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THE ASYMPTOTIC BEHAVIOUR OF THE NORMALIZING FACTOR FOR RANDOM MATRIX-VALUED EVOLUTION GIVEN BY A TRANSPORT EQUATION

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A random matrix-valued evolution $T^\varepsilon(t)$ given by a transport equation is considered. We investigate an asymptotic behaviour of a normalizing factor ρ^ε that determines a scale of time on which the asymptotic representation for the mean-value $MT^\varepsilon(t)$ of the random evolution has been found.

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Рассматривается случайная матричнозначная эволюция $T^\varepsilon(t)$, задающаяся уравнением переноса. Изучается асимптотическое поведение нормирующего множителя ρ^ε, определяющего масштаб времени, в котором найдено асимптотическое представление математического ожидания $MT^\varepsilon(t)$ случайной эволюции.

Let $T^\varepsilon(t)$ be a matrix-valued random evolution given as a solution of the differential equation

$$\frac{dT^\varepsilon(t)}{dt} = T^\varepsilon(t)A^\varepsilon\left(x(t)\right) \tag{1}$$

with the initial condition $T^\varepsilon(0) = I$, where $x(t)$ is a regenerative process on the probabilistic space $(\Omega, \mathcal{F}, \mathcal{P})$ with a moments of regeneration $\tau_1 \equiv \tau, \tau_2, \ldots, \tau_n, \ldots$ [2]; ε is a small parameter; $A^\varepsilon(x)$ is a given family of $d \times d-$ matrix-valued functions; I is the unit matrix. Suppose that τ has a finite mean $M\tau < \infty$.

According to [4] the solution of equation (1) can be represented as

$$T^\varepsilon(t) = \begin{cases} \xi^\varepsilon(t), & 0 \leqslant t \leqslant \tau_1, \\ \xi^\varepsilon(\tau_1)\xi^{\varepsilon(1)}(t-\tau_1), & \tau_1 < t \leqslant \tau_2, \\ \qquad\vdots & \\ \xi^\varepsilon(\tau_1)\ldots\xi^{\varepsilon(k)}(t-\tau_k), & \tau_k < t \leqslant \tau_{k+1}, \\ \qquad\vdots & \end{cases} \tag{2}$$

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Here a matrix-valued process $\xi^{\varepsilon(k)}(t)$, $k \in \mathbb{N}$, is an independent copy of the process $\xi^\varepsilon(t)$, $0 \leqslant t \leqslant \tau$, satisfying the differential equation

$$\frac{d\xi^{\varepsilon(k)}(t)}{dt} = \xi^{\varepsilon(k)}(t) A^\varepsilon\left(x^k(t)\right)$$

with the initial condition $\xi^{\varepsilon(k)}(0) = I$, where the processes $x^k(t) = x(\tau_k + t)$, $0 \leqslant t < \tau_{k+1} - \tau_k$, are independent copies of the process $x(t)$, $0 \leqslant t \leqslant \tau$.

In [3] the asymptotic representation for the mean-value $MT^\varepsilon(t)$ of the random evolution (2) was found on a scale of time t/ρ^ε where $\rho^\varepsilon \to 0$ as $\varepsilon \to 0$, under the assumption that the matrix $\lim\limits_{\varepsilon \to 0} MT^\varepsilon(\tau)$ is reducible and the following conditions are fulfilled:

1) the sequence of the matrices $M(T^\varepsilon(t),\, t < \tau)$ is uniformly directly Riemann integrable on $[0, \infty)$ [1];
2) The matrix $\lim\limits_{\varepsilon \to 0} M(T^\varepsilon(\tau),\, \tau \in dy)$ is non-lattice [1];
3) $\lim\limits_{t \to \infty} \sup_\varepsilon \| \int_t^\infty y M(T^\varepsilon(\tau),\, \tau \in dy) \| = 0$.

In this paper we investigate the asymptotic behaviour of the parameter ρ^ε under the assumption that $A^\varepsilon(x)$ is of the form

$$A^\varepsilon(x) = A + \delta_1(\varepsilon)\, B^1(x) + \delta_2(\varepsilon)\, B^2(x) + \cdots + \delta_n(\varepsilon)\, B^n(x) + o(\delta_n(\varepsilon)), \tag{3}$$

where A is a reducible matrix, i. e., the set $I = \{1, 2, \ldots, d\}$ can be decomposed into a finite number of pairwise disjoint sets $I_1, \ldots, I_r$ such that $A_{ij} = 0$ when $i \in I_k$, $j \in I_l$, $k \neq l$. Assume further that the restriction $A^{(s)}$ of the matrix A to the set I_s is an irreducible matrix with non-negative non-diagonal elements and the Perron root of $A^{(s)}$ is equal to 0. This ensures the existence and uniqueness (up to a positive multiplier) of positive right and left eigenvectors $\vec{u}^{(s)}$, $\vec{v}^{(s)}$ of the matrix $A^{(s)}$ respectively such that $A^{(s)}\vec{u}^{(s)} = \vec{0}$, $\vec{v}^{(s)} A^{(s)} = \vec{0}$, and without loss of generality we may suppose that $A^{\varepsilon(s)}$ is also an irreducible matrix.

The matrix-valued functions $B^1(x), \ldots, B^n(x)$ in (3) are supposed to be x-measurable and bounded; the sequence of functions $\delta_1(\varepsilon), \ldots, \delta_n(\varepsilon)$ forms a scale of infinitely small quantities as $\varepsilon \to 0$ with the property $\delta_{i+1}(\varepsilon) = o(\delta_i(\varepsilon))$, $(\varepsilon \to 0)$. Let us denote $K^\varepsilon(du) = M\left(T^\varepsilon(\tau),\, \tau \in du\right)$, $K(du) = M\left(e^{A\tau},\, \tau \in du\right)$. Then, under assumptions we have made, we may assert that $K^\varepsilon(t)$ converges weakly to $K(t)$ as $\varepsilon \to 0$; the restriction $K^{(s)}$ of the matrix $K = K[0, \infty) = Me^{A\tau}$ to the set I_s is an irreducible matrix with 1 as its Perron root and with the vectors $\vec{u}^{(s)}$, $\vec{v}^{(s)}$ as its right and left eigenvectors corresponding to the eigenvalue 1, i. e., $K^{(s)}\vec{u}^{(s)} = \vec{u}^{(s)}$, $\vec{v}^{(s)} K^{(s)} = \vec{v}^{(s)}$.

According to [1], to define parameter ρ^ε we have to fix r coordinates $w_1 \in I_1, \ldots, w_r \in I_r$ and introduce functions P^ε_{ij}, $i, j = \overline{1, d}$ that are solutions of the equations

$$P^\varepsilon_{ij} = K^\varepsilon_{ij} + \sum_{n=1}^{r} \sum_{l \in I_n \setminus w_n} K^\varepsilon_{il} P^\varepsilon_{lj} \tag{4}$$

Then we put

$$\rho^\varepsilon = \sum_{s=1}^{r} \frac{1 - P^\varepsilon_{w_s w_s}}{m_s}, \quad \text{where } m_s = \frac{\sum_{i,j \in I_s} v_i^{(s)} \int_0^\infty t K_{ij}(dt)\, u_j^{(s)}}{v^{(s)}_{w_s}\, u^{(s)}_{w_s}}; \quad s = \overline{1, r}.$$

In order to formulate a statement on the asymptotic behaviour of the parameter ρ^ε we need the following denotation

$$
\begin{aligned}
&H^{k\varepsilon}(s) = M\int_0^s T^\varepsilon(z)B^k\left(x(z)\right)e^{A(s-z)}dz, \quad H^k(s) = Me^{As}\int_0^s B^k\left(x(z)\right)dz,\\
&D^{lk}(s) = M\int_0^s H^l(z)B^k\left(x(z)\right)e^{A(s-z)}ds, \quad R^{lk}(\tau) = (R^{lk}_{ij}(\tau))_{i,j\in I},\\
&R^{lk}_{ij}(\tau) = \sum_{m=1}^r \frac{1}{\Pi^{(m)}_{w_m\,w_m}} \sum_{\substack{n\in I_m\\ p\in I_m\setminus w_m}} H^l_{ip}(\tau)V^{(m)}_{pn}H^k_{nj}(\tau);\\
&a_s = \sum_{i,j\in I_s} v^{(s)}_i \int_0^\infty tK_{ij}(dt)\,u^{(s)}_j; \quad h^l_s = \sum_{i,j\in I_s} v^{(s)}_i H^l_{ij}(\tau)\,u^{(s)}_j;\\
&d^{lk}_s = \sum_{i,j\in I_s} v^{(s)}_i\left(D^{lk}_{ij}(\tau) + R^{lk}_{ij}(\tau)\right)u^{(s)}_j; \quad s = \overline{1,r},
\end{aligned}
\tag{5}
$$

where $V^{(s)}$ is the generalized inverse matrix to $I - K^{(s)}$, i. e.,

$$V^{(s)}(I - K^{(s)}) = (I - K^{(s)})V^{(s)} = I - \Pi^{(s)};$$

$\Pi^{(s)} = [\vec{u}^{(s)} \otimes \vec{v}^{(s)}]$ is the proper projector of $K^{(s)}$.

Theorem 1. *Let $l = \min\{l_1, \dots, l_r\}$, where l_s is the first number for which $h^{l_s}_s \neq 0$. If there exists s, $1 \leqslant s \leqslant r$, such that $d^{11}_s \neq 0$, then*

i)

$$\rho^\varepsilon \sim -\delta_l(\varepsilon)\sum_{s=1}^r \frac{h^l_s}{a_s} \qquad \textit{if} \quad \delta^2_1(\varepsilon) = o(\delta_l(\varepsilon));$$

ii)

$$\rho^\varepsilon \sim -\delta^2_1(\varepsilon)\sum_{s=1}^r \frac{\alpha\, h^l_s + d^{11}_s}{a_s} \qquad \textit{if} \quad \delta_l(\varepsilon) = \alpha\delta^2_1(\varepsilon);$$

iii)

$$\rho^\varepsilon \sim -\delta^2_1(\varepsilon)\sum_{s=1}^r \frac{d^{11}_s}{a_s} \qquad \textit{if} \quad \delta_l(\varepsilon) = o(\delta^2_1(\varepsilon)).$$

Proof. A solution of differential equation (1) in view of representation (3) can be written as

$$T^\varepsilon(t) = e^{At} + \sum_{k=1}^n \delta_k(\varepsilon)\int_0^t T^\varepsilon(s)B^k\left(x(s)\right)e^{A(t-s)}ds. \tag{6}$$

For the mean-value $MT^\varepsilon(t)$ we have then

$$MT^\varepsilon(t) = Me^{At} + \sum_{k=1}^n \delta_k(\varepsilon)M\int_0^t T^\varepsilon(s)B^k\left(x(s)\right)e^{A(t-s)}ds.$$

Putting $t = \tau$ and using denotation (5) we obtain the equality

$$K^\varepsilon = K + \sum_{k=1}^n \delta_k(\varepsilon)H^{k\varepsilon}(\tau). \tag{7}$$

If $P=\lim\limits_{\varepsilon\to 0}P^{\varepsilon}$ then the elements $P_{i\,w_s}$ $(i\in I, s=\overline{1,r})$ of the matrix P satisfy the following equations

$$P_{i\,w_s}=K_{i\,w_s}+\sum_{l=1}^{r}\sum_{j\in I_l\setminus w_l}K_{i\,j}P_{j\,w_s}. \tag{8}$$

The direct verification convinces us that

$$P_{i\,w_s}=\begin{cases}u_i^{(s)}/\,u_{w_s}^{(s)}, & \text{when } i\in I_s\\ 0 & \text{when } i\notin I_s\end{cases}$$

is the solution of (8).

Taking into account equality (7), equation (4) can be rewritten as (we suppose here that $i\in I_s$)

$$P^{\varepsilon}_{i\,w_s}=K_{i\,w_s}+\sum_{j\in I_s\setminus w_s}K_{i\,j}P^{\varepsilon}_{j\,w_s}+\sum_{k=1}^{n}\delta_k(\varepsilon)H^{k\varepsilon}_{i\,w_s}(\tau)+\sum_{k=1}^{n}\sum_{l=1}^{r}\sum_{j\in I_l\setminus w_l}\delta_k(\varepsilon)H^{k\varepsilon}_{i\,j}(\tau)P^{\varepsilon}_{j\,w_s}. \tag{9}$$

Multiplying (9) by $v_i^{(s)}$ and summing up for all $i\in I_s$ gives

$$\begin{aligned}v^{(s)}_{w_s}(P^{\varepsilon}_{w_s\,w_s}-1)&=\sum_{k=1}^{n}\delta_k(\varepsilon)\sum_{i,\,j\in I_s}v_i^{(s)}H^{k\varepsilon}_{i\,j}(\tau)P_{j\,w_s}+\\&+\sum_{k=1}^{n}\delta_k(\varepsilon)\sum_{l=1}^{r}\sum_{\substack{i\in I_s\\ j\in I_l\setminus w_l}}v_i^{(s)}H^{k\varepsilon}_{i\,j}(\tau)\,[P^{\varepsilon}_{j\,w_s}-P_{j\,w_s}].\end{aligned} \tag{10}$$

Now we are going to find an asymptotic representations for the differences $H^{k\varepsilon}(\tau)-H^{k}(\tau)$ and $P^{\varepsilon}_{i\,w_s}-P_{i\,w_s}$.

$$\begin{aligned}H^{k\varepsilon}(\tau)-H^{k}(\tau)&=M\int_0^{\tau}(T^{\varepsilon}(s)-e^{As})B^k\left(x(s)\right)e^{A(\tau-s)}ds=\\&=\sum_{l=1}^{n}\delta_l(\varepsilon)M\int_0^{\tau}\left(M\int_0^{s}T^{\varepsilon}(z)B^l\left(x(z)\right)e^{A(s-z)}dz\right)B^k\left(x(s)\right)e^{A(\tau-s)}ds=\\&=\sum_{l=1}^{n}\delta_l(\varepsilon)M\int_0^{\tau}H^{l\varepsilon}(s)B^k\left(x(s)\right)e^{A(\tau-s)}ds.\end{aligned} \tag{11}$$

Under conditions of Theorem 1 it suffices to suppose that

$$\begin{aligned}H^{k\varepsilon}(\tau)-H^{k}(\tau)&=\delta_1(\varepsilon)M\int_0^{\tau}H^{1}(s)B^k\left(x(s)\right)e^{A(\tau-s)}ds+o(\delta_1(\varepsilon))=\\&=\delta_1(\varepsilon)D^{1k}(\tau)+o(\delta_1(\varepsilon)).\end{aligned} \tag{12}$$

Let us now write down a representation for $P^{\varepsilon}_{i\,w_s}-P_{i\,w_s}$, $i\in I_l$.

$$\begin{aligned}P^{\varepsilon}_{i\,w_s}-P_{i\,w_s}&=\sum_{j\in I_l\setminus w_l}K_{i\,j}\,[P^{\varepsilon}_{j\,w_s}-P_{j\,w_s}]+\sum_{k=1}^{n}\sum_{j\in I_s}\delta_k(\varepsilon)H^{k\varepsilon}_{i\,j}(\tau)P_{j\,w_s}+\\&+\sum_{k=1}^{n}\sum_{m=1}^{r}\sum_{j\in I_m\setminus w_m}\delta_k(\varepsilon)\,H^{k\varepsilon}_{i\,j}(\tau)\,[P^{\varepsilon}_{j\,w_s}-P_{j\,w_s}]\end{aligned} \tag{13}$$

It suffices to confine (13) by the terms that converge to 0 not faster then $\delta_1(\varepsilon)$. Then we get the equation

$$\sum_{j\in I_l\backslash w_l} (\delta_{i\,j} - K_{i\,j})\,[P^{\varepsilon}_{j\,w_s} - P_{j\,w_s}] = \delta_1(\varepsilon)\sum_{j\in I_s} H^1_{i\,j}(\tau)\,P_{j\,w_s} + o(\delta_1(\varepsilon)).$$

Multiplying this equation by $V^{(l)}_{w_l\,i}$ and summing up for all $i \in I_l$, we obtain

$$\sum_{j\in I_l\backslash w_l} \Pi^{(l)}_{w_l\,j}\,[P^{\varepsilon}_{j\,w_s} - P_{j\,w_s}] = -\delta_1(\varepsilon)\sum_{\substack{i\in I_l\\ j\in I_s}} V^{(l)}_{w_l\,i}\,H^1_{i\,j}(\tau)\,P_{j\,w_s} - o(\delta_1(\varepsilon)). \tag{14}$$

Let us show that for all $i \in I_l\backslash w_l$ the solution of (14) is given by

$$P^{\varepsilon}_{i\,w_s} - P_{i\,w_s} = \delta_1(\varepsilon)\frac{1}{\Pi^{(l)}_{w_l\,w_l}}\sum_{\substack{j\in I_l\\ m\in I_s}} V^{(l)}_{i\,j}\,H^1_{j\,m}(\tau)P_{m\,w_s} - o(\delta_1(\varepsilon)) \tag{15}$$

Indeed, since $\Pi^{(l)}V^{(l)} = (0)$ for all $l = \overline{1,r}$, we easily get

$$\sum_{\substack{i,j\in I_l\\ m\in I_s}} \Pi^{(l)}_{w_l\,i}V^{(l)}_{i\,j}H^1_{j\,m}(\tau)P_{m\,w_s} = 0$$

and

$$\sum_{\substack{i\in I_l\backslash w_l\\ j\in I_l, m\in I_s}} \Pi^{(l)}_{w_l\,i}V^{(l)}_{i\,j}H^1_{j\,m}(\tau)P_{m\,w_s} = -\sum_{\substack{j\in I_l\\ m\in I_s}} \Pi^{(l)}_{w_l\,w_l}V^{(l)}_{w_l\,j}H^1_{j\,m}(\tau)P_{m\,w_s}.$$

Hence

$$\sum_{i\in I_l\backslash w_l} \Pi^{(l)}_{w_l\,i}[P^{\varepsilon}_{i\,w_s} - P_{i\,w_s}] = \delta_1(\varepsilon)\frac{1}{\Pi^{(l)}_{w_l\,w_l}}\sum_{\substack{i\in I_l\backslash w_l\\ j\in I_l, m\in I_s}} \Pi^{(l)}_{w_l\,i}V^{(l)}_{i\,j}H^1_{j\,m}(\tau)P_{m\,w_s} - o(\delta_1(\varepsilon)) =$$

$$= -\delta_1(\varepsilon)\frac{1}{\Pi^{(l)}_{w_l\,w_l}}\sum_{\substack{j\in I_l\\ m\in I_s}} \Pi^{(l)}_{w_l\,w_l}V^{(l)}_{w_l\,j}H^1_{j\,m}(\tau)P_{m\,w_s} - o(\delta_1(\varepsilon)) =$$

$$= -\delta_1(\varepsilon)\sum_{\substack{j\in I_l\\ m\in I_s}} V^{(l)}_{w_l\,j}H^1_{j\,m}(\tau)P_{m\,w_s} - o(\delta_1(\varepsilon))$$

and this coincides with the right-hand side of (14) .

Now in view of representations (12), (15) we can rewrite (10) as

$$\begin{aligned} v^{(s)}_{w_s}(P^{\varepsilon}_{w_s\,w_s} - 1) &= \sum_{k=1}^{n}\delta_k(\varepsilon)\sum_{i,\,j\in I_s} v^{(s)}_i H^k_{i\,j}(\tau)P_{j\,w_s} + \\ &+ \delta_1^2(\varepsilon)\sum_{i,j\in I_s} v^{(s)}_i\left(D^{11}_{i\,j}(\tau) + R^{11}_{i\,j}(\tau)\right)\,P_{j\,w_s} + o(\delta_1^2(\varepsilon)) \end{aligned} \tag{16}$$

where

$$R^{11}_{i\,j}(\tau) = \sum_{l=1}^{r}\frac{1}{\Pi^{(l)}_{w_l\,w_l}}\sum_{\substack{m\in I_l\\ k\in I_l\backslash w_l}} H^1_{i\,k}(\tau)V^{(l)}_{k\,m}H^1_{m\,j}(\tau).$$

Suppose that l_s is the first number for which $\sum\limits_{i,\,j\in I_s} v_i^{(s)}\, H_{i\,j}^{l_s}(\tau) u_j^{(s)} \neq 0$, $s = \overline{1,r}$. Then it follows from (16) that

$$P^{\varepsilon}_{w_s\, w_s} - 1 = \delta_{l_s}(\varepsilon)\frac{h_s^{l_s}}{v_{w_s}^{(s)} u_{w_s}^{(s)}} + o(\delta_{l_s}(\varepsilon)) + \delta_1^2(\varepsilon)\frac{d_s^{11}}{v_{w_s}^{(s)} u_{w_s}^{(s)}} + o(\delta_1^2(\varepsilon)). \tag{17}$$

There are three possibilities:

1) $\delta_1^2(\varepsilon) = o(\delta_{l_s}(\varepsilon))$. Then (17) implies $P^{\varepsilon}_{w_s\, w_s} - 1 = \delta_{l_s}(\varepsilon)\dfrac{h_s^{l_s}}{v_{w_s}^{(s)} u_{w_s}^{(s)}} + o(\delta_{l_s}(\varepsilon))$;

2) $\delta_{l_s}(\varepsilon) = \alpha\delta_1^2(\varepsilon)$. Then we have $P^{\varepsilon}_{w_s\, w_s} - 1 = \delta_1^2(\varepsilon)\dfrac{\alpha\, h_s^{l_s} + d_s^{11}}{v_{w_s}^{(s)} u_{w_s}^{(s)}} + o(\delta_1^2(\varepsilon))$;

3) $\delta_{l_s}(\varepsilon) = o(\delta_1^2(\varepsilon))$. In this case $P^{\varepsilon}_{w_s\, w_s} - 1 = \delta_1^2(\varepsilon)\dfrac{d_s^{11}}{v_{w_s}^{(s)} u_{w_s}^{(s)}} + o(\delta_1^2(\varepsilon))$.

To complete the proof it remains to recall that

$$\rho^{\varepsilon} = \sum_{s=1}^{r}(1 - P^{\varepsilon}_{w_s\, w_s})\frac{v_{w_s}^{(s)} u_{w_s}^{(s)}}{a_s},$$

and to put $l = \min\{l_1, \dots, l_r\}$. □

Asymptotic of ρ^{ε} when $d_s^{11} = 0$, $s = \overline{1,r}$ is found also, but we omit consideration of this case because of awkwardness of computations.

REFERENCES

1. Феллер В. Введение в теорию вероятностей и её приложения, т.2, М.: Мир, 1967.
2. Шуренков В. М. Эргодические процессы Маркова, М.: Наука, 1989.
3. Єлейко Я. І., Ніщенко І. І. *Гранична теорема для матричнозначої випадкової еволюції,* Вісник Львівського університету, сер. механіко-матем. (1999), №53, 102–107.
4. Єлейко Я. І., Шуренков В. М. *Деякі властивості випадкових еволюцій,* Укр. мат. журнал, **47** (1995), №10, 1333–1338.

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