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FREE TOPOLOGICAL INVERSE SEMIGROUPS

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In the paper we consider free objects $I(X)$, $IC(X)$, $IA(X)$, and $SL(X)$ in the category of topological inverse semigroups and its subcategories of topological inverse Clifford, inverse Abelian semigroups, and topological semilattices, respectively. We prove that these objects exist and are algebraically free over functionally Hausdorff spaces, they are (local) k_ω-spaces if and only if X is a (local) k_ω-space. We investigate also the question of preservation of embeddings by these free constructions.

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В статье рассматриваются свободные объекты в категории топологических инверсных полугрупп и ее подкатегориях топологических инверсных клиффордовых, инверсных абелевых полугрупп и топологических полурешеток. Доказывается, что эти свободные объекты существуют и являются алгебраически свободными для любого функционально хаусдорфова пространства X. Они являются (локальными) k_ω-пространствами, если таковым есть пространство X. Изучается также вопрос сохранения вложений этими свободными конструкциями.

Introduction

The paper is devoted to free topological inverse semigroups. For better understanding the obtained results, we briefly describe the situation in the related realm of free topological groups. The conception of a free topological group $F(X)$ over a topological space X was introduced by A. A. Markov in [36], [37]. He proved that for any Tychonoff space X a free topological group $F(X)$ exists and is unique, algebraically free and completely regular, see also [39], [32], [27]. In [27] M. Graev described the topology of a free topological group $F(X)$ over a compact space X and proved that in this case, topologically, $F(X)$ is a k_ω-space. A description of the topology of a free topological group over an arbitrary Tychonoff space was given only in the 1980-s in [41], [48], [46]. Another important question concerning free topological groups was as follows: under which conditions a free topological group $F(Y)$ of a subspace $Y \subset X$ is a subgroup in $F(X)$? It turned out that the answer to this question was not trivial, see [42], [48], [49], [47].

In this paper we consider free objects $I(X)$, $IC(X)$, $IA(X)$, $SL(X)$ in the category of topological inverse semigroups and its subcategories of topological inverse Clifford, inverse

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Abelian semigroups, and topological semilattices, respectively. We prove that for a functionally Hausdorff space X the free topological inverse semigroups $I(X)$, $IC(X)$, $IA(X)$, $SL(X)$ exist, are algebraically free and functionally Hausdorff (see Theorems 1 and 2). Unlike to the situation with free topological groups it is not clear whether the semigroups $I(X)$, $IC(X)$, $IA(X)$, $SL(X)$ are Tychonoff for Tychonoff spaces X. It is so, whenever X is a regular local k_ω-space. Moreover, in this case the semigroups $I(X)$, $IC(X)$, $IA(X)$, $SL(X)$ are local k_ω-spaces too, see Theorem 6. Local k_ω-spaces are particular cases of k-spaces. For which spaces X the semigroups $I(X)$, $IC(X)$, $IA(X)$, $SL(X)$ are k-spaces? It turns out (see Theorem 7) that for metrizable X this takes place if and only if the space X is locally compact, cf. [4].

Next, we consider the question of when for a subspace X of a functionally Hausdorff space Y the induced homomorphisms $I(X) \to I(Y)$, $IC(X) \to IC(Y)$, $IA(X) \to IA(Y)$, and $SL(X) \to SL(Y)$ are topological embedding. In Theorem 4 we give some sufficient conditions on the spaces X, Y which guarantee that these homomorphisms are topological embeddings. Moreover, in Theorem 5 we prove that for a metrizable space Y the mentioned homomorphisms are topological embeddings if and only if X is open in its closure in Y.

The conceptions of M- and A-equivalences of topological spaces were introduced by Graev [27] and afterwards were investigated by many authors in various situations, see [1], [2], [43], [40], [5]. In the case of free semigroups $I(X)$, $IC(X)$, $IA(X)$, $SL(X)$ their isomorphic classification coincides with the topological classification of the spaces X (see Theorem 10). That is not true for topological classification of these semigroups, e.g. for any finite-dimensional non-degenerate Peano continua X, Y their free topological semilattices $SL(X)$ and $SL(Y)$ are homeomorphic (see [50] for the corresponding result on free topological groups).

Finally, applying the obtained results, we prove Theorem 12 which can be considered as a counterpart of a known theorem of Franklin [25]. We pose also some open questions concerning the considered theory of free objects in the categories of topological inverse (Clifford, Abelian) semigroups and topological (Lawson) semilattices.

We follow the terminology of [20], [23], [44], [30], [15].

By $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ we denote the sets of natural, rational, real, and complex numbers, respectively. As usual, $\overline{A}$ or $\mathrm{cl}_X(A)$ denotes the closure of a subset A in a topological space X while $\mathrm{Int}(A)$ stands for the interior of A in X. Under a *neighborhood* of a point x of a topological space X we understand any subset $U \subset X$ whose interior contains the point x.

All maps considered in this paper are continuous. A topological space X is defined to be *functionally Hausdorff* if for any two distinct points x, x' of X there is a continuous map $f\colon X \to [0,1]$ such that $f(x) \neq f(x')$. It is well known that a topological space X is functionally Hausdorff if and only if it admits a continuous bijective map onto a Tychonoff space.

Definitions

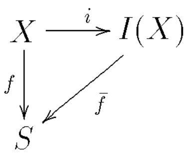
A *topological inverse semigroup* is, by definition, a Hausdorff topological space X equipped with a continuous binary associative operation $(\cdot)\colon X \times X \to X$ such that every element $x \in X$ has a unique inverse element x^{-1} and the map $(\cdot)^{-1}\colon X \to X$ assigning to each $x \in X$ its inverse x^{-1} is continuous (let us recall that x^{-1} is *inverse* to x if $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$). A topological inverse semigroup X is defined to be a *topological inverse*

Clifford semigroup (resp. a *topological inverse Abelian semigroup*) if $xx^{-1} = x^{-1}x$ for every $x \in X$ (resp. $xy = yx$ for each $x, y \in X$).

The class of topological inverse Clifford semigroups contains both the class of topological groups and the class of topological semilattices (in semilattices each element coincides with its inverse). Let us recall that a *topological semilattice* is, by definition, a topological space equipped with a continuous reflexive commutative associative binary operation. In the sequel we shall also need the conception of a *Lawson semilattice* [34], that is a topological semilattice admitting a base of the topology consisting of subsemilattices.

Under a *homomorphism* between topological semigroups X, Y we understand any continuous map $f\colon X \to Y$ such that $f(xx') = f(x)f(x')$ for any $x, x' \in X$. It is known that any homomorphism $f\colon X \to Y$ between inverse semigroups preserves the inversion, i.e. $f(x^{-1}) = (f(x))^{-1}$ for each $x \in X$. A bijective map f between topological semigroups is an *isomorphism* if both f and f^{-1} are homomorphisms.

A *free topological inverse semigroup over* a topological space X is a pair $(I(X), i)$ consisting of a topological inverse semigroup $I(X)$ and a topological embedding $i\colon X \to I(X)$ such that for every map $f\colon X \to S$ into a topological inverse semigroup S there exists a unique homomorphism $\bar{f}\colon I(X) \to S$ making the diagram



commutative.

Similarly, a free topological inverse Clifford semigroup $(IC(X), i)$, a free topological inverse Abelian semigroup $(IA(X), i)$, a free topological semigroup $(S(X), i)$, a free topological semilattice $(SL(X), i)$, and a free Lawson semilattice over a topological space X can be defined.

It follows from the definition that if a free topological inverse semigroup over X exists then it is unique up to isomorphism. The same concerns the other free objects over X. Next, we shall show that free topological inverse semigroups exist. For this we firstly recall some information about

Free Lawson semilattices

It is easily observed that any free Lawson semilattice over a topological space X can be identified with the hyperspace $\exp_\omega(X)$ of all finite non-empty subsets of X, equipped with the Vietoris topology [38]. The continuous semilattice operation on $\exp_\omega(X)$ is the union of subsets. Recall that the Vietoris topology on $\exp_\omega(X)$ is generated by the base

$$\langle U_1, \dots, U_n \rangle = \{A \in \exp_\omega(X) : A \subseteq U_1 \cup \dots \cup U_n, A \cap U_i \neq \varnothing \text{ for all } i = 1, \dots, n\},$$

where $U_1, \dots, U_n$ run over all open subsets of X. Remark that each base set $\langle U_1, \dots, U_n \rangle$ is an open subsemilattice in $\exp_\omega(X)$. We shall use the following well-known facts about $\exp_\omega(X)$:

1) for every subspace $Y \subseteq X$ the natural map $\exp_\omega(Y) \to \exp_\omega(X)$ is an embedding; moreover, if Y is open (closed) in X then so is $\exp_\omega(Y)$ in $\exp_\omega(X)$;

2) the natural map $i\colon X \to \exp_\omega(X)$ assigning to each $x \in X$ the one-point set $\{x\} \in \exp_\omega(X)$ is a topological embedding;
3) if $\mathcal{K}$ is a compact subset of $\exp_\omega(X)$, then its union $\bigcup \mathcal{K} = \bigcup_{\mathcal{F}\in\mathcal{K}} \mathcal{F}$ is a compact subset of X;
4) if the space X is Hausdorff then $\exp_\omega(X)$ is Hausdorff too, and the image $i(X)$ of X is closed in $\exp_\omega(X)$;
5) if the space X is Tychonoff or functionally Hausdorff then so is the space $\exp_\omega(X)$.

It is also worth having in mind the following simple fact (see [19]): for a topological space X the topological sum $S(X) = \bigoplus_{n=1}^\infty X^n$ is a free topological semigroup over X. The multiplication "$*$" in $S(X)$ is defined by the rule:

$$(x_1, \dots, x_n) * (y_1, \dots, y_m) = (x_1, \dots, x_n, y_1, \dots, y_m).$$

The space X is identified with the closed subset X^1 of $S(X)$. For any $n \in \mathbb{N}$ let $S_n(X) = \bigoplus_{k=1}^n X^k$ be the set of words of length $\le n$.

The construction of free topological inverse semigroups

We shall follow the idea of Kakutani (see [32]). Let X be a topological space. To construct a free topological inverse semigroup $(I(X), i)$ of X consider the set $\mathcal{F}$ of all possible pairwise non-isomorphic continuous maps $f_S\colon X \to S$ of X into topological inverse semigroups such that the set $f_S(X)$ algebraically generates S, that is S coincides with the smallest inverse subsemigroup in S containing $f_S(X)$ (here two maps $f_1\colon X \to S_1$, $f_2\colon X \to S_2$ are called *isomorphic* if $h \circ f_1 = f_2$ for some isomorphism $h\colon S_1 \to S_2$). Remark that the set $\mathcal{F}$ is not empty because it contains the canonical map $e\colon X \to \exp_\omega(X)$ of X into the free Lawson semilattice over X. Now consider the diagonal product

$$i = \triangle_{f_S\in\mathcal{F}} f_S\colon X \to \prod_{f_S\in\mathcal{F}} S$$

of maps belonging to $\mathcal{F}$. It is easy to see that the Tychonoff product $\prod_{f_S\in\mathcal{F}} S$ is a topological inverse semigroup. Let $I(X)$ denote the smallest inverse subsemigroup of $\prod_{f_S\in\mathcal{F}} S$ containing the set $i(X)$. Notice that $I(X) = \bigcup_{n=1}^\infty I_n(X)$, where $I_n(X)$ is the set of all products $i(x_1)^{\varepsilon_1} \cdots i(x_k)^{\varepsilon_k}$, where $k \le n$, $x_1, \dots, x_k \in X$ and $\varepsilon_i = \pm 1$. We claim that $(I(X), i)$ is a free topological inverse semigroup over X. Indeed, since the embedding $e\colon X \to \exp_\omega(X)$ belongs to $\mathcal{F}$, we have $\mathrm{pr} \circ i = e$, where $\mathrm{pr}\colon \prod_{f_S\in\mathcal{F}} S \to \exp_\omega(X)$ is the natural projection. Since e is an embedding, so is the map i. Moreover, if X is Hausdorff then e is a closed embedding and consequently, i is a closed embedding too.

Next, let $f\colon X \to S'$ be any continuous map into a topological inverse semigroup. Let $\langle f(X)\rangle$ denote the smallest inverse subsemigroup of S' containing the set $f(X)$. Then the map $f\colon X \to \langle f(X)\rangle$ is isomorphic to some map $f_G\colon X \to G$ from the set $\mathcal{F}$, i.e. there is an isomorphism $h\colon G \to \langle f(X)\rangle$ such that $h \circ f_G = f$. Let $\mathrm{pr}_G\colon \prod_{f_S\in\mathcal{F}} S \to G$ be the projection onto the G-coordinate. Then $\mathrm{pr}_G \circ i = f_G$. It is clear that the restriction $\mathrm{pr}_G|_{I(X)}\colon I(X) \to G$ is a continuous homomorphism of topological inverse semigroups. Hence

$$\bar{f} = h \circ \mathrm{pr}_G|_{I(X)}\colon I(X) \to \langle f(X)\rangle \subset S'$$

is a homomorphism with the property $\bar{f} = h \circ \mathrm{pr}_G \circ i = h \circ f_G = f$. To see that the map $\bar{f}$ is unique recall that each element $a \in I(X)$ can be written as a product $i(x_1)^{\varepsilon_1} \cdots i(x_n)^{\varepsilon_n}$, where $x_i \in X$ and $\varepsilon_i = \pm 1$. Since $\bar{f}$ is a homomorphism, $\bar{f}(a)$ is uniquely determined and must be equal to $f(x_1)^{\varepsilon_1} \cdots f(x_n)^{\varepsilon_n} \in S'$. Hence $(I(X), i)$ is a free topological inverse semigroup over X.

Let us remark that the construction I of free topological inverse semigroup is functorial: for any continuous map $f\colon X \to Y$ let $I(f)\colon I(X) \to I(Y)$ be a unique homomorphism making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ I(X) & \xrightarrow[I(f)]{} & I(Y) \end{array}$$

commutative. One can easily prove that $I(f \circ g) = I(f) \circ I(g)$ for any continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$.

Repeating these arguments we may also construct the free topological inverse Clifford semigroup $(IC(X), i)$, the free topological inverse Abelian semigroup $(IA(X), i)$, and the free topological semilattice $(SL(X), i)$ over each topological space X.

Summarizing, we obtain

Theorem 1. *For every topological space X there exist a free topological inverse semigroup $I(X)$, a free topological inverse Clifford semigroup $IC(X)$, a free topological inverse Abelian semigroup $IA(X)$, and a free topological semilattice $SL(X)$ over X. Moreover, if the space X is Hausdorff then the mentioned semigroups contain X as a closed subspace.*

Now let us look at the structure of the constructed free semigroups.

Free topological semilattice. Consider a (unique) homomorphism $h\colon SL(X) \to \exp_\omega(X)$ extending the identity map $X \to X$ (it exists according to the definition of $SL(X)$ as a free topological semilattice). It is known that algebraically, $\exp_\omega(X)$ is a free semilattice over the set X [45]. This implies that the map h is bijective. Thus, algebraically $SL(X)$ is a free semilattice of X. Moreover, if the space X is (functionally) Hausdorff then so is the space $SL(X)$.

Free topological inverse Clifford semigroup. We shall exploit the construction of free inverse semigroups due to Petrich [45]. Let $F(X)$ and $A(X)$ denote respectively the free topological group and the free topological Abelian group over a topological space X. It is known that free topological groups $F(X)$ and $A(X)$ exist for every topological space [32], [36]. Moreover, for a functionally Hausdorff space X the groups $F(X)$, $A(X)$ are known to be Tychonoff and algebraically free. For such X the natural maps $X \to F(X)$, $X \to A(X)$ are injective. This allows us to identify X with the set of generators of $F(X)$ or $A(X)$. So, from now on, X is a functionally Hausdorff space. For any $a \in F(X)$ let $\mathrm{supp}(a)$ denote the support of a, that is the smallest subset $A \subset X$ such that a lies in the image of the group $F(A)$ under the natural homomorphism $F(A) \to F(X)$. Analogously, the support $\mathrm{supp}(a)$ of a point $a \in A(X)$ can be defined. In fact, $\mathrm{supp}(a)$ is nothing else but the set of all letters in the reduced word of a, i.e. a word of the smallest length, representing the element a.

In the product $F(X) \times \exp_\omega(X)$ consider the subset

$$IC_X = \{(a, A) : \mathrm{supp}(a) \subset A\}.$$

Evidently, $F(X)\times\exp_\omega(X)$ is a topological inverse Clifford semigroup (as the product of topological inverse Clifford semigroups $F(X)$ and $\exp_\omega(X)$). It is easy to see that IC_X is an inverse subsemigroup in $F(X)\times\exp_\omega(X)$ and the map $j\colon X\to IC_X$ assigning to each $x\in X$ the pair $(x,\{x\})$ is a closed embedding. Since IC_X is a topological inverse semigroup, there exists a unique homomorphism $h\colon IC(X)\to IC_X$ such that $h\circ i=j$ where $i\colon X\to IC(X)$ is the embedding of X into $IC(X)$. According to [45], algebraically IC_X is a free inverse Clifford semigroup over X. Consequently, the homomorphism h is bijective and $IC(X)$ is algebraically a free inverse Clifford semigroup over X. Moreover, since $IC(X)$ maps injectively onto the functionally Hausdorff space IC_X, the topological semigroup $IC(X)$ is functionally Hausdorff.

Free topological inverse Abelian semigroup. Replacing the free topological group $F(X)$ by the free topological Abelian group $A(X)$ over X and repeating the preceding arguments we get that algebraically $IA(X)$ is a free inverse Abelian semigroup over X and $IA(X)$ is a functionally Hausdorff topological semigroup which can be identified with the semigroup

$$IA_X=\{(a,A):\operatorname{supp}(a)\subset A\}\subset A(X)\times\exp_\omega(X)$$

retopologized by the strongest inverse semigroup topology inducing on X its original topology.

Free topological inverse semigroup. We shall identify $I(X)$ with a subset of the product $F(X)\times\exp_\omega(F(X))$. For an element $a\in F(X)$ let $r(a)$ denote the reduced word of a, i.e. the word of the smallest length representing the element a. The identity 1 of $F(X)$ is identified with the empty word. For any reduced word $a=a_1\dots a_n\in F(X)$ let

$$\hat{a}=\{1,a_1,a_1a_2,\dots,a_1a_2\cdots a_n\}.$$

We call a subset A of $F(X)$ *saturated* if $a\in A$ implies $\hat{a}\subset A$. Finally, we put

$$I_X=\{(a,A):A\neq\{1\}\text{ is saturated and }r(a)\in A\}\subset F(X)\times\exp_\omega(F(X)).$$

with the multiplication $(a,A)\cdot(b,B)=(ab,A\cup aB)$, where both ab and aB are products taken in $F(X)$. As noted in [45], $(a^{-1},a^{-1}A)$ is the inverse of (a,A) in I_X. One can easily verify that the so-defined inverse semigroup operations of I_X are continuous with respect to the topology inherited from $F(X)\times\exp_\omega(F(X))$. Thus I_X is a topological inverse semigroup. Let $j\colon X\to I_X$ be the map assigning to each $x\in X$ the pair $(x,\{1,x\})\in I_X$. Evidently, j is a closed embedding. Let $h\colon I(X)\to I_X$ be a (unique) homomorphism such that $h\circ i=j$, where $i\colon X\to I(X)$ is the embedding of X into $I(X)$. It is known that algebraically I_X is a free inverse semigroup over X, see [45]. Because of this, the map h must be bijective. Consequently, algebraically, $I(X)$ is a free inverse semigroup over X; moreover the underlying topological space of $I(X)$ is functionally Hausdorff.

Let us summarize all said above in

Theorem 2. *For any functionally Hausdorff space X the free topological inverse semigroup $I(X)$, the free topological inverse Clifford semigroup $IC(X)$, the free topological inverse Abelian semigroup $IA(X)$, and the free topological semilattice $SL(X)$ are functionally Hausdorff and algebraically free.*

Question 1 (I. V. Protasov) Are the semigroups $I(X)$, $IC(X)$, $IA(X)$, and $SL(X)$ Tychonoff for a Tychonoff space X?

Free topological inverse semigroups over k_ω-spaces

Recall that a Hausdorff space X is defined to be a *k_ω-space* provided the topology of X is generated by a countable collection $\mathcal{K}$ of compact subsets of X in the sense that $X = \cup\mathcal{K}$ and a subset $U \subset X$ is open if and only if the intersection $U \cap K$ is open in K for every compactum $K \in \mathcal{K}$. According to [26], every k_ω-space is normal.

It is known that the free topological (Abelian) group over a k_ω-space is a k_ω-space too [35]. An analogous result holds also for free topological inverse semigroups (see also [16], [17], [18] for further generalizations).

Theorem 3. *If X is a k_ω-space then the topological semigroups $I(X)$, $IC(X)$, $IA(X)$, and $SL(X)$ are k_ω-spaces.*

Proof. Suppose X is a k_ω-space and $\{X_n\}_{n\in\mathbb{N}}$ is a countable collection of compact subsets of X generating the topology of X. Without loss of generality, $X_n \subset X_{n+1}$ for each $n \in \mathbb{N}$. We shall prove that $I(X)$ is a k_ω-space. Let $S\left(X \sqcup X^{-1}\right) = \bigoplus_{n\in\mathbb{N}}\left(X \sqcup X^{-1}\right)^n$ be the free topological semigroup over the topological sum of X and its copy X^{-1}. Let $p\colon S\left(X \sqcup X^{-1}\right) \to I(X)$ be the continuous map assigning to each sequence $(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n}) \in S\left(X \sqcup X^{-1}\right)$ the product $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ taken in $I(X)$ (here $\varepsilon_i = \pm 1$). Evidently, $I(X) = \bigcup_{n\in\mathbb{N}} I_n(X_n)$, where each $I_n(X_n)$ is compact as a continuous image of the compactum $S_n(X_n) = \bigoplus_{k=1}^{n}\left(X_n \sqcup X_n^{-1}\right)^k$. Let τ be the topology on X generated by the collection $\{I_n(X_n)\}_{n=1}^{\infty}$, that is a subset $U \subset I(X)$ is open in τ if and only if the intersection $U \cap I_n(X_n)$ is open in $I_n(X_n)$ for every $n \in \mathbb{N}$. We shall show that the semigroup $I(X)$ equipped with this topology is a topological inverse semigroup. First, notice that $\left(I_n(X_n)\right)^{-1} = I_n(X_n)$ and $I_n(X_n) \cdot I_n(X_n) \subset I_{2n}(X_{2n})$ for each $n \in \mathbb{N}$. By the definition of the topology τ, to prove the continuity of the inverse $(\cdot)^{-1}\colon (I(X), \tau) \to (I(X), \tau)$ it suffices to verify that this map is continuous on each $I_n(X_n)$. But this is obvious since $\left(I_n(X_n)\right)^{-1} \subset I_n(X_n)$ and on $I_n(X_n)$ the topology τ coincides with the original one. To prove that the multiplication is continuous with respect to the topology τ we shall use the well known fact stating that the product of k_ω-spaces is a k_ω-space [26]. To be more precise, the topology of the product $(I(X), \tau) \times (I(X), \tau)$ is generated by the collection $\{I_n(X_n) \times I_n(X_n)\}_{n\in\mathbb{N}}$. Since $I_n(X_n) \cdot I_n(X_n) \subset I_{2n}(X_{2n})$, the multiplication $(\cdot)\colon I_n(X_n) \times I_n(X_n) \to (I(X), \tau)$ is continuous for each $n \in \mathbb{N}$. Consequently, it is continuous and $(I(X), \tau)$ is a topological inverse semigroup.

Now consider the embedding $i\colon X \to I(X)$. Since $i(X_n) \subset I_n(X_n)$, $n \in \mathbb{N}$, we conclude that the map $i\colon X \to (I(X), \tau)$ is continuous. Then there exists a (unique) homomorphism $h\colon I(X) \to (I(X), \tau)$ such that $h \circ i = i$. Since $I(X)$ is a free inverse semigroup over X, h is the identity mapping. We claim that h is a homeomorphism. To prove this it suffices to show that the map $h^{-1} = \mathrm{id}\colon (I(X), \tau) \to I(X)$ is continuous. Fix any open set $U \subset I(X)$. Since $U \cap I_n(X_n)$ is open in $I_n(X_n)$ for each n, we get that U is open in the topology τ. Thus, the space $I(X)$ is homeomorphic to $(I(X), \tau)$ which is a k_ω-space.

Similar arguments show that the semigroups $IC(X)$, $IA(X)$, and $SL(X)$ are k_ω-spaces too. □

Embeddings of free topological inverse semigroups

In this section we investigate the action of the constructions I, IC, IA, and SL on maps. Especially, we shall be interested in the question of preservation of topological embeddings by these constructions. Let us remark that for free topological groups this question was resolved in [42], [49], and [47]. In particular, for a subspace X of a metrizable space Y the induced group homomorphism $F(X) \to F(Y)$ is a topological embedding if and only if the set X is closed in Y.

A similar result will be proven for free inverse topological semigroups: for a subset X of a metrizable space Y the induced homomorphisms $I(X) \to I(Y)$, $IC(X) \to IC(Y)$, $IA(X) \to IA(Y)$, and $SL(X) \to SL(Y)$ are topological embeddings if and only if the set X is locally closed in Y.

Here we call a subset X of a topological space Y *locally closed* if every point $x \in X$ has a neighborhood U in Y such that the intersection $U \cap X$ is closed in U. Equivalently, X is locally closed if X is open in its closure $\overline{X}$ in Y. We begin from the following statement resulting from Theorem 2.

Proposition 1. *If $f\colon X \to Y$ is a continuous injective (surjective) map between functionally Hausdorff spaces then so are the maps $I(f)\colon I(X) \to I(Y)$, $IC(f)\colon IC(X) \to IC(Y)$, $IA(f)\colon IA(X) \to IA(Y)$, and $SL(f)\colon SL(X) \to SL(Y)$.* □

This proposition allows us for a subset X of a functionally Hausdorff space Y to identity $I(X)$ with a subset (not a subspace!) of $I(Y)$. The same concerns the free semigroups $IC(X)$, $IA(X)$, and $SL(X)$.

Proposition 2. *Suppose X is an open (closed) subset of a functionally Hausdorff space Y. Then the set $I(X)$ (resp. $IC(X)$, $IA(X)$, $SL(X)$) is open (closed) in $I(Y)$ (resp. in $IC(Y)$, $IA(Y)$, $SL(Y)$).*

Proof. Using the definition of $I(Y)$ as a free topological inverse semigroup, define $\mathrm{supp}\colon I(Y) \to \exp_\omega(Y)$ to be a (unique) continuous homomorphism extending the identity map $\mathrm{id}\colon Y \to Y$. Using the fact that $I(X)$ is generated by the set X, one may easily prove that $I(X) = \mathrm{supp}^{-1}(\exp_\omega(X))$. Now if X is open (closed) in Y then $\exp_\omega(X)$ is open (closed) in $\exp_\omega(Y)$. Consequently, $I(X)$ is open (closed) in $I(Y)$. Similar arguments work also for the semigroups $IC(X)$, $IA(X)$, and $SL(X)$. □

Let us define a space X to be a *local k_ω-space* provided any point $x \in X$ has a neighborhood which is a k_ω-space.

Theorem 4. *Suppose X is a subspace of a functionally Hausdorff space Y. The inclusions $I(X) \subset I(Y)$, $IC(X) \subset IC(Y)$, $IA(X) \subset IA(Y)$, and $SL(X) \subset SL(Y)$ are topological embeddings provided one of the following conditions is satisfied:*

1. *X is a retract of Y;*
2. *Y is metrizable and X is locally closed in Y;*
3. *Y is a k_ω-space and X is locally closed in Y;*
4. *X is compact;*
5. *Y is regular and X is locally compact;*
6. *Y is regular and X is an open local k_ω-subspace in Y.*

Proof. We shall consider the inclusion $I(X) \subset I(Y)$ only; for the other inclusions the proofs are analogous. To prove the items 2, 3, and 5 we will need the following

Lemma 1. *Suppose X is a subset of a functionally Hausdorff space Y. The homomorphism $I(X) \to I(Y)$ is a topological embedding, provided every finite subset of X has a neighborhood F in X such that F is closed in Y and the homomorphism $I(F) \to I(Y)$ is a topological embedding.*

Proof. Let $e\colon X \to Y$ denote the natural embedding. Denote by $\mathrm{supp}\colon I(X) \to \exp_\omega(X)$ a unique homomorphism extending the identity embedding $X \to \exp_\omega(X)$. To prove that the map $I(e)$ is a topological embedding, fix an element $a \in I(X)$ and consider its support $\mathrm{supp}(a) \subset X$. By the hypothesis, $\mathrm{supp}(a)$ has a neighborhood $F \subset X$ such that F is closed in Y and the inclusion $I(F) \to I(Y)$ is a topological embedding.

Consider the chain of embeddings $F \xrightarrow{e_F} X \xrightarrow{e} Y$ and the chain of the induced homomorphisms $I(F) \xrightarrow{I(e_F)} I(X) \xrightarrow{I(e)} I(Y)$. By Proposition 2, the set $V = I(e_F)(I(F))$ is a neighborhood of a in $I(X)$ and the set $I(e)(V) = I(e \circ e_F)(I(F))$ is a neighborhood of $I(e)(a)$ in $I(e)(I(X)) \subset I(Y)$. By the hypothesis, the composition $I(e) \circ I(e_F) = I(e \circ e_F)\colon I(F) \to I(Y)$ is an embedding. This implies that the restriction $I(e)|_V$ is a homeomorphism onto its image.

Therefore each element $a \in I(X)$ has a neighborhood $V \subset I(X)$ such that the restriction $I(e)|_V\colon V \to I(e)(V)$ of $I(e)$ onto V is a homeomorphism onto a neighborhood of $I(e)(a)$ in $I(e)(I(X))$. This fact together with the injectivity of $I(e)$ imply that $I(e)\colon I(X) \to I(Y)$ is a topological embedding. □

Now let us return to the proof of Theorem 4.

1. If $r\colon Y \to X$ is a retraction, i.e. $r \circ e = \mathrm{id}_X$, where $e\colon X \to Y$ is the embedding, then, by the functoriality of the construction I, we obtain $I(r) \circ I(e) = \mathrm{id}\,|_{I(X)}$ which implies $I(e)$ is a closed topological embedding.

2. Using the regularity of metrizable spaces and applying Lemma 1 we reduce our task to considering the particular case when the set X is closed in Y. We will exploit the Hartman-Mycielski construction $HM(X)$ over X, see [28]. Recall that $HM(X)$ is the set of all maps $f\colon [0,1) \to X$ for which there exist $n \in \mathbb{N}$ and a sequence $0 = a_0 < a_1 < \cdots < a_n = 1$ such that f is constant on each interval $[a_{i-1}, a_i)$, $1 \le i \le n$. If d is a bounded metric on X then the formula $\hat{d}(f,g) = \int_0^1 d(f(t), g(t))dt$ defines a metric on $HM(X)$. The topology on $HM(X)$ generated by this metric does not depend on the choice of the bounded metric d generating the topology of X. Moreover, a reasonable topology on $HM(X)$ can be defined for any (not necessarily metrizable) topological space X, see [14]. Remark that the space X can be identified with the subspace of constant functions in $HM(X)$. We shall use the following important extension property of the space $HM(X)$ (proven in [7, Proposition 3]): in the case where X is a closed subset of a metrizable (more generally stratifiable) space Y, the identity embedding $X \to HM(X)$ extends to a continuous map $\xi\colon Y \to HM(X)$.

Now consider the free topological inverse semigroup $I(X)$ of X and the Hartman-Mycielski construction $HM(I(X))$ over $I(X)$. Since X is a subspace of $I(X)$, we may consider $HM(X)$ as a subspace of $HM(I(X))$, see Proposition 2 of [14]. Notice that the operation of pointwise multiplication of functions turns $HM(I(X))$ into an inverse topological semigroup (cf. Corollary 2 of [14]). Hence, the map $\xi\colon Y \to HM(X) \subset HM(I(X))$ can be uniquely extended to a continuous homomorphism $\bar{\xi}\colon I(Y) \to HM(I(X))$. Denote by $e\colon X \to Y$ the natural embedding and let $I(e)\colon I(X) \to I(Y)$ be the continuous injective homomorphism extending the map e. Notice that the composition $\bar{\xi} \circ I(e)\colon I(X) \to HM(I(X))$ coincides with the natural topological embedding $I(X) \subset HM(I(X))$. This immediately implies that $I(e)\colon I(X) \to I(Y)$ is a topological embedding.

3. As in the preceding case, it is enough to prove the particular case of a closed subset X of a k_ω-space Y. But this case can be easily derived from the proof Theorem 3 describing the topological structure of free topological inverse semigroups over k_ω-spaces.

4. Suppose X is compact. Denote by $\mathcal{F}$ the collection of all continuous maps from Y into the segment $[0,1]$ and consider the map $\xi\colon Y \to [0,1]^{\mathcal{F}}$ of Y into the Tychonoff cube $[0,1]^{\mathcal{F}}$ defined by $\xi(y) = (f(y))_{f\in\mathcal{F}}$, $y \in Y$. Denote by $e\colon X \to Y$ the natural embedding. Since Y is functionally Hausdorff, the map ξ is injective. Then the compactness of X implies that the composition $\xi \circ e$ is a closed embedding. By item 3, the homomorphism $I(\xi \circ e) = I(\xi) \circ I(e)\colon I(X) \to I([0,1]^{\mathcal{F}})$ is a topological embedding. This implies that the map $I(e)\colon I(X) \to I(Y)$ is a topological embedding too.

5. This easily follows from the preceding item and Lemma 1.

6. Finally, suppose X is an open local k_ω-subspace of a regular space Y. Denote by $e\colon X \to Y$ the embedding. Since the map $I(e)\colon I(X) \to I(Y)$ is injective, to prove that $I(e)$ is an embedding, it suffices for every point $a \in I(X)$ to find a neighborhood $W \subset I(X)$ of a such that $I(e)(W)$ is open in $I(Y)$ and $I(e)|_W$ is a topological embedding. So, fix any point $a \in I(X)$. Since X is a local k_ω-space, the finite set $\mathrm{supp}(a) \subset X$ has an open neighborhood U which is a k_ω-space. Using the regularity of Y, find an open neighborhood $V \subset Y$ of $\mathrm{supp}(a)$ such that $\mathrm{cl}_Y(V) \subset U$. Notice that the quotient space $Y/(Y\backslash V)$ coincides with the quotient space $U/(U\backslash V)$ and the latter is a k_ω-space. Then we have the following chain of continuous maps

$$V \overset{i_V}{\hookrightarrow} U \overset{i_U}{\hookrightarrow} X \overset{i}{\hookrightarrow} Y \overset{\pi}{\longrightarrow} Y/(Y\backslash V)$$

which induces the chain of continuous homomorphisms

$$I(V) \overset{I(i_V)}{\hookrightarrow} I(U) \overset{I(i_U)}{\longrightarrow} I(X) \overset{I(i)}{\longrightarrow} I(Y) \overset{I(\pi)}{\longrightarrow} I\left(Y/(Y\backslash V)\right).$$

Remark that $W = I(i_U \circ i_V)(I(V))$ is an open neighborhood of a and $I(i)(W)$ is open in $I(Y)$. Since the composition $\pi \circ i \circ i_U \circ i_V$ embeds V as an open subset into the k_ω-space $Y/(Y\backslash V)$, the composition $I(\pi) \circ I(i) \circ I(i_U) \circ I(i_V)$ embeds $I(V)$ into $I(Y/(Y\backslash V))$ (see item 3). Consequently, $I(i)$ embeds W into $I(Y)$. □

Remark 1. From the proof it follows that the requirement of the metrizability of the space Y in item 2 of Theorem 4 can be weakened to the stratifiability of Y (for definition and properties of stratifiable spaces, see [13]). In the meantime, the following theorem shows that for metrizable Y the condition of local closedness of X in Y is the best possible.

Theorem 5. *Let Y be a metrizable space and X be a subset of Y. The homomorphisms $I(X) \to I(Y)$, $IC(X) \to IC(Y)$, $IA(X) \to IA(Y)$ and $SL(X) \to SL(Y)$ are topological embeddings if and only if the subset X is locally closed in Y.*

Proof. The sufficiency follows from item 2 of Theorem 4. To prove the necessity, suppose the subset X is not locally closed. Then X is not open in its closure $\overline{X}$ and we may find a compact subset $K \subset Y$ such that the intersection $K \cap X$ is not locally compact (cf. [22, 8.3] or [8, Lemma 7]). Now the necessity follows from the subsequent □

Lemma 2. *Let Y be a functionally Hausdorff space and X a normal subspace of Y. If Y contains a compact subset K such that $K \cap X$ is metrizable and non-locally compact, then the* *homomorphisms $I(X) \to I(Y)$, $IC(X) \to IC(Y)$, $IA(X) \to IA(Y)$ and $SL(X) \to SL(Y)$ are not topological embeddings.*

Proof. Suppose $K \subset Y$ is a compact subset such that $K \cap X$ is metrizable and non-locally compact. Consider the following subsets of the complex plane:

$$T = \{re^{i\varphi} : r \leq 1,\ 0 < \varphi \leq 1\} \text{ and } T_0 = \{0\} \cup \Big\{\frac{1}{n}e^{i/m} : n, m \in \mathbb{N}\Big\}.$$

According to [22, 8.3] or [8, Lemma 7] any non-locally compact metrizable space contains a closed topological copy of the space T_0. Hence there exists a closed embedding $h\colon T_0 \to X$ such that $h(T_0) \subset X \cap K$. By Theorem 2, the free topological semilattice $SL(X)$ is algebraically free and thus each point of $SL(X)$ can be identified with a non-empty finite subset of X. The same concerns the other considered free topological semilattices. In $SL(T_0)$ consider the subset

$$F = \Big\{A \in SL(T_0) : A \subset \Big\{\frac{1}{|A|}e^{i/m} : m \in \mathbb{N}\Big\}\Big\}$$

Let $f\colon X \to Y$ denote the natural embedding. According to Theorem 4, $SL(K)$ can be identified with a subspace of $SL(Y)$.

Let us show that $f \circ h(0) \in K \subset SL(K) \subset SL(Y)$ is a cluster point of the set $SL(f \circ h)(F) \subset SL(K) \subset SL(Y)$. For this, fix any neighborhood U of $f \circ h(0)$ in $SL(K)$. $SL(K)$, being a k_ω-space, is regular. Hence there exists a neighborhood $V \subset SL(K)$ of $f \circ h(0)$ such that $\overline{V} \subset U$. Then $(f \circ h)^{-1}(V)$ is a neighborhood of 0 in T_0 and hence there is an $n \in \mathbb{N}$ such that $\frac{1}{n}e^{i/m} \in (f \circ h)^{-1}(V)$ for every $m \in \mathbb{N}$. Since $K \cap \overline{V}$ is compact, the set $f \circ h(\{\frac{1}{n}e^{i/m} : m \in \mathbb{N}\}) \subset \overline{V} \cap K$ has a cluster point $x \in \overline{V} \cap K$. Since $x^n = x$, there is a neighborhood W of x in $SL(K)$ such that $W^n \subset U$. Because $W \cap K$ is a neighborhood of x in K and x is a cluster point of the set $f \circ h(\{\frac{1}{n}e^{i/m} : m \in \mathbb{N}\})$ we may find n distinct numbers $m_1, \dots, m_n \in \mathbb{N}$ such that $f \circ h(\frac{1}{n}e^{i/m_k}) \in W$ for each $1 \leq k \leq n$. Then $f \circ h(\{\frac{1}{n}e^{i/m_k} : 1 \leq k \leq n\}) \in W^n \subset U$. Evidently, $f \circ h(\{\frac{1}{n}e^{i/m_k} : 1 \leq k \leq n\}) \in SL(f \circ h)(F)$, i.e. $U \cap SL(f \circ h)(F) \neq \varnothing$ and thus $f \circ h(0)$ is a cluster point of $SL(f \circ h)(F)$.

Now consider the (injective continuous) semilattice homomorphism $SL(f)\colon SL(X) \to SL(Y)$. To prove that $SL(f)$ is not an embedding, it suffices to verify that $h(0) \in X \subset SL(X)$ is not a cluster point of the set $SL(f)^{-1}(SL(f \circ h)(F)) = SL(h)(F)$ in $SL(X)$. The space T, being a convex G_δ-subset of the complex plane, is an absolute extensor for normal spaces, see [31, Ch.II, §14 and §16]. Hence the embedding $h^{-1}\colon h(T_0) \to T$ can be extended to a map $g\colon X \to T$. Because $SL(h)(F) \subset SL(g)^{-1}(F)$, to show that $h(0)$ is not a cluster point of the set $SL(h)(F)$ in $SL(X)$, it is enough to verify that 0 is not a cluster point of the set F in $SL(T)$.

For this we shall construct a special semilattice S over T as follows. In the free Lawson semilattice $\exp_\omega(T)$ over T consider the subset

$$S = \bigcup_{0 \leq r \leq 1} \{A \in \exp_\omega(T) : |z| = r \text{ for any } z \in A\}.$$

The semilattice operation "$*$" on S is defined by the rule:

$$A * B = (\min_{z \in A \cup B} |z|) \cdot \{e^{i\arg(z)} : z \in A \cup B\} \subset \mathbb{C}$$

for any $A, B \in S$. Remark that this operation is continuous with respect to the Vietoris topology $\mathcal{V}$ on S (that is, the topology on S inherited from $\exp_\omega(T)$).

Now we enrich the topology of S at the point 0. For every $k \in \mathbb{N}$ consider the subset

$$F_k = \left\{ A \in S : A \subset \{re^{i/m} : m \in \mathbb{N}\} \quad \text{for some } r \geq \frac{1}{k|A|} \right\} \subset S.$$

Define a topology τ on S letting $\{U \backslash F_k \mid U \in \mathcal{V}, k \in \mathbb{N}\}$ be its neighborhood base at 0; at other points of S the topology τ coincides with the Vietoris topology $\mathcal{V}$. We claim that the semilattice operation "$*$" is continuous with respect to the topology τ. Obviously, it is enough to prove the continuity of the operation "$*$" at pairs $(A_0, B_0) \in S \times S$, where $A_0 = 0$.

Let $U \backslash F_k$, $U \in \mathcal{V}$, $k \in \mathbb{N}$, be any neighborhood of $0 = A_0 * B_0$ in the topology τ. Since the operation "$*$" is continuous in the Vietoris topology, there are neighborhoods $V, W \in \mathcal{V}$ of A_0, B_0, respectively, such that $V * W \subset U$. We consider separately two cases: $B_0 = 0$ and $B_0 \neq 0$. In the first case we claim that $(V \backslash F_{2k}) * (W \backslash F_{2k}) \subset U \backslash F_k$. Indeed, suppose on the contrary that $A * B \in F_k$ for some $A \in V \backslash F_{2k}$ and $B \in W \backslash F_{2k}$. Then $A * B \subset \{re^{i/m} : m \in \mathbb{N}\}$ for some $r \geq \frac{1}{k|A*B|}$. By the definition of the operation "$*$", $|A * B| \leq |A| + |B| \leq 2 \max\{|A|, |B|\}$. Without loss of generality, $|A| \geq |B|$ and hence $|A * B| \leq 2|A|$. Then $A * B \subset \{re^{i/m} : m \in \mathbb{N}\}$ for $r \geq \frac{1}{k|A*B|}$ implies $A \subset \{r'e^{i/m} : m \in \mathbb{N}\}$, where $r' \geq \frac{1}{k|A*B|} \geq \frac{1}{2k|A|}$, i.e., $A \in F_{2k}$, a contradiction. Therefore, $(V \backslash F_{2k}) * (W \backslash F_{2k}) \subset U \backslash F_k$.

Now consider the case $B_0 \neq 0$. Find $m_0 \in N$ such that $\arg(z) > \frac{1}{m_0}$ and $|z| > \frac{1}{m_0}$ for every $z \in B_0$. We may choose a neighborhood W of B_0 such that $\arg(z) > \frac{1}{m_0}$ and $|z| > \frac{1}{m_0}$ for each $z \in B \in W$. We claim that $(V \backslash F_{2km_0}) * W \subset U \backslash F_k$. Suppose on the contrary that $A * B \subset F_k$ for some $A \in V \backslash F_{2km_0}$ and $B \in W$. Then $A * B \subset \{re^{i/m} : m \in \mathbb{N}\}$ for some $r \geq \frac{1}{k|A*B|}$. By the definition of the operation "$*$" this implies $B \subset \{r_B e^{i/m} : m \in \mathbb{N}\}$ for some $r_B \geq \frac{1}{k|A*B|}$ and $A \subset \{r_A e^{i/m} : m \in \mathbb{N}\}$ for some $r_A \geq \frac{1}{k|A*B|}$. Since $\arg(z) > \frac{1}{m_0}$ for each $z \in B$, we get $|B| \leq m_0$. Then

$$k|A * B| \leq k(|A| + |B|) \leq k(|A| + m_0) \leq 2km_0|A|.$$

Hence

$$r_A \geq \frac{1}{k|A * B|} \geq \frac{1}{2km_0|A|}$$

and $A \in F_{2km_0}$, a contradiction.

Therefore $(S, \tau, *)$ is a topological semilattice. Since the natural inclusion $\alpha \colon T \to S$ is continuous with respect to the topology τ on S and $SL(T)$ is a free topological semilattice, the (unique) semilattice homomorphism $\bar{\alpha} \colon SL(T) \to S$ extending the map α is continuous. By the definition of the topology τ, the point 0 is not cluster for the set $F_1 \subset S$. Consequently, 0 is not a cluster point for the subset $\bar{\alpha}^{-1}(F_1)$ in $SL(T)$. Because $F \subset \bar{\alpha}^{-1}(F_1)$ this implies that 0 is not a cluster point for the set F in $SL(T)$. Consequently, the map $SL(f)$ is not an embedding.

Considering the semilattices of idempotents, by similar arguments it can be shown that the maps $I(f)$, $IC(f)$ and $IA(f)$ neither are embeddings. □

Question 2. Is Theorem 5 true for all functionally Hausdorff spaces Y? (Notice that the proof can be adapted in order to show that the theorem holds for first countable stratifiable spaces Y).

Some topological properties of free topological inverse semigroups

First, we derive from Theorem 4 the following important

Theorem 6. *If X is a Tychonoff local k_ω-space then so are the spaces $I(X)$, $IC(X)$, $IA(X)$, and $SL(X)$.*

Proof. Suppose X is a Tychonoff local k_ω-space. Show firstly that $I(X)$ is a local k_ω-space. Fix any point $a \in I(X)$. Since X is a local k_ω-space, the finite set $\mathrm{supp}(a)$ has an open neighborhood U which is a k_ω-space. According to Theorem 3, $I(U)$ is a k_ω-space and by Proposition 2 and item 6 of Theorem 4, $I(U)$ is an open subspace of $I(X)$. Since $a \in I(U)$, we see that $I(U)$ is a k_ω-neighborhood of a. Thus $I(X)$ is a local k_ω-space.

To show that $I(X)$ is Tychonoff let F be a closed subset in $I(X)$ such that $a \notin F$. Since X is Tychonoff, we may find a neighborhood $V \subset X$ of $\mathrm{supp}(a)$ such that $\overline{V} = \mathrm{cl}_X(V) \subset U$. Then by Proposition 2, $I(\overline{V})$ is a closed subset of $I(X)$. The space $I(U)$, being a k_ω-space, is Tychonoff. Thus, there is a continuous function $f\colon I(U) \to [0,1]$ such that $f(a) = 1$ and $f((I(U)\setminus I(V)) \cup F) = 0$. Extend f over all $I(X)$ letting $f|_{I(X)\setminus I(V)} \equiv 0$. Obviously, the so extended map is continuous and has the properties: $f(a) = 1$ and $f(F) = 0$. Thus the space $I(X)$ is Tychonoff. □

It follows from the proof of Theorem 3 that in the case of a k_ω-space X the structure of compact subsets of $I(X)$ is quite understandable: every such a subset $K \subset I(X)$ lies in $I_n(C)$ for some compact subset $C \subset X$ and some $n \in \mathbb{N}$. It turns out that the same is true for any functionally Hausdorff space X.

Proposition 3. *For a functionally Hausdorff space X and a compact subset K of $I(X)$ (resp. of $IC(X)$, $IA(X)$, $SL(X)$) there are a compact subset $C \subset X$ and $n \in \mathbb{N}$ such that K lies in $I_n(C)$ (resp. $IC_n(C)$, $IA_n(C)$, $SL_n(C)$).*

Proof. Let $\mathrm{supp}\colon I(X) \to \exp_\omega(X)$ be a unique homomorphism extending the identity embedding $X \subset \exp_\omega(X)$. By the continuity of the homomorphism supp, the subset $\mathrm{supp}(K)$ of $\exp_\omega(X)$ is compact. Consequently, its union $C = \cup_{a\in K}\mathrm{supp}(a)$ is a compact subset of X. By Theorem 4, the natural homomorphism $I(C) \to I(X)$ is a topological embedding. Thus we can consider K as a compact subset of the k_ω-semigroup $I(C)$. Since the collection $\{I_n(C)\}_{n\in\mathbb{N}}$ generates the topology of $I(C)$ (see the proof of Theorem 3), we conclude that $K \subset I_n(C)$ for some $n \in \mathbb{N}$.

The same argument works in the case of the semigroups $IC(X)$, $IA(X)$, and $SL(X)$. □

Remark that each local k_ω-space is a k-space. According to [4] the free topological group $F(X)$ over a metrizable space X is a k-space if and only if either X is discrete or X is a k_ω-space; the free topological Abelian group $A(X)$ over a metrizable space X is a k-space if and only if X is a topological sum of a discrete space and a k_ω-space. What can be said about free topological inverse semigroups?

Proposition 4. *Suppose X is a functionally Hausdorff space. If the free topological inverse semigroup $I(X)$ (or $IC(X)$, $IA(X)$, $SL(X)$) is a k-space, then every closed metrizable subspace in X is locally compact.*

Proof. Suppose X contains a closed non-locally compact metrizable subspace. Then X contains a closed topological copy of the above-mentioned non-locally compact space $T_0 = \{0\} \cup \{\frac{1}{n}e^{i/m} : n, m \in \mathbb{N}\} \subset \mathbb{C}$, see [22, 8.3] or [8]. Let $f\colon T_0 \to X$ be the corresponding closed embedding. Then the map $g = f \cdot f^{-1}\colon T_0 \to E(I(X))$, $g : x \mapsto f(x)(f(x))^{-1}$ is a closed embedding of T_0 into the subset $E(I(X))$ of all idempotents of $I(X)$. (This follows from the equality $\mathrm{supp} \circ g = \mathrm{supp} \circ f$ and the fact that $\mathrm{supp} \circ f = f$ is a closed embedding, where $\mathrm{supp} : I(X) \to \exp_\omega(X)$ is a unique continuous homomorphism extending the natural embedding $X \subset \exp_\omega(X)$). Let $a_0 = g(0)$ and $a_{n,m} = g\left(\frac{1}{n}e^{i/m}\right)$ for $n, m \in \mathbb{N}$ and put $b_{n,m} = a_{m,m} \cdot a_{m+1,m+1} \cdot \dots \cdot a_{m+n,m+n}$ for $n, m \in \mathbb{N}$. Consider the set

$$Z = \{a_{n,m} \cdot b_{n,m} : m > n\} \subset E(I(X)).$$

We claim that Z is a closed set in $I(X)$. Since $I(X)$ is a k-space, it suffices to verify that the intersection $Z \cap K$ is closed in K for any compact subset $K \subset I(X)$. So, fix a compactum $K \subset I(X)$. By Proposition 3, $K \subset I_{n_0}(C)$ for some $n_0 \in \mathbb{N}$ and some compact subset $C \subset X$. Since $f\colon T_0 \to X$ is a closed embedding, $f^{-1}(C)$ is a compact subset of T_0. Thus there exists $m_0 \in \mathbb{N}$ such that $\frac{1}{n}e^{i/m} \notin f^{-1}(C)$, provided $n \le n_0$ and $m > m_0$. Fix any point $a_{n,m} \cdot b_{n,m} \in Z \cap K$. Then $a_{n,m} \cdot b_{n,m} \in I_{n_0}(C)$ and thus $n \le n_0$ and $\mathrm{supp}\,(a_{n,m} \cdot b_{n,m}) \subset C$. Since $f\left(\frac{1}{n}e^{i/m}\right) \in \mathrm{supp}\,(a_{n,m} \cdot b_{n,m}) \subset C$ we get $m \le m_0$. This implies that the intersection $Z \cap K \subset \{a_{n,m} \cdot b_{n,m} : n \le n_0, m \le m_0\}$ is finite and, consequently, closed in K. Therefore the set Z is closed in $I(X)$. Now use the continuity of the multiplication to find a neighborhood $U \subset I(X)$ of $a_0 = g(0)$ such that $(U \cdot U) \cap Z = \varnothing$ (remark that $a_0 \notin Z$ and $a_0 \cdot a_0 = a_0$). Since $g^{-1}(U)$ is a neighborhood of 0 in T_0, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n}e^{i/m} \in g^{-1}(U)$ for all $n \ge n_0$ and $m \in \mathbb{N}$. This implies $a_{n,m} \in U$ for any $n \ge n_0$ and $m \in \mathbb{N}$. Since $a_0^{n_0} = a_0$ there is a neighborhood V of a_0 such that $V^{n_0} \subset U$. Since the sequence $\{a_{m,m}\}$ tends to a_0 as $m \to \infty$, there is $m_0 \ge n_0$ such that $a_{m,m} \in V$ for any $m \ge m_0$. This implies $b_{n_0,m_0} \in U$ and thus $a_{n_0,m_0} \cdot b_{n_0,m_0} \in (U \cdot U) \cap Z$, a contradiction with $(U \cdot U) \cap Z = \varnothing$. □

Theorem 6 and Proposition 4 imply the following theorem characterizing metrizable spaces whose free topological inverse semigroups are k-spaces.

Theorem 7. *For a metrizable space X the following conditions are equivalent:*

1) *one of the spaces $I(X)$, $IC(X)$, $IA(X)$, or $SL(X)$ is a k-space;*
2) *the semigroups $I(X)$, $IC(X)$, $IA(X)$, and $SL(X)$ are Tychonoff local k_ω-spaces;*
3) *the space X is locally compact.*

Now we consider some dimension questions connected with free topological inverse semigroups. Recall that a space X is defined to be *totally disconnected* if for every distinct points $x, y \in X$ there is a clopen (= closed-and-open) subset $U \subset X$ such that $x \in U$ but $y \notin U$.

Theorem 8. *If X is a totally disconnected space then so are the spaces $I(X)$, $IC(X)$, $IA(X)$, and $SL(X)$.*

Proof. Suppose X is totally disconnected. Fix two distinct points $a, b \in I(X)$ and let $C = \mathrm{supp}(a) \cup \mathrm{supp}(b)$. Since C is a finite subset of X, there is a retraction $r\colon X \to C$ which induces the retraction $I(r)\colon I(X) \to I(C)$. According to statement 1 of Theorem 4, $I(C)$ is a discrete subspace in $I(X)$. Thus the set $\{a\}$ is a clopen neighborhood of a in $I(C)$

not containing the point b. Consequently, $U = (I(r))^{-1}(\{a\})$ is a clopen neighborhood of a in $I(X)$ such that $b \notin U$. Therefore, $I(X)$ is totally disconnected. By analogy, prove that the spaces $IC(X)$, $IA(X)$, and $SL(X)$ are totally disconnected too. □

Under a *zero-dimensional space* we understand a topological space X admitting a base of topology consisting of clopen subsets (of course, this is equivalent to $\operatorname{ind} X \leq 0$). Evidently, every zero-dimensional Hausdorff space is totally disconnected. For compact spaces the converse is also true, see [23, 7.1.12]. Moreover, since each k_ω-space is normal (see [26]), Theorem 7.2.1 [23] implies that the same is true for k_ω-spaces: any totally disconnected k_ω-space is zero-dimensional.

Since the zero-dimensionality is a local property, we get that a regular local k_ω-space is zero-dimensional if and only if it is totally disconnected. This fact and Theorems 6 and 8 imply

Theorem 9. *If X is a zero-dimensional local k_ω-space then so are the spaces $I(X)$, $IC(X)$, $IA(X)$, and $SL(X)$.*

Question 3 Are the spaces $I(X)$, $IC(X)$, $IA(X)$, and $SL(X)$ zero-dimensional for a zero-dimensional space X?

Isomorphisms of free topological inverse semigroups

The main result of this section reads as follows (for related results, see [2], [3], [6], [40], [43]).

Theorem 10. *For functionally Hausdorff spaces X, Y the following conditions are equivalent:*

1) *X and Y are homeomorphic;*
2) *the free Lawson semilattices $\exp_\omega(X)$ and $\exp_\omega(Y)$ are isomorphic;*
3) *the free topological semilattices $SL(X)$ and $SL(Y)$ are isomorphic;*
4) *the free topological inverse Abelian semigroups $IA(X)$ and $IA(Y)$ are isomorphic;*
5) *the free topological inverse Clifford semigroups $IC(X)$ and $IC(Y)$ are isomorphic;*
6) *the free topological inverse semigroups $I(X)$ and $I(Y)$ are isomorphic.*

Proof. The implications $1) \Rightarrow i)$, $i = 2, \dots, 6$, are trivial.

To prove $2) \Rightarrow 1)$ and $3) \Rightarrow 1)$ notice that points of X can be identifies with the minimal elements of the semilattices $\exp_\omega(X)$ and $SL(X)$ (every semilattice is partially ordered by the relation $s \leq s'$ if and only if $s \cdot s' = s'$).

To prove $4) \Rightarrow 1)$ and $5) \Rightarrow 1)$ notice that the map assigning to each $x \in X$ the idempotent xx^{-1} is a homeomorphism of X onto the set of minimal elements of the semilattice of idempotents of $IC(X)$ (or $IA(X)$).

Finally, consider the implication $6) \Rightarrow 1)$. First, remark that each minimal idempotent of the semigroup $I(X)$ is of the form xx^{-1} or $x^{-1}x$ for some $x \in X$. This can be easily seen analyzing the Petrich construction of a free inverse semigroup I_X over X [45]. Moreover, one may prove that the map $i_X : X \sqcup X^{-1} \to I(X)$ assigning to each $x \in X \sqcup X^{-1}$ the idempotent xx^{-1} is a homeomorphism of the topological sum $X \sqcup X^{-1}$ onto the subspace of all minimal idempotents of the semilattice $I(X)$.

Observe that $i_X(x^{-1}) = (i_X(x))^{-1}$ for each $x \in X \sqcup X^{-1}$. Define the retraction $|\cdot|_X : X \sqcup X^{-1} \to X$ letting $|x|_X = x$ if $x \in X$ and $|x|_X = x^{-1}$ if $x \in X^{-1}$ (we suppose that $(x^{-1})^{-1} = x$). By analogy, define the maps $i_Y : Y \sqcup Y^{-1} \to I(Y)$ and $|\cdot|_Y : Y \sqcup Y^{-1} \to Y$.

Now suppose $h : I(X) \to I(Y)$ is a topological isomorphism. Then it induces a homeomorphism between the sets of minimal idempotents. Hence $h \circ i_X(X \sqcup X^{-1}) = i_Y(Y \sqcup Y^{-1})$. Consider the maps $f : X \to Y$ and $g : Y \to X$, where $f(x) = \left|i_Y^{-1} \circ h \circ i_X(x)\right|_Y$ for $x \in X$ and $g(y) = \left|i_X^{-1} \circ h^{-1} \circ i_Y(y)\right|_X$ for $y \in Y$. We claim that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$, i.e. f is a homeomorphism. Indeed, fix any $x \in X$ and let $y = f(x)$. Then either $yy^{-1} = h(xx^{-1})$ or $y^{-1}y = h(xx^{-1})$. In the first case,

$$g(y) = \left|i_X^{-1} \circ h^{-1}(yy^{-1})\right|_X = \left|i_X^{-1} \circ h^{-1} \circ h(xx^{-1})\right|_X = \left|i_X^{-1}(xx^{-1})\right|_X = |x|_X = x.$$

Now consider the second case: $y^{-1}y = h(xx^{-1})$; $h(x^{-1}x)$ being a minimal idempotent of $I(Y)$ is of the form $h(x^{-1}x) = zz^{-1}$ for some $z \in Y \sqcup Y^{-1}$. The equality

$$\begin{aligned} y^{-1}yzz^{-1}y^{-1}yzz^{-1} &= h\left((xx^{-1})(x^{-1}x)(xx^{-1})(x^{-1}x)\right) = \\ &= h\left((x(x^{-1}x^{-1})(xx)(x^{-1}x^{-1})x\right) = h\left((xx^{-1}x^{-1}x\right) = y^{-1}yzz^{-1} \end{aligned}$$

implies $z = y$ or $z = y^{-1}$. In fact, the case $z = y^{-1}$ not possible because $zz^{-1} = h(x^{-1}x) \neq h(xx^{-1}) = y^{-1}y$. Consequently, $h(x^{-1}x) = yy^{-1}$. Then

$$g(y) = \left|i_X^{-1} \circ h^{-1}(yy^{-1})\right|_X = \left|i_X^{-1} \circ h^{-1} \circ h(x^{-1}x)\right|_X = \left|i_X^{-1}(x^{-1}x)\right|_X = \left|x^{-1}\right|_X = x.$$

Thus $g \circ f = \mathrm{id}_X$. By analogy we may verify that $f \circ g = \mathrm{id}_Y$. □

Therefore, the isomorphic classification of the free inverse semigroups $I(X)$ (resp. $IC(X)$, $IA(X)$, $SL(X)$, and $\exp_\omega(X)$) coincides with the topological classification of the spaces X. What can be said about topological classification of these semigroups? The situation here is more complicated. We begin with three characterizations.

Let l^2_f denote the linear hull of the standard orthonormal basis of the separable Hilbert space l^2. Evidently, l^2_f may be written as a countable union $l^2_f = \bigcup_{n=1}^\infty \mathbb{R}^n$, where

$$\mathbb{R}^n = \{(x_i) \in l^2 : x_i = 0 \text{ for } i > n\} \subset l^2.$$

Besides the topology inherited from l^2, there is another natural topology on l^2_f, that generated by the collection $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$. The obtained topological space is denoted by $\mathbb{R}^\infty$ (thus, a subset U is open in $\mathbb{R}^\infty$ if and only if $U \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for all $n \in \mathbb{N}$).

Theorem A [21]. *The free Lawson semilattice $\exp_\omega(X)$ is homeomorphic to l^2_f if and only if X is a connected locally path-connected σ-compact strongly countable-dimensional metrizable space.*

Theorem B [9]. *The free topological semilattice $SL(X)$ is homeomorphic to $\mathbb{R}^\infty$ if and only if the topology of X is generated by a countable increasing collection $\{X_n\}_{n=1}^\infty$ of finite-dimensional non-degenerate Peano continua.*

Recall that a space X is *strongly countable-dimensional*, provided X can be expressed as a countable union of closed finite-dimensional subspaces; a *Peano continuum* is, by definition, any connected locally connected metrizable compact space.

To give the third characterization we need to recall the definition of a *bitopological space* that is a triple (X, τ, τ') consisting of a set X and two topologies τ and τ' on X. Often a bitopological space (X, τ, τ') is identified with the pair $((X, \tau), (X, \tau'))$ of topological spaces. Two bitopological spaces (X, τ_X, τ'_X) and (X, τ_Y, τ'_Y) are called *homeomorphic* if there is a bijective map $h\colon X \to Y$ which is both a homeomorphism of (X, τ_X) and (Y, τ_Y) and a homeomorphism of (X, τ'_X) and (Y, τ'_Y). The pairs $(l^2_f, \mathbb{R}^\infty)$ and $(\exp_\omega(X), SL(X))$ are just examples of bitopological spaces.

Theorem C [10]. *The bitopological space* $(\exp_\omega(X), SL(X))$ *is homeomorphic to* $(l^2_f, \mathbb{R}^\infty)$ *if and only if* X *is a connected locally connected locally compact locally finite-dimensional metrizable space.*

These characterizations imply the following remarks.

Remark 2. There are spaces X, Y such that $\exp_\omega(X)$ and $\exp_\omega(Y)$ are homeomorphic, but $SL(X)$ and $SL(Y)$ are not (e.g. $X = [0, 1]$, $Y = l^2_f$).

Remark 3. There are spaces X, Y such that $SL(X)$ and $SL(Y)$ are homeomorphic, but $\exp_\omega(X)$ and $\exp_\omega(Y)$ are not (e.g. $X = [0, 1]$, $Y = \mathbb{R}^\infty$).

Remark 4. There are spaces X, Y such that the bitopological spaces $(\exp_\omega(X), SL(X))$ and $(\exp_\omega(Y), SL(Y))$ are homeomorphic, but X and Y are not (e.g. $X = [0, 1]$, $Y = (0, 1]$).

Remark 5. If $\exp_\omega(X)$ is homeomorphic to l^2_f and $SL(X)$ is homeomorphic to $\mathbb{R}^\infty$ then the bitopological space $(\exp_\omega(X), SL(X))$ is homeomorphic to $(l^2_f, \mathbb{R}^\infty)$.

Question 4. Do there exist X, Y such that

1) $\exp_\omega(X)$ is homeomorphic to $\exp_\omega(Y)$;
2) $SL(X)$ is homeomorphic to $SL(Y)$;
3) $(\exp_\omega(X), SL(X))$ is not homeomorphic to $(\exp_\omega(Y), SL(Y))$?

Question 5. Do there exist Peano continua X and Y such that $SL(X)$ and $SL(Y)$ are homeomorphic, but $\exp_\omega(X)$ and $\exp_\omega(Y)$ are not?

Some partial results are known also for the semigroups $IC(X)$ and $IA(X)$. We recall that a space X is called a *Euclidean retract* if X is homeomorphic to a retract of a finite-dimensional Euclidean space, see [11, §1] or [12].

Theorem D [29]. *If* X *is a Euclidean absolute retract, then the semigroups* $IC(X \times [0, 1])$ *and* $IA(X \times [0, 1])$ *are homeomorphic to* $\mathbb{R}^\infty \times \mathbb{Z}$.

In light of Theorems B and D the following question seems to be natural (cf. [50]).

Problem. *Characterize the topological spaces* X *whose free topological inverse semigroup* $I(X)$ *(resp.* $IC(X)$, $IA(X)$*) is locally homeomorphic to* $\mathbb{R}^\infty$.

Since for non-homeomorphic X, Y their free topological semilattices may be homeomorphic let us ask the following

Question 6. Describe the classes $\mathcal{P}$ of topological spaces, having the following property: if $I(X)$ is homeomorphic to $I(Y)$ and $X \in \mathcal{P}$, then $Y \in \mathcal{P}$.

Let us define two spaces X, Y to be *I-equivalent* if $I(X)$ is homeomorphic to $I(Y)$. Similarly, define IC-, IA-, SL-, and $\exp_\omega$-equivalent spaces.

We define a class $\mathcal{P}$ of topological spaces to be

- *topological* if any topological copy of a space $X \in \mathcal{P}$ belongs to $\mathcal{P}$;
- *closed-hereditary* if for every space $X \in \mathcal{P}$ every closed subspace of X belongs to the class $\mathcal{P}$;

- *closed under finite products* if $X \times Y \in \mathcal{P}$ for every $X, Y \in \mathcal{P}$;
- *closed under images of closed maps of finite order* if a Tychonoff space X belongs to $\mathcal{P}$, whenever there exists a surjective closed map $f\colon Y \to X$ of a space $Y \in \mathcal{P}$ such that $\sup_{x \in X} |f^{-1}(x)| < \aleph_0$;
- *σ-closed* if $X \in \mathcal{P}$ whenever X can be written as a countable union $X = \bigcup_{n=1}^{\infty} X_n$ of closed subsets $X_n \subset X$ with $X_n \in \mathcal{P}$;
- *weakly σ-closed* if $X \in \mathcal{P}$, whenever X can be written as a countable union $X = \bigcup_{n=1}^{\infty} X_n$ of subsets $X_n \subset X$ with $X_n \in \mathcal{P}$, such that $X = \bigcup_{n=1}^{\infty} \mathrm{Int}(X_n)$.

The following general theorem answers Question 6.

Theorem 11. *Suppose $\mathcal{P}$ is a topological closed-hereditary weakly σ-closed class of topological spaces, closed with respect to finite products and images of closed maps of finite order.*

1. *Suppose X and Y are I-, IC-, IA-, SL-, or $\exp_\omega$-equivalent Tychonoff spaces. If the class $\mathcal{P}$ is σ-closed and $X \in \mathcal{P}$, then $Y \in \mathcal{P}$.*
2. *Suppose X and Y are I-, IC-, IA- or SL-equivalent functionally Hausdorff spaces. If $X \in \mathcal{P}$ and Y is first countable then $Y \in \mathcal{P}$.*

Proof. Recall that $S(X) = \bigoplus_{k=1}^{\infty} X^k$ is a free topological semigroup over X and $S_n(X) = \bigoplus_{k=1}^{n} X^k$, $n \in \mathbb{N}$. Now suppose $X \in \mathcal{P}$ is a Tychonoff space. Let $p_X : S(X \sqcup X^{-1}) \to I(X)$ be the natural homomorphism. Let us show that the restriction $p_X|_{S_n(X \sqcup X^{-1})}$ is a closed map. For this consider the commutative diagram

$$\begin{array}{ccc}
S_n(X \sqcup X^{-1}) & \hookrightarrow & S_n(\beta X \sqcup \beta X^{-1}) \\
\downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_{\beta X}} \\
I(X) & \xrightarrow[I(i)]{} & I(\beta X)
\end{array}$$

where $I(i)$ is the homomorphism induced by the embedding $i\colon X \to \beta X$ of X into its Stone-Čech compactification. To show that $p_X|_{S_n(X \sqcup X^{-1})}$ is a closed map, fix any closed set $F \subset S_n(X \sqcup X^{-1})$ and let $\overline{F}$ be the closure of F in $S_n(\beta X \sqcup \beta X^{-1})$ which is, obviously, compact. Then $p_{\beta X}\left(\overline{F}\right)$ is compact, and consequently, $p_X(F) = (I(i))^{-1}\left(p_{\beta X}\left(\overline{F}\right)\right)$ is closed in $I(X)$. This proves that $p_X|_{S_n(X \sqcup X^{-1})}$ is a closed map of finite order (the latter being obvious). This implies $I_n(X) = p_X\left(S_n\left(X \sqcup X^{-1}\right)\right) \in \mathcal{P}$ is closed in $I(X)$.

Now suppose a Tychonoff space Y is I-equivalent to X. Since Y embeds as a closed subset into $I(Y)$, we get that Y is homeomorphic to a closed subset $\tilde{Y}$ in $I(X)$. Then $\tilde{Y} = \bigcup_{n=1}^{\infty}\left(\tilde{Y} \cap I_n(X)\right)$ is a countable union of closed subsets from the class $\mathcal{P}$ (which is closed-hereditary). In the case of a σ-closed class $\mathcal{P}$ this implies $\tilde{Y} \in \mathcal{P}$ and $Y \in \mathcal{P}$ because $\mathcal{P}$ is a topological class. Similar arguments work also in the case of IC-, IA-, SL-, and $\exp_\omega$-equivalences.

Suppose now $X \in \mathcal{P}$ and Y is first countable. We shall show that

$$\tilde{Y} = \bigcup_{n=1}^{\infty} \mathrm{Int}\left(\tilde{Y} \cap I_n(X)\right).$$

Fix any point $y \in \tilde{Y}$. We have to prove that y has a neighborhood $U \subset \tilde{Y}$ which lies in $I_n(X)$ for some $n \in \mathbb{N}$. Assuming the converse and using the fact that $\tilde{Y}$ is first countable at y, we could construct a sequence $(y_n) \subset \tilde{Y}$ convergent to y and such that $y_n \notin I_n(X)$ for every $n \in \mathbb{N}$.

Since $K = \{y_0\} \cup \{y_n : n \in \mathbb{N}\}$ is compact in $I(X)$, Proposition 3 implies that $K \subset I_n(X)$ for some $n \in \mathbb{N}$, a contradiction with the choice of the sequence (y_n). Thus $\tilde{Y} = \bigcup_{n=1}^{\infty} \operatorname{Int}\left(\tilde{Y} \cap I_n(X)\right)$ and $\tilde{Y} \in \mathcal{P}$ (because $\mathcal{P}$ is weakly σ-closed). The same arguments work also in the cases of IC-, IA-, and SL-equivalences (but not $\exp_\omega$-equivalences). □

Remark 6. Theorem 11 implies that in the class of metrizable (separable) spaces many important topological properties are preserved by I-, IC-, IA- and SL- equivalences. Among such properties there are dimension properties such as the local finite-dimensionality and the (strong) countable-dimensionality (see [24, Ch.5]) as well as many descriptive properties such as belonging to a given Borel or projective class (see [33]).

One application

It is known that each topological group is a quotient-group of a zero-dimensional group [3]. In this section we prove a semigroup counterpart of this result. Recall that a topological space X is called *sequential* if a subset $F \subset X$ is closed if and only if for every convergent in X sequence $(x_n) \subset F$ we have $\lim x_n \in F$ [23, §1.6].

Theorem 12. *For every topological inverse semigroup X which is a k-space (resp. a sequential space) there is a quotient homomorphism $h\colon Y \to X$ of a topological inverse semigroup Y which is a topological sum of zero-dimensional (resp. countable) k_ω-spaces. Moreover if X is Clifford, Abelian or a semilattice, then the same is Y.*

Proof. Consider firstly the case where X is sequential. Then by Franklin Theorem [25], there is a quotient map $f\colon Y \to X$ of a topological sum of all convergent sequences of X onto the space X. By the definition of a free topological inverse semigroup, there is a unique homomorphism $h\colon I(Y) \to X$ extending the map f. Evidently, h is a quotient map. Theorems 3, 4 and Propositions 1, 2 imply that $I(Y)$ is a topological sum of countable k_ω-spaces.

If X is a k-space then there is a quotient map $f\colon Y \to X$ of a topological sum of zero-dimensional compacta onto X. Using Theorems 3, 4, and 9, and Propositions 1, 2 we get X is a quotient semigroup of the free topological inverse semigroup which is a topological sum of zero-dimensional k_ω-spaces.

If X is Clifford, Abelian or a semilattice, replace in the preceding arguments $I(Y)$ by the corresponding free semigroups. □

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