments are different. Then there exist singular α-points of the solutions $p(t)$, $\gamma(t)$ with the asymptotic behaviour*

$$\begin{cases} p(t) &= \dfrac{\tilde{p}^0}{t} + \alpha_1 u_1 + \sum_0^2 \psi_i t \operatorname{Ln}^i t + \sum_2^4 \alpha_i v_i t + o(t), \\ \gamma(t) &= \dfrac{\alpha_1}{t} v_1 + \kappa_1 \overset{\sim 0}{p}\ Lnt + \kappa_0 v_1 + \alpha_4 \overset{\sim 0}{p} + \sum_0^2 \chi_i t \operatorname{Ln}^i t + \alpha_5 v_{-1} t + o(t), \end{cases} \tag{1.6}$$

$$t \to 0,\ Arg\, t = const;$$

here $\alpha_1, ..., \alpha_5$ are free parameters,

$$v_1 = A\tilde{p}^0,\ \tilde{p}^0 \times v_{-1} = -v_{-1},\ \langle v_{-1}, v_1 \rangle = 1,$$

$\{v_2, v_3\}$ is the eigenspace of the operator

$$\xi \overset{\mathbf{D}}{\to} A^{-1}\big(A\tilde{p}^0 \times \xi + A\xi \times \tilde{p}^0\big), \tag{1.7}$$

$$v_4 = \mu \tilde{p}^0,\ A^{-1}\big(\tilde{p}^0 \times r - 3\mu A\tilde{p}^0\big) \in \{v_2, v_3\},$$

$$u_1 = -\mathbf{D}^{-1} A^{-1} (v_1 \times r), \tag{1.8}$$

$$\kappa_1 = \alpha_1^2 \langle v_1, r \rangle,\ \kappa_0 = \alpha_1^2 \langle \tilde{p}^0, u_1 \rangle,\ \psi_2 = \frac{\alpha_1^2 \langle v_1, r \rangle^2}{6 \langle A v_1, v_1 \rangle} \Big(\Big(\sum\nolimits_\sigma A_1 \Big) \tilde{p}^0 - 3 v_1 \Big),$$

$$\chi_2 = \frac{\alpha_1}{2} v_1 \times \psi_2 + \frac{\alpha_1 \kappa_1}{2} \langle v_1, r \rangle v_{-1},$$

$$\psi_1 = -\frac{2}{3}\psi_2 + \frac{1}{3} A^{-1} \left(\alpha_1^2 \langle v_1, r \rangle \tilde{p}^0 \times r \right) + \tag{1.9}$$

$$\operatorname{pr}{}^{\{\tilde{p}^0\}}_{\{v_2, v_3\}} A^{-1} \Big(\alpha_1^2 A u_1 \times u_1 + \alpha_1^2 \langle \overset{\sim 0}{p}, u_1 \rangle v_1 \times r + \alpha_4 \overset{\sim 0}{p} \times r \Big),$$

where $\operatorname{pr}{}^{\{v_1\}}_{\{v_2\}}$ *is the projection on the plane v_2 parallel to the plane v_1.*

$$\chi_1 = -\alpha_1 \langle v_1, \psi_1 \rangle \overset{\sim 0}{p} + \Big(\frac{\alpha_1}{2} \langle \overset{\sim 0}{p}, \psi_1 \rangle - \langle v_{-1}, \chi_2 \rangle - \frac{\alpha_1}{2} \kappa_1 \langle v_{-1}, u_1 \rangle \Big) v_1 + \alpha_1 \alpha_4 \langle v_1, r \rangle v_{-1}.$$

Theorem 3. *There exist singular β-points of the solutions of Euler-Poisson's problem with the asymptotic behaviour*

$$\begin{cases} p(t) &= \dfrac{\tilde{p}^0}{t} + \beta_2 u_2 t + \beta_3 u_3 t^2 + \beta_4 u_4 t^3 + ... + \beta_0 u_0 t^{\lambda_0 - 1} + \\ & + \beta^0 u^0 t^{\lambda^0 - 1} + \sum \beta_0^i (\beta^0)^j \psi_{kl} t^{k\lambda^0 + l\lambda_0 - 1} + ... \\ \gamma(t) &= \dfrac{\tilde{\gamma}^0}{t^2} + \beta_2 u_2 + \beta_3 u_3 t + \beta_4 u_4 t^2 + ... + \beta_0 v_0 t^{\lambda_0 - 2} + \\ & + \beta^0 v^0 t^{\lambda^0 - 2} + \sum \beta_0^i (\beta^0)^j \chi_{kl} t^{k\lambda^0 + l\lambda_0 - 2} + ... \end{cases} \tag{1.10}$$

here $\beta_2, \beta_3, \beta_4, \beta_0, \beta^0$ are free parameters and $k, l \in \mathbb{N}$, $t = e^{i(\bar{\lambda}_0 - \bar{\lambda}^0)\theta}$, $\theta \to +\infty$, $\operatorname{Im} \lambda_0 < 0$, (u_k, v_k), $k = 2, 3, 4$, $(u_0, v_0), (u^0, v^0)$ are the eigenvectors of the operator

$$\mathbf{H} : (Ap, \gamma) \to \Big(A\tilde{p}^0 \times p + Ap \times \tilde{p}^0 + \gamma \times r + Ap, \tilde{\gamma}^0 \times p + \gamma \times \overset{\sim 0}{p} + 2\gamma \Big)$$

with the eigenvalue k, λ_0, λ^0, $\lambda_0^{(0)} = \frac{1}{2} \overset{(+)}{-} \sqrt{\frac{1}{4} - S}$, *where*

$$S = \frac{\dfrac{(2\langle A\gamma,\delta\rangle + \langle Ap,p\rangle)(\langle A\gamma,\delta\rangle + \langle Ap,p\rangle)}{\langle A\gamma,\gamma\rangle} - \dfrac{3\,\langle p,r\rangle^2}{\langle\gamma,r\rangle} - 2\,\langle A\delta,\delta\rangle + 2\,\langle\delta,r\rangle}{\dfrac{\langle p,r\rangle^2}{2\,\langle\gamma,r\rangle} - \dfrac{\langle A\gamma,\delta\rangle^2}{\langle A\gamma,\gamma\rangle} + \langle A\delta,\delta\rangle}.$$

§ **2. The factorization of Euler-Poisson's flow.** Let $p(t), \gamma(t)$ be a solution of Euler-Poisson's flow. Then $\alpha p(\alpha t)$, $\alpha^2\gamma(\alpha t)$ is also a solution. This circumstance gives a possibility for the factorization on the set of the trajectories.

At first, project the trajectories (of Euler-Poisson's flow) onto the 11-dimensional sphere.

Let $p(t) = \alpha(t)\breve{p}(t)$, $\gamma(t) = \alpha^2(t)\breve{\gamma}(t)$, where $p(t), \gamma(t)$ is a solution of (0.1) and the point $(\breve{p}(t), \breve{\gamma}(t))$ moves on sphere S_C^{11} :

$$(\breve{p},\breve{p}) + (\breve{\gamma},\breve{\gamma}) = C^2, \quad ((x,y) = \mathrm{Re}\langle x, \bar{y}\rangle).$$

System (0.1) has the following form after changing the variables

$$\begin{cases} A\,\dot{\breve{p}} &= \alpha(A\breve{p}\times\breve{p} + \breve{\gamma}\times r - f\,(\breve{p},\breve{\gamma})A\breve{p}), \\ \dot{\breve{\gamma}} &= \alpha(\breve{\gamma}\times\breve{p} + 2f\,(\breve{p},\breve{\gamma})\breve{\gamma}), \\ \dot{\alpha} &= \alpha^2 f\,(\breve{p},\breve{\gamma}), \end{cases} \tag{2.1}$$

where

$$f(\breve{p},\breve{\gamma}) = \frac{(A\breve{p}\times\breve{p} + \breve{\gamma}\times r, A^{-1}\breve{p}) + (\breve{\gamma}\times\breve{p},\breve{\gamma})}{(\breve{p},\breve{p}) + 2\,(\breve{\gamma},\breve{\gamma})}.$$

If we do not care of time correspondence between the trajectory $p(t), \gamma(t)$ and its projection $\breve{p}(t)$, $\breve{\gamma}(t)$, then we can obtain the following representation of the projection $\breve{p}(\theta)$, $\breve{\gamma}(\theta)$

$$\begin{cases} A\,\dot{\breve{p}} &= A\breve{p}\times\breve{p} + \breve{\gamma}\times r - f\,A\breve{p}, \\ \dot{\breve{\gamma}} &= \breve{\gamma}\times\breve{p} - 2f\,\breve{\gamma}. \end{cases} \tag{2.2}$$

There is a natural question: what is behaviour of the trajectories $\breve{p}(t)$, $\breve{\gamma}(t)$ when t approaches a singular point t_*?

Let

$$\begin{cases} \mathcal{H}(z) &= \frac{1}{2}\,\langle Ap,p\rangle + \langle\gamma,r\rangle\,, \\ \mathcal{M}(z) &= \langle Ap,\gamma\rangle\,, \\ \mathcal{T}(z) &= \langle\gamma,\gamma\rangle\,. \end{cases} \tag{2.3}$$

Due to relation z and $\breve{z}$ we have $\mathcal{H}(z(t)) = \alpha^2(t)\mathcal{H}(\breve{z}(t))$, $\mathcal{M}(z(t)) = \alpha^3(t)\mathcal{M}(\breve{z}(t))$, $\mathcal{T}(z(t)) = \alpha^4(t)\mathcal{T}(\breve{z}(t))$ and $||z(t)|| \to \infty \Longleftrightarrow \mathcal{H}(\breve{z}(t)) \to 0$, $\mathcal{M}(\breve{z}(t)) \to 0, \mathcal{T}(\breve{z}(t)) \to 0$.

So we see that the investigation of the singular points is connected with the structure of flow (0.1) on zero-level surface of the function $\mathcal{H}, \mathcal{M}, \mathcal{T}$.

Theorem 4. *Let* $\mathbb{C}$ *act as a transformation group on* $\mathbb{C}^n$ *in the following way*

$$\alpha : (z_1, ..., z_n) \to (\alpha^{k_1} z_1, ..., \alpha^{k_n} z_n),$$

$k = (k_1, ..., k_n) \in \mathbb{N}^n$. *Then the quotient-space* $P_k^{n-1} = \left\{(z_1^{(k_1)} : ... : z_n^{(k_n)})\right\}$ *is a compact holomorphic manifold [7] by this action.*

Proof. Let π_0 and π be canonical projections $\pi_0\colon \mathbb{C}^n \to P^{n-1}$, $\pi\colon \mathbb{C}^n \to P_k^{n-1}$. Then consider the mapping of the complex projective space P^{n-1} $f\colon P^{n-1} \to P_k^{n-1}$ which is induced by mapping f^0:

$$f^0\colon (z_1, ..., z_n) \to (z_1^{k_1}, ..., z_n^{k_n}), \quad f = \pi \circ f^0 \circ \pi_0^{-1}.$$

Evidently, the mapping f is correct, but it has singular points if $z_i = 0$. In these singular points the mapping f is open, and therefore, any atlas of P^{n-1} induces the atlas P_k^{n-1}, consequently P_k^{n-1} is a holomorphic manifold. The compactness of the manifold P_k^{n-1} follows from the compactness of P^{n-1}. □

Proposition 2. *The canonical projection $\pi\colon \mathbb{C}^6 \to P_*^5$, where $* = (1,1,1,2,2,2), \mathbb{C}^6 = \{(p_1, p_2, p_3, \gamma_1, \gamma_2, \gamma_3)\}$ maps the foliation [8], induced by flow (0.1) onto the foliation $\mathcal{F}$ of the compact holomorphic manifold P_*^5.*

Proof. This proposition follows from the invariantness of the trajectories $(p(t), \gamma(t))$ by the action $\alpha\colon (p(t), \gamma(t)) \to (\alpha p(\alpha t), \alpha^2\gamma(\alpha t))$ and from the definition of the manifold P_*^5. □

Proposition 3. *The singular points of the foliation F of the compact holomorphic manifold M^5 are the π-projection of the singular points z_0 of the flow (0.1) and π-projection of the solutions of characteristic system (1.3).*

Proof. Evidently, the singular points of differential equation (0.1) are projected onto the singular points of the foliation $\mathcal{F}$. Moreover, if the vector $\dot{z}$ touches the π-pre-image of $\pi(z)$ then the point $\pi(z)$ is a singular point of the foliation too. Such the points satisfy the system

$$\begin{cases} Ap \times p + \gamma \times r = \lambda p, \\ \gamma \times p = 2\lambda\gamma, \ \lambda \neq 0. \end{cases}$$

The solutions of this system are $p = \lambda\tilde{p}^0, \gamma = \lambda^2\tilde{\gamma}^0$, i.e. $(p, \gamma) = \pi\left(\tilde{p}^0, \tilde{\gamma}^0\right)$. □

Remark 2. The foliation F is integrable since there exists the fibre invariant mapping

$$J\colon P_*^5 \setminus X^2 \to P^2, \ J\colon (p_1^{(1)} : p_2^{(1)} : p_3^{(1)} : \gamma_1^{(2)} : \gamma_2^{(2)} : \gamma_3^{(2)}) \to \left(\mathcal{H}^6(p,\gamma) : \mathcal{M}^4(p,\gamma) : \mathcal{T}^3(\gamma)\right),$$

where $X^2 = \{\pi(z) : \mathcal{H}(z) = \mathcal{M}(z) = \mathcal{T}(z) = 0\}$. Moreover, the surface X^2 is fibre invariant for the foliation $\mathcal{F}$ too.

Proposition 4. *The singular points (p_0, γ_0) of the equations (0.1) are equal to $p_0 = 0$, $\gamma_0 = \nu r$, $\nu \in \mathbb{C}$ or satisfy the system*

$$\begin{cases} Ap_0 - \mu p_0 &= \nu r, \\ \gamma_0 &= \nu p_0, \end{cases} \tag{2.4}$$

where μ, ν are any complex numbers.

One can prove it by straightforward calculations.

Proposition 5. *The parameters (p_0, γ_0) of the singular points of the foliation $\mathcal{F}|_{X^2}$ satisfy the following relations*

$$\begin{cases} \gamma_0 &= \nu p_0, \\ \langle Ap_0, p_0\rangle &= \langle p_0, p_0\rangle = \langle p_0, r\rangle = 0. \end{cases} \tag{2.5}$$

Proof. In order to obtain these relations (2.5) one can substitute the solutions of (2.4) into (2.3). □

Remark 3. System (2.5) is overdetermined. Its solutions are

$$(p_{01} : p_{02} : p_{03}) = \left(\sqrt{B_{23}} : \sqrt{B_{31}} : \sqrt{B_{12}}\right).$$

In addition, we get restriction $\sum_\sigma r_1\sqrt{B_{23}} = 0$. If $\nu = 0$ we get also the overdetermined system

$$\begin{cases} Ap_0 = \mu p_0, \\ \langle Ap_0, p_0\rangle = 0, \end{cases}$$

which has a nonzero solution only if $\prod_\sigma B_{12} = 0$.

Theorem 5. *Every non-constant solution $z(t)$ of Euler-Poisson's equations have a singular point $t_* \in \mathbb{C}$ if*

$$\prod_\sigma B_{12} \sum_\sigma r_1\sqrt{B_{23}} \neq 0.$$

Proof. Let $z(t)$ be an arbitrary solution of problem (0.1) and $Y = \pi(z(t))$ be the leaf of the foliation $\mathcal{F}$. Assume, that the solution $z(t)$ has no singular points $t_* \in \mathbb{C}$. Then the leaf Y has no singular points for otherwise the leaf Y would have a singular point $\pi(\tilde{z}^0)$ and the solution $z(t)$ would have singular α or β-points (see (1.6), (1.10)).

Let $y \in Y$ be an arbitrary point and $\pi(z(t_0)) = y$, $\|z(t_0)\| = 1$. One can move along a path $\gamma \subset \mathbb{C}$ from the point t_0 to the point t_1 where $\|z(t_1)\| > 2$.

And what is more, there exists a neighbourhood U in P^5_* such as for all $y' \in U$ if $\pi(z(t_0)) = y'$, $\|z(t_0)\| = 1$ then along γ we have $\|z(t_1)\| > 2$. So we have an open covering of the closure of Y and let us choose a finite subcovering U_i. Let $|t_{0i} - t_{1i}| < T$ for all i. Now construct a path Γ which contains the points $t_0, t_1, ..., t_n, \ldots$ where t_k may be found by t_{k-1} in the same way as t_1 was found by t_0. Then the points $t_0, t_1, ..., t_n, ...$ must be into the disc with the radius $T + \frac{T}{2} + \frac{T}{2^2} + ... = 2T$ and there exists the limit point $t_* \in \mathbb{C}$, which must be a singular point. □

Theorem 6. *All singular points of the solutions of Euler-Poisson's equations (0.1) are α, β-points or the limit points of the α, β-points, or are approximated by asymptotic (1.10) if* $\operatorname{Im}\lambda_0 \to 0, \operatorname{Re}\lambda_0 \leq 0$.

Proof. Suppose that there exists a solution of system (0.1) with a singular point which does not satisfy the conditions of the theorem. We will call this point a point with unknown asymptotic type or unknown point.

Let us consider the evolution of system (0.1) to the case of Euler: $A_i(\varepsilon) = A_i$, $r_i(\varepsilon) = \varepsilon r_i$. If $\varepsilon = 0$ then the solutions $p(t), \gamma(t)$ have only α-points and the β-points go to infinity. The disappearing of an unknown point in this process has two explanations : a) the unknown point transforms into α (β)-point (or points); b) the unknown point goes to infinity.

In case a) we see that if parameter ε increases from 0 to 1 then some α (β)-point transforms into unknown point. But it is impossible because the asymptotics of α (β)-point depends on five parameters or in other words is a general asymptotics.

Let us consider case b). Let $z(t)$ be the solution of (0.1) with unknown point t_* and $Y = \pi(z(t))$. If $t \to t_*$ then the leaf Y has a limit leaf $Y_0 \subset \mathcal{F}|_{X^2}$.

Let the leaf Y_0 correspond to the solution $z^*(t)$. The leaf Y_0 can not enter a singular point of the foliation $\mathcal{F}|_{X^2}$. In another case (let $t_*^{'}$ be α-(β-)point we move along the solution $z^*(t)$, $t \in [t_0, t_*]$ and simultaneously along the leaf $\pi(z^*(t)) = Y$. Due to the compactness of X^2 there exist $t_1, ..., t_n, \in [t_0, t_*]$ such that $z^*(t_n)$ approximately equals $(\lambda_n p_0, \lambda_n^2 \gamma_0)$, where $\lambda_n \to \infty$ $(n \to \infty)$, and (p_0, γ_0) is some fixed vector.

Consequently, there exist singular points $t_n + (t_*^{'} - t_1)\lambda_1 \lambda_n^{-1}$ (we use invariance of solutions (0.1) by the translation $(p(t), \gamma(t) \to (\alpha p(\alpha t), \alpha^2 \gamma(\alpha t)))$. Therefore, the singular point t_* is a limit point of the α, β-points. So the leaf Y_0 does not enter the singular points $\pi(\tilde{z}^0)$ of the foliation $\mathcal{F}|_{X^2}$.

Let $t_* \to \infty$ if $\varepsilon \to \varepsilon_0$. Then there exists a limit leaf $Y_0(\varepsilon_0)$ (due to the compactness of X^2) which does not enter the singular points $\pi(\tilde{z}^0)$. According to the proof of Theorem 5 the solution $z^*(t, \varepsilon_0) \in \pi^{-1}(Y_0(\varepsilon_0))$ has the unknown singular point $t_* \in C$ which contradicts assumption b). □

§ 3. The necessary condition of the existence of the single-valued solution to Euler-Poisson's equation.

Proposition 6. *Any solution of Euler-Poisson's equation is multivalued in a neighbourhood of an α-point if and only if $\langle A\tilde{p}^0, r\rangle \neq 0$.*

Proof. 1. Suppose that $\langle v_1, r\rangle \neq 0$ $(v_1 = A\tilde{p}^0)$ but some solution (0.1) is single-valued in a neighbourhood of an α-point (see (1.6)). Then

$$\kappa_1 = \psi_1 = \psi_2 = 0, \tag{3.1}$$

$$\kappa_1 = \alpha_1^2 \langle v_1, r\rangle \Longrightarrow \alpha_1 = 0, \tag{3.2}$$

$$\psi_1 = -\tfrac{2}{3}\psi_2 + \tfrac{1}{3}A^{-1}\big(\alpha_1^2 \langle v_1, r\rangle \tilde{p}^0 \times r\big) +$$
$$+ \operatorname{pr}_{\{v_2, v_3\}}^{\{\tilde{p}^0\}} A^{-1}\left(\alpha_1^2 A u_1 \times u_1 + \alpha_1^2 \langle \tilde{p}^0, u_1\rangle v_1 \times r + \alpha_4 \tilde{p}^0 \times r\right) = \alpha_4 \operatorname{pr}_{\{v_2, v_3\}}^{\{\tilde{p}^0\}} A^{-1}\big(\tilde{p}^0 \times r\big). \tag{3.3}$$

In the basis $(\tilde{p}^0, v_2, v_3)$ we have the relations

$$A^{-1}\big(\tilde{p}^0 \times r\big) = \lambda_1 \tilde{p}^0 + \lambda_2 v_2 + \lambda_3 v_3 \Longleftrightarrow \tilde{p}^0 \times r = \lambda_1 A\tilde{p}^0 + \lambda_2 A v_2 + \lambda_3 A v_3 \Longleftrightarrow$$

$$\Longleftrightarrow -\langle v_1, r\rangle = \langle \tilde{p}^0 \times r, v_1\rangle = \lambda_2 \langle A v_2, v_1\rangle + \lambda_3 \langle A v_3, v_1\rangle \neq 0 \Longleftrightarrow$$

$$\Longleftrightarrow \operatorname{pr}_{\{v_2, v_3\}}^{\{\tilde{p}^0\}} A^{-1}\big(\tilde{p}^0 \times r\big) \neq 0 \overset{\text{see } (3.1),(3.3)}{\Longrightarrow} \alpha_4 = 0. \tag{3.4}$$

According to asymptotic behaviour (1.6) in the α-point we have

$$\mathcal{T} = \langle \gamma, \gamma\rangle = 2\alpha_1\alpha_5 - \alpha_4^2 = 0 \Longrightarrow \gamma = 0 \quad (\text{see } (3.2), (3.4)).$$

So we have the case of Euler and in this case $r = 0 \Rightarrow \langle v_1, r\rangle = 0$.

2. Suppose $\langle v_1, r\rangle = 0$. According to (1.6) $\kappa_1 = 0$,

$$\psi_2 = \frac{\alpha_1^2 \langle v_1, r\rangle}{6\langle A v_1, v_1\rangle}\left(\Big(\sum\nolimits_\sigma A_1\Big)\tilde{p}^0 - 3v_1\right) = 0, \tag{3.5}$$

$$\chi_2 = \frac{1}{2}\alpha_1 v_1 \times \psi_2 + \frac{1}{2}\alpha_1 \kappa_1 \langle v_1, r\rangle v_{-1} = 0.$$

Now we want to prove that $\psi_1 = 0$. According to (1.9), (3.5)

$$\psi_1 = 0 \Leftrightarrow \alpha_1^2 Au_1 \times u_1 + \alpha_1^2 \langle \tilde{p}^0, u_1 \rangle v_1 \times r + \alpha_4 \tilde{p}^0 \times r = \lambda' A\tilde{p}^0 \Leftrightarrow \alpha_1^2 Au_1 \times u_1 = \lambda v_1,$$

because $A\tilde{p}^0 = -A\tilde{p}^0 \times \tilde{p}^0$ and

$$(v_1 \times r) \times v_1 = \langle v_1, v_1 \rangle r - \langle r, v_1 \rangle v_1 = 0, \tag{3.6}$$

$$(\tilde{p}^0 \times r) \times v_1 = \langle \tilde{p}^0, v_1 \rangle r - \langle r, v_1 \rangle \tilde{p}^0 = 0. \tag{3.7}$$

So we must prove that

$$\begin{cases} \langle Au_1 \times u_1, v_1 \rangle = 0, \\ \langle Au_1 \times u_1, \tilde{p}^0 \rangle = 0 \end{cases} \tag{3.8}$$

According to (1.8) $u_1 = \mathbf{D}^{-1} A^{-1} (v_1 \times r)$, $\mathbf{D}(\tilde{p}^0) = -2\tilde{p}^0, \mathbf{D}(v_i) = v_i, i = 2, 3$. Consequently, $\mathbf{D}^{-1}(x) = \mu \tilde{p}^0 + x$, $\mu \in \mathbb{C}$, and then

$$u_1 = -A^{-1}(v_1 \times r) - \mu \tilde{p}^0, \quad Au_1 = -v_1 \times r - \mu A\tilde{p}^0,$$

$$\begin{gathered} \langle Au_1 \times u_1, v_1 \rangle = \left\langle (v_1 \times r + \mu v_1) \times \left(A^{-1}(v_1 \times r) + \mu \tilde{p}^0\right), v_1 \right\rangle = \\ = \left\langle (v_1 \times r) \times \left(A^{-1}(v_1 \times r) + \mu \tilde{p}^0\right), v_1 \right\rangle = 0, \quad \text{(see (3.6))} \end{gathered}$$

$$\begin{gathered} \left\langle Au_1 \times u_1, \tilde{p}^0 \right\rangle = \left\langle (v_1 \times r + \mu v_1) \times \left(A^{-1}(v_1 \times r) + \mu \tilde{p}^0\right), \tilde{p}^0 \right\rangle = \\ = \langle (v_1 \times r + \mu v_1) \times A^{-1}(v_1 \times r), \tilde{p}^0 \rangle = \langle \tilde{p}^0 \times \mu v_1, A^{-1}(v_1 \times r) \rangle = \\ = \langle \mu v_1, A^{-1}(v_1 \times r) \rangle = 0 \quad \text{(see (3.7))} \end{gathered}$$

So, $\psi_1 = 0$ and hence $\chi_1 = 0$. We considered all resonance terms of asymptotics (1.6), consequently the solution of (0.1) is single-valued in a neighbourhood of the α-point if $\langle A\tilde{p}^0, r \rangle = 0$. □

Proposition 7. *Any solution of Euler-Poisson's equations is multivalued in a neighbourhood of the β-point if and only if*

$$\begin{cases} \lambda_0 \notin \mathbb{Z}, \\ |\beta_0|^2 + |\beta^0|^2 \neq 0. \end{cases}$$

The proof is evident (see (1.10)).

Theorem 7. *If the parameters A_i, r_i of the solid satisfy the following conditions*

1. $\prod_\sigma B_{12} \sum_\sigma r_1 \sqrt{B_{23}} \neq 0.$
2. $\sum_\sigma A_1 r_1 \sqrt{\dfrac{A_2 A_3}{B_{12} B_{31}}} \neq 0;$
3. $\lambda_0 \notin \mathbb{Z};$

then all single-valued solutions of Euler-Poisson's equations, if any, have the representation

$$\begin{cases} p(t) = \sum_k \tilde{p}_k^0 \zeta(t - t_k) + p_0, \\ \gamma(t) = \sum_k \tilde{\gamma}_k^0 \varrho(t - t_k) + \gamma_0, \end{cases} \tag{3.9}$$

where $\{(\tilde{p}_k^0, \tilde{\gamma}_k^0)\}$ are the β-solutions of the characteristic system.

Proof. Condition 1) guarantees the existence of the singular points $t_* \in C$ of solutions (0.1). Condition 2) implies that all α-points of solutions (0.1) are multivalued. Due to condition 3) all β-points of solutions (0.1) are multivalued if $|\beta_0|^2 + |\beta^0|^2 \neq 0$.

Let now $\beta_0 = \beta^0 = 0$. Then

$$\mathcal{H} = \beta_2 \left(\langle \tilde{p}^0, u_2 \rangle + \langle v_2, r \rangle \right), \quad \mathcal{M} = \beta_3 \left(\langle v_1, v_3 \rangle + \langle u_3, A\tilde{\gamma}^0 \rangle \right),$$

$$\mathcal{T} = \beta_2^2 \left(\langle v_2, v_2 \rangle + \langle \chi_0, \tilde{\gamma}^0 \rangle \right) + \beta_4 \langle v_4, \tilde{\gamma}^0 \rangle$$

and hence all singular points

$$p(t) = \frac{\tilde{p}_k^0}{t} + ..., \quad \gamma(t) = \frac{\tilde{\gamma}_k^0}{t^2} + ...$$

have the same asymptotics.

It may be only if the functions $p(t), \gamma(t)$ are periodic or doubly periodic.

The periodic functions $p(t), \gamma(t)$ are bounded outside of some band because in the opposite case $p(t), \gamma(t)$ would be doubly periodic (the proof is similar to the proof of Th.5). Such functions have the representation

$$p(t) = \sum_{1 \leq k \leq 8, n \in \mathbb{Z}} \frac{\tilde{p}_k^0}{t - t_k + nT}, \quad \gamma(t) = \sum_{1 \leq k \leq 8, n \in \mathbb{Z}} \frac{\tilde{\gamma}_k^0}{(t - t_k + nT)^2}, \quad T^2 \in \mathbb{R}.$$

and do not satisfy (0.1).

The doubly periodic functions $p(t), \gamma(t)$ have representation (3.9). □

In conclusion we formulate this theorem in an affirmative form.

Theorem 8. *If there exist the single-valued solutions of Euler-Poisson's equations (0.1) then the parameters A_i, r_i satisfy one of the following conditions:*

1. $\prod_\sigma B_{12} \sum_\sigma r_1 \sqrt{B_{23}} = 0.$
2. $\sum_\sigma A_1 r_1 \sqrt{\dfrac{A_2 A_3}{B_{12} B_{31}}} = 0;$
3. $\lambda_0 \in \mathbb{Z};$
4. *the solution $(p(t), \gamma(t))$ have the representation*

$$\begin{cases} p(t) = \sum_k \tilde{p}_k^0 \zeta(t - t_k) + p_0, \\ \gamma(t) = \sum_k \tilde{\gamma}_k^0 \varrho(t - t_k) + \gamma_0, \end{cases}$$

where $\{(\tilde{p}_k^0, \tilde{\gamma}_k^0)\}$ are the β-solutions of the characteristic system.

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